Infinite dimensional port Hamiltonian systems

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CAS Seminar - Topic: PDE
1 Introduction
   - Context
   - Outline

2 Infinite dimensional linear 1-D case
   - A simple example
   - Considered class of systems
   - Dirac structure and PHS on Hilbert Space
   - Boundary control systems
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3 Model reduction
   - Mixed finite element method
   - Spectral properties

4 Conclusion
Recent technological progresses and physical knowledge allow to go toward the use of complex systems:

- Highly nonlinear.
- Involving numerous physical domains and possible heterogeneity.
- With distributed parameters or organized in network.
Recent technological progresses and physical knowledge allow to go toward the use of complex systems:

- Highly nonlinear.
- Involving numerous physical domains and possible heterogeneity.
- With distributed parameters or organized in network.

New issue for system control theory

Modelling step is important → the physical properties can be advantageously used for analysis, control or simulation purposes
Example 1: Ionic Polymer Metal Composite

- Electromechanical system.
- 3 scales: Polymer-electrode interface, diffusion in the polymer, beam bending.
Example 2: Nanotweezer for DNA manipulation

- Multiphysic system.
- Infinite dimensional system.
Example 3: Adsorption process

- Multiscale heterogeneous system.
- Dynamic behavior driven by irreversible thermodynamic laws.
Example 3: Adsorption process

- Multiscale heterogeneous system.
- Considered phenomena:
  - Fluid scale: convection, dispersion.
  - Pellet scale: diffusion (Stephan-Maxwell).
  - Microscopic scale: Knudsen law.
A unified approach: the port Hamiltonian framework

Emphasis the geometric structure related to power exchanges $\rightsquigarrow$ powerful for analysis, model reduction and control.

Today: Infinite dimensional linear 1-D case and discretization
A unified approach: the port Hamiltonian framework

Emphasis the geometric structure related to power exchanges $\rightsquigarrow$ powerful for analysis, model reduction and control.

Today: Infinite dimensional linear 1-D case and discretization

\[
\begin{align*}
\dot{x}(t) &= \mathcal{J} x(t), \\
\mathcal{B} x(t) &= u(t).
\end{align*}
\]
A unified approach : the port Hamiltonian framework

Emphasis the geometric structure related to power exchanges \( \sim \) powerful for analysis, model reduction and control.

Today : Infinite dimensional linear 1-D case and discretization

Other works :

- Infinite dimensional linear 2-D, 3-D cases (joint work with B. Maschke and H. Zwart).
- Infinite dimensional port Hamiltonian systems 2D, 3D cases (cf A.v.d. Schaft and B. Maschke).
- Infinite dimensional NL systems : Passivity and PHS vs. Stability and Riemann Invariants (Joint work : V. Dos Santos, B. Maschke).
- Non linear control : IDA-PBC, Entropy based control of Chemical reactors (Joint work : F. Couenne).
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Vibrating string:

\[
\frac{\partial^2 u(z, t)}{\partial t^2} = \frac{1}{\mu(z)} \frac{\partial}{\partial z} \left( T(z) \frac{\partial u(z, t)}{\partial z} \right)
\]

The classical modelling is based on the wave equation: Newton’s law + Hooke’s law (restoring force proportional to the deformation).

The structure of the model is not apparent. **How to choose the boundary conditions?**
Let choose as state variables the energy variables:

- the strain \( \epsilon = \frac{\partial u(z,t)}{\partial z} \)
- the elastic momentum \( p = \mu(z) v(z, t) \)

The **total energy** is given by: \( H(\epsilon, p) = U(\epsilon) + K(p) \)

- \( U(\epsilon) \) is the **elastic potential energy**:
  \[
  U(\epsilon) = \int_{a}^{b} \frac{1}{2} T(z) \left( \frac{\partial u(z, t)}{\partial z} \right)^2 = \int_{a}^{b} \frac{1}{2} T \epsilon(z, t)^2 
  \]
  where \( T(z) \) denotes the elastic modulus.

- \( K(v) \) is the **kinetic co-energy**:
  \[
  K(p) = \int_{a}^{b} \frac{1}{2} \mu(z) v(z, t)^2 = \int_{a}^{b} \frac{1}{2} \frac{1}{\mu(z)} p^2(z, t) 
  \]
  where \( \mu(z) \) denotes the string mass.
The vector of fluxes is given by:

$$ \beta = \begin{pmatrix} \nu(t, z) \\ \sigma \end{pmatrix} $$

where $\nu(z, t)$ is the velocity and $\sigma(z, t) = T(z)\epsilon(z, t)$ the stress. The vector of fluxes $\beta$ may be expressed in term of the generating forces:

$$ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \epsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} $$

canonical inerdomain coupling generating forces
From the conservation laws:

\[ \frac{\partial}{\partial t} \begin{pmatrix} \epsilon \\ p \end{pmatrix} + \frac{\partial}{\partial z} \begin{pmatrix} v \\ \sigma \end{pmatrix} = 0 \]

Consequently

\[ \frac{\partial}{\partial t} \begin{pmatrix} \epsilon \\ p \end{pmatrix} = - \frac{\partial}{\partial z} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \epsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \]

and

**PDEs:**

\[ \frac{\partial}{\partial t} \begin{pmatrix} \epsilon \\ p \end{pmatrix} = \begin{pmatrix} 0 & - \frac{\partial}{\partial z} \\ - \frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \epsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \Leftrightarrow \frac{\partial^2 u(z, t)}{\partial t^2} = \sigma^2 \frac{\partial^2 u(z, t)}{\partial z^2} + BC \]
Underlying structure:

\[
\frac{\partial}{\partial t} \begin{pmatrix} \epsilon \\ p \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} T(z) & 0 \\ 0 & \frac{1}{\mu(z)} \end{pmatrix} \begin{pmatrix} \epsilon \\ p \end{pmatrix}
\]

\[f = \text{matrix differential operator}
\]

\[e = \text{driving force}\]

Hamiltonian operator \(\mathcal{J}\) is **skew-symmetric only for function with compact domain strictly** in \(Z\):

\[
\int_a^b \begin{pmatrix} e_1 & e_2 \end{pmatrix} \mathcal{J} \begin{pmatrix} e'_1 \\ e'_2 \end{pmatrix} + \begin{pmatrix} e'_1 & e'_2 \end{pmatrix} \mathcal{J} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = -[e_1 e'_2 + e_2 e'_1]_a^b
\]
Power balance equation:

\[
\frac{d}{dt} H(\epsilon, p) = \int_a^b \left( \frac{\delta H}{\delta \epsilon} \frac{\partial \epsilon}{\partial t} + \frac{\delta H}{\delta p} \frac{\partial p}{\partial t} \right) dz
\]

\[
= - \int_a^b \left( \frac{\delta H}{\delta \epsilon} \frac{\partial}{\partial z} \frac{\delta H}{\delta p} + \frac{\delta H}{\delta p} \frac{\partial}{\partial z} \frac{\delta H}{\delta \epsilon} \right) dz
\]

\[
= - \left[ \frac{\delta H}{\delta \epsilon} \frac{\delta H}{\delta p} \right]_a^b
\]

If driving forces are zero at the boundary, the total energy is conserved, else there is a flow of power at the boundary. Define two port boundary variables as follows:

\[
\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \left( \frac{\delta H}{\delta \epsilon} \right)_{a,b}
\]
A simple example

The linear space $\mathcal{D} \ni (f_1, f_2, e_1, e_2, f_\partial, e_\partial)$

- $\begin{pmatrix} f_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$
- $\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} |_{a,b}$

defines a **Dirac structure** $:\mathcal{D} = \mathcal{D}^\perp$ with respect to the symmetric pairing:

$$\int_a^b e_1 f_1 dz + \int_a^b e_2 f_2 dz + [f_\partial e_\partial]_a^b$$

**Port Hamiltonian system**

$$\left( \frac{\partial}{\partial t} \alpha, \frac{\delta H}{\delta \alpha}, f_\partial, e_\partial \right) \in \mathcal{D}$$
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Considered class of systems:

\[
\frac{\partial x}{\partial t}(t, z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) \mathcal{L}(z) x(t, z), \quad x(0, z) = x_0(z),
\]

\[
\uparrow
\]

\[
\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{pmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{pmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}
\]

with \( e_p = S f_p \) where \( S \) is a coercive operator.

\[
\begin{pmatrix} f \\ f_p \end{pmatrix} \in \mathcal{F}, \quad \begin{pmatrix} e \\ e_p \end{pmatrix} \in \mathcal{E} \quad \text{and} \quad \mathcal{E} = \mathcal{F} = L_2((a, b), \mathbb{R}^n) \times L_2((a, b), \mathbb{R}^n)
\]

Covers models of: beams, wave, plates, (with or without damping) and also systems of diffusion/convection, chemical reactors ...
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The system is defined by:

\[ f = \mathcal{J} e \]

- Let the space of flow variables, \( \mathcal{F} \), and the space of effort variables, \( \mathcal{E} \), be real Hilbert spaces.
- Define the space of bond variables as \( \mathcal{B} = \mathcal{F} \times \mathcal{E} \) endowed by the natural inner product

\[ \langle b^1, b^2 \rangle = \langle f^1, f^2 \rangle_\mathcal{F} + \langle e^1, e^2 \rangle_\mathcal{E}, \quad b^1 = (f^1, e^1), \quad b^2 = (f^2, e^2) \in \mathcal{B}. \]

In order to define a Dirac structure, let us moreover endow the bond space \( \mathcal{B} \) with a canonical symmetrical pairing, i.e., a bilinear form defined as follows:

\[ \langle b^1, b^2 \rangle_+ = \langle f^1, r_\mathcal{E} f^2 \rangle_\mathcal{F} + \langle e^1, r_\mathcal{F} e^2 \rangle_\mathcal{E}, \quad b^1 = (f^1, e^1), \quad b^2 = (f^2, e^2) \in \mathcal{B}. \quad (1) \]
Denote by $\mathcal{D}^\perp$ the orthogonal subspace to $\mathcal{D}$ with respect to the symmetrical pairing:

$$\mathcal{D}^\perp = \left\{ b \in \mathcal{B} \mid \langle b, b' \rangle_+ = 0 \text{ for all } b' \in \mathcal{D} \right\}. \quad (2)$$

**Definition** : A Dirac structure $\mathcal{D}$ on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ is a subspace of $\mathcal{B}$ which is maximally isotropic with respect to the canonical symmetrical pairing, i.e.,

$$\mathcal{D}^\perp = \mathcal{D}. \quad (3)$$

$$\begin{pmatrix} f \\ e \end{pmatrix} \in \mathcal{D} \iff \text{Power conservation}$$
Definition: Let $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ be the bound space defined above and consider the Dirac structure $\mathcal{D}$ and the Hamiltonian function $\mathcal{H}(x)$ with $x$ the energy variables. Define the flow variables, $f \in \mathcal{F}$ as the time variation of the energy variables and the effort variables $e \in \mathcal{E}$ as the variational derivative of $\mathcal{H}(x)$. The system

$$(f, e) = \left( \frac{\partial x}{\partial t}, \frac{\delta \mathcal{H}}{\delta x} \right) \in \mathcal{D}$$

is a Port Hamiltonian system with total energy $\mathcal{H}(x)$.
Parametrization:

\[ J e = \sum_{i=0}^{N} P(i) \frac{d^i e}{dz^i}(z) \quad z \in [a, b], \]

where \( e \in H^N((a, b); \mathbb{R}^n) \) and \( P(i), i = 0, \ldots, N, \) is a \( n \times n \) real matrix with \( P_N \) non singular and \( P_i = P_i^T (-1)^{i+1} \). Let define

\[
Q = \begin{pmatrix}
P_1 & P_2 & \cdots & P_N \\
-P_2 & -P_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{N-1}P_N & 0 & \cdots & 0
\end{pmatrix}
\]

Back to the Vibrating string

\[
\begin{pmatrix}
\frac{\partial}{\partial t} \begin{pmatrix}
\epsilon \\
p
\end{pmatrix} \\
f
\end{pmatrix}_{P_1} = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial z} T(z) \\
0
\end{pmatrix}_{\frac{1}{\mu(z)}} \begin{pmatrix}
\epsilon \\
p
\end{pmatrix}_{e}, \quad Q = P_1
\]
We define the symmetrical pairing (not depending on $\mathcal{J}$) of and the port variables associated with $\mathcal{J}$. Let $\mathcal{F} = \mathcal{E} = L^2((a, b); \mathbb{R}^n) \times \mathbb{R}^{nN}$ and define $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ with the following canonical symmetrical pairing:

$$\langle (f^1, f^1_\partial, e^1, e^1_\partial) (f^2, f^2_\partial, e^2, e^2_\partial) \rangle = \langle e^1, f^2 \rangle_{L^2} + \langle e^2, f^1 \rangle_{L^2} - \langle e^1_\partial, f^2_\partial \rangle - \langle e^2_\partial, f^1_\partial \rangle,$$

**Definition**: The port variables $(e_\partial, f_\partial) \in \mathbb{R}^{nN}$ associated with $\mathcal{J}$ are defined by:

$$
\begin{pmatrix}
    f_\partial \\
    e_\partial
\end{pmatrix}
= R_{\text{ext}}
\begin{pmatrix}
    e(b) \\
    \vdots \\
    \frac{d^{N-1}e}{dz^{N-1}}(b) \\
    e(a) \\
    \vdots \\
    \frac{d^{N-1}e}{dz^{N-1}}(a)
\end{pmatrix},
R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
    Q & -Q \\
    I & I
\end{pmatrix}$$
The subspace $\mathcal{D}_{\mathcal{J}}$ of $\mathcal{B}$ defined as

$$
\mathcal{D}_{\mathcal{J}} = \left\{ \begin{pmatrix} f \\ f_\partial \\ e \\ e_\partial \end{pmatrix} \right| e \in H^N((a, b); \mathbb{R}^n), \mathcal{J} e = f, \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ \vdots \\ \partial_z^{N-1} e(a) \end{pmatrix} \right\}
$$

is a Dirac structure, that means that $\mathcal{D} = \mathcal{D}^\perp$.

Back to the **Vibrating string**

$$
\frac{\partial}{\partial t} \begin{pmatrix} \epsilon \\ p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon \\ p \end{pmatrix}, \quad \frac{\partial}{\partial z} \begin{pmatrix} \frac{1}{\mu(z)} \epsilon \\ p \end{pmatrix}, \quad Q = P_1
$$

$$
\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ l & l \end{pmatrix} \begin{pmatrix} e(b) \\ e(a) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} T(a)\epsilon(a) - T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} - \frac{p(b)}{\mu(b)} \end{pmatrix} + \begin{pmatrix} T(a)\epsilon(a) + T(b)\epsilon(b) \\ \frac{p(a)}{\mu(a)} + \frac{p(b)}{\mu(b)} \end{pmatrix}
$$
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Boundary Control Systems:

**Theorem:** Let $W$ be a $nN \times 2nN$ full rank matrix. The system

$$
\begin{align*}
\dot{x}(t) &= J\mathcal{L}x(t) \\
    u(t) &= Bx(t) = W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}
\end{align*}
$$

is a boundary control system, where $A_W = (J\mathcal{L})_{\ker B}$ is the generator of a contraction semigroup on $L_2((a, b), \mathbb{R}^n)$ if and only if

$$
W\Sigma W^T \geq 0 \text{ where } \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
$$
Let define the linear mapping $C : H^N((a, b), \mathbb{R}^n) \rightarrow \mathbb{R}^{nN}$ as

$$Cx(t) := \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$$

and the output as $y(t) = Cx(t)$, then for $u \in C^2((0, \infty); \mathbb{R}^{nN})$ and $x(0) - Bu(0) \in D(J_W)$ the following balance equation is satisfied:

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}.$$

where $P_W, \tilde{W} = \begin{pmatrix} W \Sigma W^T & W \Sigma \tilde{W} \\ \tilde{W}^T \Sigma W^T & \tilde{W} \Sigma \tilde{W}^T \end{pmatrix}^{-1}$
Boundary control systems

Using $W = S(I + V, I - V)$:

\[
\begin{align*}
V = 0 & \quad \left\{ \begin{array}{l}
\dot{x}(t) = Jx(t), \\
u(t) = \frac{1}{2}(f_\partial(t) + e_\partial(t)) \\
y(t) = \frac{1}{2}(f_\partial(t) - e_\partial(t))
\end{array} \right. \\
\implies & \quad \text{boundary control system, with} \\
& \quad \text{the associated semigroup} \\
& \quad \text{a contraction} \\
& \quad \frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2
\end{align*}
\]

\[
\begin{align*}
V = I & \quad \left\{ \begin{array}{l}
\dot{x}(t) = Jx(t) \\
u(t) = f_\partial(t) \\
y(t) = -e_\partial(t)
\end{array} \right. \\
\implies & \quad \text{boundary control system, with} \\
& \quad \text{the associated semigroup} \\
& \quad \text{unitary} \\
& \quad \frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t)
\end{align*}
\]

Vibrating string :

\[
V = 0 \implies u = \frac{1}{\sqrt{2}} \begin{pmatrix} T(a)\epsilon(a) \\ \frac{p(a)}{\mu(a)} \end{pmatrix} \quad \text{and} \quad y = \frac{1}{\sqrt{2}} \begin{pmatrix} -T(b)\epsilon(b) \\ -\frac{p(b)}{\mu(b)} \end{pmatrix}
\]
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Control using feedback at the boundary

Open loop system

\[
\begin{cases}
\dot{x} = \mathcal{J}_L x \\
u = W_{imp} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\
y = C_{imp} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}
\end{cases}
\]

Closed loop system

\[
\begin{cases}
\dot{x} = \mathcal{J}_L x \\
r = (W_{imp} + \alpha C_{imp}) \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\
y = C_{imp} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}
\end{cases}
\]
The closed loop system described by

\[
\begin{align*}
    \dot{x} &= J_L x \\
    r &= (W_{imp} + \alpha C_{imp}) \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \\
    y &= C_{imp} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}
\end{align*}
\]

is a boundary control system. Furthermore, the operator

\[ A_s = J_L|_{D(A_s)} \]

generates a contraction semigroup on

\[ X = L_2 ((a, b); \mathbb{R}^n) \]

where

\[
D(A_s) = \left\{ x \in D(J) \mid \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \ker W \right\}
\]

and \( W = (W_{imp} + \alpha C_{imp}) \) is a full rank \( nN \times 2nN \) matrix.
**Theorem**: (Assumption $(\lambda - A_s)^{-1} : X \to X$ is a compact operator for $\lambda > 0$). Then the system described by $(VV^T = 0)$:

\[
\begin{align*}
\dot{x} &= Jx \\
\begin{bmatrix}
r \\
y
\end{bmatrix} &= (W_{imp} + \alpha C_{imp}) \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}
\end{align*}
\]

with $r = 0$ and $\alpha > 0$ is globally asymptotically stable. For any $x(0) \in X$ the unique (classical or weak) solution $x(t) = T(t)x(0)$ of the closed loop system asymptotically approaches to zero, i.e.

\[
\lim_{\infty} \|x(t)\|_X = 0
\]
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Conclusion
Let us restrict the previous class to:

\[
\frac{\partial x}{\partial t}(t, z) = P_1 \frac{\partial z}{\partial t} (\mathcal{L}x)(t, z) + (P_0 - G_0)\mathcal{L}x(t, z)
\]

Let consider a BCS as defined previously, assuming that \( u(t) = 0, \ \forall t \geq 0 \). Then the system is **exponentially stable** if

either \( \| (\mathcal{L}x(b)) \|_\mathbb{R}^2 \leq k_1 (\langle \alpha y, y \rangle_\mathbb{R} + \langle G_0 \mathcal{L}x(t), \mathcal{L}x(t) \rangle_\mathbb{R} ) \)

or \( \| (\mathcal{L}x(a)) \|_\mathbb{R}^2 \leq k_1 (\langle \alpha y, y \rangle_\mathbb{R} + \langle G_0 \mathcal{L}x(t), \mathcal{L}x(t) \rangle_\mathbb{R} ) \)
As state variables we choose

\[
x_1 = \frac{\partial w}{\partial z} - \phi : \text{ shear displacement,}
\]
\[
x_2 = \rho \frac{\partial w}{\partial t} : \text{ transverse momentum distribution,}
\]
\[
x_3 = \frac{\partial \phi}{\partial z} : \text{ angular displacement,}
\]
\[
x_4 = I\rho \frac{\partial \phi}{\partial t} : \text{ angular momentum distribution.}
\]

Then the model of the beam can be rewritten as

\[
\frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} K x_1 \\ \frac{1}{\rho} x_2 \\ EI x_3 \\ \frac{1}{l_\rho} x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} K x_1 \\ \frac{1}{\rho} x_2 \\ EI x_3 \\ \frac{1}{l_\rho} x_4 \end{pmatrix} \end{pmatrix}. 
\]
One can define the boundary port variables:

\[
\begin{pmatrix}
    f_{\partial} \\
e_{\partial}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
P_1 & -P_1 \\
1 & 1
\end{bmatrix} \begin{pmatrix}
    (Lx)(b) \\
    (Lx)(a)
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
    (\rho^{-1}x_2)(b) - (\rho^{-1}x_2)(a) \\
    (Kx_1)(b) - (Kx_1)(a) \\
    (l_\rho^{-1}x_4)(b) - (l_\rho^{-1}x_4)(a) \\
    (Elx_3)(b) - (Elx_3)(a) \\
    (Kx_1)(b) + (Kx_1)(a) \\
    (\rho^{-1}x_2)(b) + (\rho^{-1}x_2)(a) \\
    (Elx_3)(b) + (Elx_3)(a) \\
    (l_\rho^{-1}x_4)(b) + (l_\rho^{-1}x_4)(a)
\end{pmatrix}.
\] (4)

Let us consider stabilization by applying velocity feedback i.e. following BC:

\[
\frac{1}{\rho(a)} x_2(a) = 0, \quad \frac{1}{l_\rho(a)} x_4(a) = 0,
\]

\[
K(b)x_1(b, t) = -\alpha_1 \frac{1}{\rho(b)} x_2(b, t), \quad El(b)x_3(b, t) = -\alpha_2 \frac{1}{l_\rho(b)} x_4(b)
\]
Input mapping:

\[ W = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \alpha_1 & 1 & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 & 0 & 1 & \alpha_2 \end{bmatrix} \]

Then \[ W \Sigma W^T = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{bmatrix} \]

As output we can choose

\[ y = \begin{bmatrix} -K(a)x_1(a) \\ -(EI)(a)x_3(a) \\ \frac{1}{\rho(b)}x_2(b) \\ \frac{1}{I_\rho(b)}x_4(b) \end{bmatrix}, \quad \text{with} \quad \tilde{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]

Then

\[ P_{W,W}^{-1} = \begin{bmatrix} 2\alpha & I \\ I & 0 \end{bmatrix}, P_{W,W} = \begin{bmatrix} 0 & I \\ I & -2\alpha \end{bmatrix} \]
Energy balance:

\[
\frac{d}{dt} E(t) = \frac{d}{dt} \|x(t)\|_{\mathcal{L}}^2 = \langle u(t), y(t) \rangle u - \langle \alpha y(t), y(t) \rangle_{\mathbb{R}}
\]

where

\[
\langle \alpha y(t), y(t) \rangle_{\mathbb{R}} = \alpha_1 |(\rho^{-1} x_2)(b, t)|^2 + \alpha_2 |(l^{-1} x_4)(b, t)|^2
\]

Then

\[
\| (\mathcal{L} x(b)) \|_{\mathbb{R}}^2 = |(k x_1)(b)|^2 + |(\rho^{-1} x_2)(b)|^2 + |(E l x_3)(b)|^2 + |(l^{-1} x_4)(b)|^2
\]

\[
= (\alpha_1^2 + 1)||\rho^{-1} x_2)(b, t)|^2 + (\alpha_2^2 + 1)||l^{-1} x_4)(b)|^2
\]

\[
\leq \kappa \langle \alpha y(t), y(t) \rangle_{\mathbb{R}}
\]

\[\Rightarrow\] Stability
1 Introduction
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   • Considered class of systems
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   • Boundary control systems
   • Stability, control feedback

3 Model reduction
   • Mixed finite element method
   • Spectral properties

4 Conclusion
Back to the example of the adsorption column:

Classical formulamtion.

\[
\begin{align*}
\frac{\partial q(z,t)}{\partial t} + \frac{\partial N}{\partial z} + \alpha F_{ads} &= 0 \\
\frac{\partial q_p(z,t)}{\partial t} - F_{ads} &= 0
\end{align*}
\]

$q$ mole density in the fluid phase, $q_p$ in the adsorbed phase, total flux

\[N = N_{conv} + N_{disp} = \nu q - D \frac{\partial \mu}{\partial z},\ F_{ads} = k_1 a (\mu - \mu_p)\]

with Dankwert boundary conditions $N|_0 = \nu q_{in}$ et $\frac{\partial q}{\partial z}|_L = 0$
Back to the example of the adsorption column:

**Classical formulation.**

\[
\begin{align*}
\frac{\partial q(z,t)}{\partial t} + \frac{\partial N}{\partial z} + \alpha F_{ads} &= 0 \\
\frac{\partial q_p(z,t)}{\partial t} - F_{ads} &= 0
\end{align*}
\]  

with Dankwert boundary conditions \( N|_0 = vq_{in} \) et \( \frac{\partial q}{\partial z}|_L = 0 \)
Back to the example of the adsorption column:

**Classical formulation.**

\[
\frac{\partial q(z,t)}{\partial t} + \frac{\partial N}{\partial z} + \alpha F_{ads} = 0 \\
\frac{\partial q_p(z,t)}{\partial t} - F_{ads} = 0
\]  

(5)

with Dankwert boundary conditions \( N|_0 = v q_{in} \) et \( \frac{\partial q}{\partial z}|_L = 0 \)

**Port Hamiltonian formulation**

\[
\begin{bmatrix}
f_q \\
F_{disp} \\
\alpha f_{qp} \\
F_{ads}
\end{bmatrix} = 
\begin{bmatrix}
0 & d & 0 & -1 \\
d & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0
\end{bmatrix} 
\begin{bmatrix}
\mu \\
N \\
\mu_p \\
N_{ads}
\end{bmatrix}
\]  

(6)

with constitutive relations

\[
N = -D \ast F_{disp} + v \ast q = -D \ast F_{disp} + \frac{v}{b} \mu \\
N_{ads} = \alpha a k_1 F_{ads} \\
\mu = bq
\]  

(7)
Back to the example of the adsorption column:

### Classical formulation.

\[
\begin{align*}
\frac{\partial q(z,t)}{\partial t} + \frac{\partial N}{\partial z} + \alpha F_{ads} &= 0 \\
\frac{\partial q_p(z,t)}{\partial t} - F_{ads} &= 0
\end{align*}
\]

(5)

with Dankwert boundary conditions \( N|_0 = v q_{in} \) et \( \frac{\partial q}{\partial z}|_L = 0 \)

### Dirac structure: interconnection structure

\[
\begin{bmatrix}
  f_q \\
  F_{disp} \\
  \alpha f_{qp} \\
  F_{ads}
\end{bmatrix} =
\begin{bmatrix}
  0 & d & 0 & -1 \\
  d & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
  \mu \\
  N \\
  \mu_p \\
  N_{ads}
\end{bmatrix}
\]

(6)

with the boundary port variables

\[
\begin{pmatrix}
  f_{\partial} \\
  e_{\partial}
\end{pmatrix} =
\begin{pmatrix}
  N(0) \\
  N(L) \\
  \mu(0) \\
  -\mu(L)
\end{pmatrix}
\]
Constitutive thermodynamical law

\[ \mu = f(q) \]

Constitutive dispersion law

\[ N_0 \rightarrow \frac{\partial q}{\partial t} \rightarrow N \rightarrow \mu \rightarrow \frac{dN}{dt} \rightarrow \mu_0 \rightarrow N_{\text{disp}} \rightarrow -d\mu \rightarrow N_{\text{conv}} \rightarrow \bar{f}(d\mu) \]

Accumulation

Interconnection

Dissipation
Goal of the discretization scheme: preserve the structure
Remark 3.1 The Hodge star operator $\ast$ is a linear operator mapping $p$ forms on $V$ to $(n - p)$ forms i.e. :

$$\ast : \Lambda^p(V) \rightarrow \Lambda^{n-p}(V)$$

In cartesian coordinates, consider the functions $g(z)$ and the 1-form $f(z) = g(z) \, dz$ then

$$\ast f(z) = g(z).$$

$q_L$s and $q_L$ being 1-forms, $\ast q_L$ and $\ast q_L$ are 0-forms.

(3) Closure equation associated with the diffusion - We use Knudsen law [19] to present the diffusion in the adsorbed phase (microporous scale). That is to say we only consider the friction exerted by the solid on the adsorbed species. So the constitutive relation representing the diffusion is:

$$f_{mic}^2 = -D_{mic} \ast q_{LRT} \ast e_{mic}^2$$

(15)

where $D_{mic}$ is the diffusion constant.

4 Model reduction based on geometrical properties

4.1 General concept

As previously mentioned, the port based modelling of the adsorption column is based on basic elements having well defined energetic behavior as already depicted in Figure 3. The distributed aspect of this port based model is essentially supported by the Dirac structure which links power exchanges within the spatial domain and through the boundaries. The proposed discretization method consists in splitting the initial structured infinite dimensional model into $N$ finite dimensional sub-models (finite elements) with the same energetic behavior (cf Figure 4). Furthermore, the support functions used for the interpolation of both effort and flow variables are different to have enough degrees of freedom to guarantee the conservation of the structural properties.

Figure 4: Principle of the spatial discretization
The variables $dN = N_d$ and $d\mu = \mu_d$ are one-forms and are approximated on $\Omega_{ab}$ by:

\[ \overline{N}_d(t, z) = N_{ab}^d(t) w_{ab}^N(z) \]
\[ \overline{e}_2(t, z) = \mu_{ab}^d(t) w_{ab}^\mu(z) \]

where the support one-forms $w_{ab}^N$ and $w_{ab}^\mu$ satisfy: $\int_{\Omega_{ab}} w_{ab}^N = \int_{\Omega_{ab}} w_{ab}^\mu = 1$.

The variables $\mu$ and $N$ are zero-forms and are approximated on $\Omega_{ab}$ by:

\[ \overline{\mu}(t, z) = \mu(a) w_a^\mu(z) + \mu(b) w_b^\mu(z), \]
\[ \overline{N}(t, z) = N(a) w_a^N(z) + N(b) w_b^N(z) \]

where the support zero-forms $w_a^\mu, w_b^\mu, w_a^N$ et $w_b^N$ satisfy:

\[ w_a^\mu(a) = 1, \quad w_a^\mu(b) = 0, \quad w_b^\mu(a) = 0, \quad w_b^\mu(b) = 1 \]
\[ w_a^N(a) = 1, \quad w_a^N(b) = 0, \quad w_b^N(a) = 0, \quad w_b^N(b) = 1, \]
The approximated variables must verify the relation induced by the interconnection structure:

\[
\begin{align*}
\overline{N_d} &= dN \\
\overline{\mu_d} &= d\mu
\end{align*}
\]

\[
\begin{align*}
N_{ab}^d &= N(b) - N(a) = N_\partial^b(t) - N_\partial^a(t) \\
\mu_{ab}^d &= \mu(b) - \mu(a) = -\mu_\partial^b(t) - \mu_\partial^a(t)
\end{align*}
\]

Write the net power using the previous interpolation formulae:

\[
\overline{P}_{ab}^{\text{net}} = \mu_{ab}^d \overline{N}_{ab}^d + \mu_{ab}^d \overline{\mu}_{ab}^d + \left[\mu_\partial^b N_\partial^b - \mu_\partial^a N_\partial^a\right]
\]

and identify with the real one, we obtain for \(\mu_{ab}^{\text{ab}}\) and \(N_{ab}^{\text{ab}}\):

\[
\begin{align*}
\mu_{ab}^{\text{ab}} &= \alpha_{ab} \mu_\partial^a(t) - (1 - \alpha_{ab}) \mu_\partial^b(t) \\
N_{ab}^{\text{ab}} &= (1 - \alpha_{ab}) N_\partial^a(t) + \alpha_{ab} N_\partial^b(t)
\end{align*}
\]

where the parameter \(\alpha_{ab} = \int_{\Omega_{ab}} w_\mu^a w_{ab}^N \in [0, 1]\).

Relations between \(f_{ab} = \left[N_{ab}^d \ N_{ab}^a \ N_\partial^a \ N_\partial^b\right]^T\), \(e_{ab} = \left[\mu_{ab}^{\text{ab}} \ \mu_{ab}^d \ \mu_\partial^a \ \mu_\partial^b\right]^T\) define again a finite dimensional Dirac structure.
Objective: compute $N_{\text{disp}}^{ab}$ from $\mu_d^{ab}$ such that the power behavior of the dissipation element (dispersion law) is preserved: $N_{\text{disp}}^{ab}\mu_d^{ab} = \int_a^b \mu_d \wedge \overline{N}_{\text{disp}}$ as well as the constitutive law of the dissipative element. Let us write the approximate instantaneous power

$$\overline{P}_R^{ab} = -K_{ab}D(\mu_d^{ab}(t))^2$$

with $K_{ab}$ depending on forms $w$ only.

$$N_{\text{disp}}^{ab} = \frac{\partial \overline{P}_R^{ab}}{\partial \mu_d^{ab}} = -2K_{ab}D\mu_d^{ab}(t)$$

Objective: compute $\mu^{ab}$ with respect to $q^{ab}$.

$q(t, z), \dot{q}$ and $N_d$ lies in the same space $\implies \overline{q}(t, z) = q^{ab}(t)w^{N_d}_{ab}(z)$.

The energy of the element and its approximation on $[a, b]$ are given by:

$$G_{ab} = \int_0^t (\int_a^b \dot{q}(t, z)\mu(t, z)) \, dt, \quad \bar{g}_{ab} = \int_0^t \dot{q}^{ab} [bq^{ab}(t)]w^{N_d}_{ab}(z)dt = \int_0^t \dot{q}^{ab}\mu^{ab} \, dt$$

$$\frac{d\bar{G}_{ab}}{dt} = \dot{q}_{ab}\mu^{ab} \Rightarrow \frac{d\bar{G}_{ab}}{dq^{ab}} = \mu^{ab}$$

So the thermodynamic relations linking the gibbs density, concentration and chemical potentials are preserved by discretization.
Global scheme
Simulations

With 10 meshes

Finite difference with N=10

Structural method N=10
Analysis on simple cases:

- Hyperbolic system (like undamped wave equation):
  \[
  \begin{pmatrix}
  f_1 \\
  f_2
  \end{pmatrix}
  =
  \begin{pmatrix}
  0 & -\frac{\partial}{\partial z} \\
  -\frac{\partial}{\partial z} & 0
  \end{pmatrix}
  \begin{pmatrix}
  e_1 \\
  e_2
  \end{pmatrix}
  \]

- Parabolic system (like diffusion):
  \[
  \begin{pmatrix}
  f_1 \\
  e_2
  \end{pmatrix}
  =
  \begin{pmatrix}
  0 & -\frac{\partial}{\partial z} \\
  -\frac{\partial}{\partial z} & 0
  \end{pmatrix}
  \begin{pmatrix}
  e_1 \\
  f_2
  \end{pmatrix}
  \]

**Question**

What about spectral properties with respect to closure equations?

- Case 1: \( f = \frac{\partial x}{\partial t}, e = \delta_x \mathcal{H}, \mathcal{H} = \int_a^b (\alpha x_1^2 + \beta x_2^2) \, dz \)
- Case 2: \( f_1 = \frac{\partial x_1}{\partial t}, e_1 = bx_1, f_2 = De_2 \)
**Case 1** : The associated PDE (without closure equation) is given by:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_1
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial z} & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + BC
\]

The spectrum associated to this operator is defined by:

\[
\begin{pmatrix}
\varphi_1 & \varphi_1 \\
\varphi_2 & \varphi_2
\end{pmatrix} \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial z} & 0
\end{pmatrix} \begin{pmatrix}
\varphi_1 & \varphi_1 \\
\varphi_2 & \varphi_2
\end{pmatrix} + BC
\]

Now let us consider the closure equation \( \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \delta_{x_1} \mathcal{H} \\ \delta_{x_1} \mathcal{H} \end{pmatrix} \)

with \( \mathcal{H}(x_1, x_2) = \int_a^b (\alpha x_1^2 + \beta x_2^2) \, dz \), then

\[
\begin{pmatrix}
\psi_1 & \psi_1 \\
\psi_2 & \psi_2
\end{pmatrix} \begin{pmatrix}
\mu_1 & 0 \\
0 & \mu_2
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial z} & 0
\end{pmatrix} \begin{pmatrix}
2\beta & 0 \\
0 & 2\alpha
\end{pmatrix} \begin{pmatrix}
\psi_1 & \psi_1 \\
\psi_2 & \psi_3
\end{pmatrix}
\]
Finally we obtain by identification:

\[
\begin{pmatrix}
\phi_1^1 & \phi_2^1 \\
\phi_1^2 & \phi_2^2 \\
\end{pmatrix}
= \begin{pmatrix}
\psi_1^1 & \psi_2^1 \\
\psi_1^2 & \psi_2^2 \\
\end{pmatrix}
\begin{pmatrix}
\sqrt{2\beta} & 0 \\
0 & \sqrt{2\alpha} \\
\end{pmatrix}
\begin{pmatrix}
\mu_1 & 0 \\
0 & \mu_2 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1^1 & \phi_2^1 \\
\phi_1^2 & \phi_2^2 \\
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
\end{pmatrix}
\begin{pmatrix}
\phi_1^1 & \phi_2^1 \\
\phi_1^2 & \phi_2^2 \\
\end{pmatrix}
\begin{pmatrix}
\psi_1^1 & \psi_2^1 \\
\psi_1^2 & \psi_2^2 \\
\end{pmatrix}
\]

Geometrical transformation:

**Homothecy** of the spectrum with factor \(2\sqrt{\alpha\beta}\)
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Spectrum of the canonical system

Spectrum of the hyperbolic system

Homothety
Case 2: In a same way

\[
\begin{pmatrix}
  f_1 \\
  e_2
\end{pmatrix}
= \begin{pmatrix}
  0 & -\frac{\partial}{\partial z} \\
  -\frac{\partial}{\partial z} & 0
\end{pmatrix}
\begin{pmatrix}
  e_1 \\
  f_2
\end{pmatrix}
\]  
(12)

with the constraint

\[f_2 = De_2\]  
(13)

With the hypothesis of separation of variables (12) can be rewritten with \(x = \psi_0 \exp \mu t, e_1 = bx\):

\[
\begin{pmatrix}
  \mu & 0 \\
  0 & b
\end{pmatrix}
\begin{pmatrix}
  \psi_0 \\
  \frac{\partial \psi_0}{\partial z}
\end{pmatrix}
= \begin{pmatrix}
  0 & -\frac{\partial}{\partial z} \\
  -\frac{\partial}{\partial z} & 0
\end{pmatrix}
\begin{pmatrix}
  b & 0 \\
  0 & Db
\end{pmatrix}
\begin{pmatrix}
  \psi_0 \\
  \frac{\partial \psi_0}{\partial z}
\end{pmatrix}
+ BC
\]  
(14)

The eigenvalues and eigenfunctions satisfy:

\[
\begin{pmatrix}
  \mu & 0 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  \tilde{\psi}_0 \\
  \tilde{e}_2
\end{pmatrix}
= \begin{pmatrix}
  0 & \frac{\partial}{\partial z} \\
  \frac{\partial}{\partial z} & 0
\end{pmatrix}
\begin{pmatrix}
  b & 0 \\
  0 & D
\end{pmatrix}
\begin{pmatrix}
  \tilde{\psi}_0 \\
  \tilde{e}_2
\end{pmatrix}
\]
which leads finally to

\[
\begin{pmatrix}
\sqrt{b} & 0 \\
0 & \sqrt{D\mu}
\end{pmatrix}
\begin{pmatrix}
\tilde{\psi}_0 \\
\tilde{e}_2
\end{pmatrix}
\frac{\mu^{1/2}}{\sqrt{bD}}
= \begin{pmatrix}
0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial z} & 0
\end{pmatrix}
\begin{pmatrix}
\sqrt{b} & 0 \\
0 & \sqrt{D\mu}
\end{pmatrix}
\begin{pmatrix}
\tilde{\psi}_0 \\
\tilde{e}_2
\end{pmatrix}
\]

which has to be identified to:

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}
\lambda = \begin{pmatrix}
0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial z} & 0
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}
\]

**Geometrical transformation**:

Dilation and rotation of the spectrum

\[
\lambda_k = |\lambda_k|e^{\pm i\frac{\pi}{2}} \longrightarrow \mu_k = Db|\lambda_k|^2 e^{\pm i\pi}
\]
Spectral properties

Canonical system associated to Stokes–Dirac structure
Parabolic system

Dilatation and rotation
Enoncé du problème discrétisé structuré (Figure reprise de Baai u2007)

\[ e_0 \rightarrow e_0^0 \rightarrow f_0^0 \rightarrow e_2^{de} \rightarrow f_2 = f_{\text{conv}} + f_{\text{diff}} \]

\[ e_1 = u_1 \quad f_1 = df_1 \]

\[ e_2 = de_1 \]

Entrée (condition frontière sur a)
\[ z_0 = a \]

Sortie
\[ z_N = a + N\Delta z = b \]

\[ e_1(1) \rightarrow e_1^1 \rightarrow f_1(1) \rightarrow f_1(i) \rightarrow f_1(i+1) \rightarrow f_1(N) \]

\[ e_2(1) \rightarrow e_2^1 \rightarrow f_2(1) \rightarrow f_2(i) \rightarrow f_2(i+1) \rightarrow f_2(N) \]

\[ z_i = a + i\Delta z \quad z_{i-1} = a + (i-1)\Delta z \]

\[ -e_0^i \rightarrow f_0^i \rightarrow -e_0^{i+1} \rightarrow f_0^{i+1} \rightarrow \ldots \]

\[ e_0 \]

Introduction

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Let first consider one element of discretization.

- **Input**: input mass flow
- **Output**: output effort

This leads to:

\[
\begin{pmatrix}
f_{1}^{ab} \\
e_{2}^{ab} \\
f_{\partial b} \\
e_{\partial a}
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{\alpha} \\
-\frac{1}{\alpha} & 0 \\
0 & \frac{1}{\alpha} \\
\frac{1}{\alpha} & 0
\end{pmatrix} \begin{pmatrix}
e_{1}^{ab} \\
f_{2}^{ab} \\
e_{2}^{ab} \\
f_{\partial b}
\end{pmatrix} + \begin{pmatrix}
0 & -\frac{1}{\alpha} \\
-\frac{1}{\alpha} & 0 \\
0 & \frac{\alpha-1}{\alpha} \\
\frac{1-\alpha}{\alpha} & 0
\end{pmatrix} \begin{pmatrix}
e_{\partial b} \\
f_{\partial a}
\end{pmatrix}
\]

This element is interconnected with the next one using continuity of flow and equality of efforts: \( e_{\partial b}^{0} = e_{\partial a}^{1} \) and \( f_{\partial b}^{0} = -f_{\partial a}^{1} \)
In the case $\alpha = 1$

\[
\begin{pmatrix}
  f_1^1 \\
  e_2^1 \\
  \vdots \\
  f_N^1 \\
  e_2^N
\end{pmatrix} = \begin{pmatrix}
  0 & M \\
  -M^T & 0
\end{pmatrix} \begin{pmatrix}
  e_1^1 \\
  f_2^1 \\
  \vdots \\
  e_1^N \\
  f_2^N
\end{pmatrix} + g \begin{pmatrix}
  e_\partial_N^1 \\
  f_0^0
\end{pmatrix}
\]

\[
\begin{pmatrix}
  f_\partial_N^1 \\
  e_\partial_0^0
\end{pmatrix} = g^T \begin{pmatrix}
  e_1^1 \\
  f_2^1 \\
  \vdots \\
  e_1^N \\
  f_2^N
\end{pmatrix}
\]

with

\[
M = \begin{bmatrix}
  1 & -1 & 0 & 0 & 0 & \cdots \\
  0 & 1 & -1 & 0 & 0 & \cdots \\
  0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & \cdots & 0 & 1 & 1 & -1 \\
\end{bmatrix}
\]
The spectrum associated with the Dirac structure:

\[
\begin{bmatrix}
0 & d \\
d & 0
\end{bmatrix}
\begin{pmatrix}
e_q \\
e_p
\end{pmatrix}
= \lambda
\begin{pmatrix}
e_q \\
e_p
\end{pmatrix}
\]

is computed using: \( \frac{d e_q}{dz^2} = \lambda^2 e_q,\ e_q = Ae^{\lambda z} + Be^{-\lambda z}\) and \(e_p = Ae^{\lambda z} - Be^{-\lambda z}\). With homogeneous boundary conditions \(e_p(0) = e_q(L) = 0\):

\[
\begin{bmatrix}
1 & -1 \\
e^{\lambda L} & e^{-\lambda L}
\end{bmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
= 0
\]

And

\[
\lambda_k = \frac{2k + 1}{2L} \pi i
\]
The interconnection of $n$ elements gives rise to the following state matrix:

\[
A = \begin{bmatrix}
0 & M \\
-M^T & 0
\end{bmatrix}
\]  

(15)

with

\[
M = \begin{bmatrix}
1 & -1 & 0 & 0 & \cdots \\
0 & 1 & -1 & 0 & \cdots \\
0 & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & -1
\end{bmatrix}
\]  

(16)

The eigenvalues of the finite dimensional system are given by:

\[
\lambda_k = \frac{n}{L} \sqrt{2 \cos \left( \frac{2k + 1}{2n + 1} \pi \right)} - 2
\]

if $k \ll n$ then:

\[
\lambda_k \approx \frac{2k + 1}{2L} \pi i
\]

= first $k$ eigenvalues of the infinite dimensional system
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4 Conclusion
Conclusion:

For 1D case:
- A powerful tool to highlight physical properties
- Parametrization of all admissible BC for a large class of systems
- Simple tools (matrix conditions) to check stability

Model reduction
- Preserve energetic properties and the geometric structure
- Link the solutions of very different systems (parabolic and hyperbolic)
- Numerically efficient
Future work:

Many extensions:

- Generalization to 2D-3D linear systems.
- Stabilization of a class of non-linear systems.
- Extensions to irreversible thermodynamic systems.
- Control using infinite dimensional Port Hamiltonian Systems (Imersion/reduction + Casimir functions).
- Model reduction of complex interconnected systems.
Advertisement:

- JDA-JNA course
- ISEM lecture notes → book.
Y. Le Gorrec, H. Zwart and B. Maschke,

Dirac structures and Boundary Control Systems associated with Skew-Symmetric Differential Operators


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