Contrôle bilinéaire d'équations de Schrödinger

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Séminaire CAS, 20 octobre 2011

The studied PDE

We consider a quantum particle, in N-D ($N = 1, 2, 3$), in a potential $V = V(x)$ and a uniform electric field $u = u(t) \in \mathbb{R}^N$. It is represented by a wave function

$$\begin{array}{rccc} \psi : & \mathbb{R} \times \Omega & \to & \mathbb{C} \\ & (t, x) & \mapsto & \psi(t, x) \end{array} \qquad \int_\Omega |\psi(t, x)|^2 dx = 1$$

that solves the Schrödinger equation

$$i \frac{\partial \psi}{\partial t}(t, x) = [-\Delta + V(x) - u(t)\mu(x)]\psi(t, x), x \in \Omega,$$

where Ω is a (possibly unbounded) domain of $\mathbb{R}^N$ and μ is the dipolar moment of the particle.

state : $\psi(t) \in \mathcal{S}$ control : $u(t)$

The control problem

Qu : Given an initial condition $\psi_0(x)$ and a target $\psi_f(x)$, does there exist $T > 0$ and $u : [0, T] \to \mathbb{R}^n$ such that the solution of

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = [-\Delta + V(x) - u(t)\mu(x)]\psi(t,x), \quad x \in \Omega \subset \mathbb{R}^n, \\ + \text{B.C.} \\ \psi(0,.) = \psi_0, \end{cases}$$

satisfies $\psi(T,.) = \psi_f$?

One may want to realize this motion

- **exactly** : $\psi(T) = \psi_f$ or **approximately** : $\|\psi(T) - \psi_f\| < \epsilon$,
- in **finite time** : $T < +\infty$ or **asymptotically** : $T = +\infty$,
- **globally** or **locally**.

Applications

- **Quantum chemistry :** induce chemical reactions using external fields.
- **Quantum information theory :** construction of quantum gates for quantum computers.

Structure of the talk

1. In finite dimension
   - Iterated Lie brackets
   - Feedback stabilization
2. Exact controllability
   - BMS's negative result
   - Local exact controllability in 1D
   - Generalization in 2D ?
   - Generalization to continuous spectrum ?
3. Feedback stabilization
   - Approximate stabilization
   - Weak stabilization in H^2
   - Strong stabilization ?
   - With polarizability
4. Approximate controllability

1st part

In finite dimension

In finite dimension

The control of bilinear systems, in finite dimension, is well understood [Rachevski-Chow('30),...].

It is still a very active field of research.

In this section, we tackle only some results in finite dimension that have been adapted to PDEs :

- iterated Lie brackets,
- feedback stabilization.

Exact controllability in finite dimension

$$i\frac{d\psi}{dt} = H_0\psi + u(t)H_1\psi \qquad \psi \in \mathbb{C}^N, H_0, H_1 \in \mathcal{M}_N(\mathbb{C})\text{hermitian}$$

Controllability $\Leftrightarrow$ Lie(H_0, H_1) is conjugated to su(N) or sp(N/2).
[Albertini-D'Alessandro(01), Rachevsky-Chow('30)]

The generalization in infinite dimension is not clear :

- No algebra structure
- Dirac masses appear in the (formal) iterated Lie brackets
  Ex : $D(H_0) = H^2 \cap H^1_0(0,1), \quad H_0 = -\partial_x^2, \quad H_1 = x^2,$
- The system may be not controllable, even if all its Galerkin approximations are controllable.
  Ex : $i\partial_t\psi = -\partial_x^2\psi + x^2\psi - u(t)x\psi, x \in \mathbb{R}$
  [Mirrahimi-Rouchon(04), Fu-Schirmer-Solomon(01)]
  Sometimes, the Galerkin approach works.
  [Boscain&al(09), Agrachev-Sarychev(05), Shirikyan(06)]

What to do with Dirac masses ?

$$\begin{cases} i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} - u(t)x^2\psi, & x \in (0,1), \\ \psi(t,0) = \psi(t,1) = 0 \end{cases}$$

We will see that this equation is locally controllable around the ground state $(\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}, u \equiv 0)$.

$$\begin{array}{ll} D(H_0) := H^2 \cap H^1_0(0,1) & H_0(\psi) := -\psi'', \\ D(H_1) := L^2(0,1) & H_1(\psi) := x^2\psi. \end{array}$$

$$[H_0, H_1](\varphi_1) = -4x\varphi_1' - 2\varphi_1$$

$$H_0(\psi) := -\psi'' - \psi(0)\delta_0' + \psi(1)\delta_1', \quad \forall \psi \in H^2(0,1)$$

$$[H_0,[H_0,H_1]](\varphi_1) = -4H_0(\varphi_1) - 4\varphi_1'(1)\delta_1' \ldots$$

The quantum harmonic oscillator

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + x^2\psi - u(t)x\psi, \quad x \in \mathbb{R}, \quad t \in (0,T)$$

$$\text{Lie}(H_0, H_1)\psi = \text{Span}\{H_0, H_1, [H_0, H_1], \text{Id}\}\psi$$

$\rightarrow$ Intuition : the system is not exactly controllable.

[Mirrahimi-Rouchon(04)] :
A 2D controllable part : - average position $q(t) := \int x|\psi(t,x)|^2 dx$
- average momentum $p(t) := -2\Im \int \overline{\partial_x \psi} \psi$
An ∞-D uncontrollable part : $\phi(t,x) := \psi(t, x+q)e^{-i\frac{p}{2}x + ir}$ where $r(t) := \int_0^t (q^2 - 3p^2/4 - uq)$ because $i\partial_t \phi = -\partial_x^2 \phi + x^2 \phi$.

However, all the Galerkin approximations are controllable [Fu-Schirmer-Solomon(01)].

About the natural strategy

1. control the Galerkin approximations
2. pass to the limit $N \to +\infty$

Used successfully to prove :
- exact controllability of dissipative equations [Agrachev-Sarychev(05), Shirikyan(06)],
- approximate controllability in L^2 of bl-Schrödinger [Boscain&al(09)] thanks to the conservation of the L^2-norm.

Not yet successful to prove :
- exact controllability for non dissipative equations,
- approximate controllability in H^s for bl-Schrödinger equations.

Feedback stabilization in finite dimension

$$i\frac{dX}{dt} = H_0 X + u(t) H_1 X \qquad X \in \mathbb{C}^N, H_0, H_1 \in \mathcal{M}_N(\mathbb{C})\text{hermitian}$$

Goal : $X(t) \to e_1$, where $H_0 e_1 = 0$.

$$\mathcal{L}[X] := 1 - |\langle X, e_1 \rangle|^2$$
$$\frac{d}{dt}\mathcal{L}[X(t)] = 2u(t)\Im\big(\langle H_1 X, e_1\rangle\langle e_1, X\rangle\big)$$

[Mirrahimi-Rouchon-Turinici(05)] : stabilization $\Leftrightarrow$ controllability of the linearized system around $(X = e_1, u = 0) \Leftrightarrow \langle H_1 e_1, e_k\rangle \neq 0, \forall k \neq 1$.

The generalization in ∞-D is not obvious because bounded and closed subsets are not necessarily closed (LaSalle).

[KB-Coron-Mirrahimi-Rouchon(07)] : implicit feedbacks when the linearized system is not controllable.

With polarizability

$$i\frac{dX}{dt} = H_0 X + u(t) H_1 X + u(t)^2 H_2 X$$

[Coron-Grigoriu-Lefter-Turinici(09)] : discontinuous feedback and periodic feedback

$$u(t, X) = \alpha(X) + \sin(t/\epsilon)\beta(X)$$

1. identify α and β st the average system converges
2. approximate stability result for the true system

2nd part

Exact controllability

Exact controllability : state of the art

$$i\partial_t\psi(t,x) = \Big[-\Delta + V(x) - u(t)\mu(x) \Big]\psi(t,x), x \in \Omega$$

Well understood :

1. exact controllability in 1D, Ω bounded,

Not well understood :

1. exact controllability in multi-D,
2. with continuous spectrum.

$\rightarrow$ results only on toy models.

Now, let us explain in detail these 3 points.

Well understood : 1D exact controllability

$$i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x) \qquad \psi(t,0)=\psi(t,1)=0$$

- **Negative results :**

  (1) $\text{Int}_{S\cap H^2}\{\psi(t;u); t \geqslant 0, u \in L^r_{loc}, r>1\} = \emptyset, \quad \forall \psi_0 \in S \cap H^2$
  [**Ball-Marsden-Slemrod(82)**,Turinici(00),Ilner-Lange-Teism.(06)]

  (2) 2D controllable part for $i\partial_t\psi = -\partial_x^2\psi + x^2\psi - u(t)x\psi, x \in \mathbb{R}$
  [Mirrahimi-Rouchon(04)]

- **Local controllability in 1D :** $\forall T > T_{min}$, $\forall \psi_0, \psi_f$ close enough to the ground state in H^a, $\exists u \in H^b(0,T)$...
  [KB(05) : (a,b)=(7,1) ; **KB-Laurent(10)** : (a,b)=(3,0),(5,1),... ; Coron(06) : $T_{min} > 0$; KB-Morancey : T_{min} in progress]

- **'Almost' global controllability 1D :** $\forall K, L \in \mathbb{N}^*$, $\exists T>0, u \in H^1_0(0,T)$ realizing $\psi(0)=\varphi_K$ and $\psi(T)=\varphi_L$.
  [KB-Coron(06)]

- **Global exact controllability** $T=\infty$ [Nersesyan(11)] : IMT for multivalued functions

Not well understood :

$$i\partial_t\psi(t,x) = \Big[-\Delta + V(x) - u(t)\mu(x) \Big] \psi(t,x), x \in \Omega$$

1. **Exact controllability in multi-D**
   - Spectral controllability of linearized systems :
     no in 3D, yes in 2D when $T > T_{min}(\Omega)$ for generic Ω
     [KB-Chitour-Khateb-Long(09)]
   - For a toy model : 1D wave equation **[KB(09)]**
2. **With continuous spectrum**
   - For a toy-model : Bloch equation
     [KB-Coron-Rouchon(09)]

Ball, Marsden and Slemrod's negative result

$$\frac{dw}{dt}(t) = \mathcal{A}w(t) + p(t)\mathcal{B}(w(t))$$

Theorem [BMS(82)] : X is a Banach space with $\dim(X) = +\infty$, $\mathcal{A}$ generates a C^0-semi group of bounded linear operators on X, $\mathcal{B} : X \to X$ is bounded, $w_0 \in X$.

$$\text{Int}_X\{w(t; p, w_0); t \geqslant 0, p \in L^r_{loc}((0,\infty),\mathbb{R}), r > 1\} = \emptyset$$

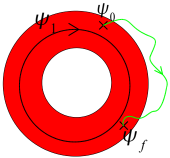
For Schrödinger : no exact controllability in $H^2 \cap H^1_0(\Omega) \cap \mathcal{S}$ with controls in L^r_{loc} [Turinici(00)], also for NLS [Ilner-Lange-Teisman(06)].
But exact controllability is still possible in smoother spaces !

Ex : $w_{tt} = w_{xx} + u(t)\mu(x)w, \qquad w_{tt} + w_{xxxx} + u(t)w_{xx} = 0$

Local exact controllability for 1D Schrödinger equations

$$i\partial_t\psi = -\partial_x^2\psi - u(t)\mu(x)\psi \qquad \psi(t,0)=\psi(t,1)=0$$

Ground state :
$\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$



Theorem[KB-Laurent(10)] : Let $T>0$, $\mu \in H^3((0,1),\mathbb{R})$ s.t.

$$\exists c>0, \qquad \frac{c}{k^3} \leqslant \left| \int_0^1 \mu(x)\sin(\pi x)\sin(k\pi x)dx \right|, \qquad \forall k \in \mathbb{N}^*. \quad (*)$$

There exists $\eta>0$ s.t. $\forall \psi_f \in \mathcal{S}\cap H^3_{(0)}(0,1)$ with $\|\psi_f - \psi_1(T)\|_{H^3} < \eta$, there exists $u \in L^2((0,T),\mathbb{R})$ s.t. the solution with $\psi(0)=\psi_1(0)$ satisfies $\psi(T)=\psi_f$.

Rk : (*) is generic $\frac{(-1)^k\mu'(1)-\mu'(0)}{(k\pi)^3} + \frac{1}{(k\pi)^3}\int_0^1 \frac{d^3[\mu(x)\sin(\pi x)]}{dx^3}\cos(k\pi x)dx$

Classical proof : the linear test

Linearized system around $(\psi_1, u \equiv 0)$ controllable in time T.

$\Downarrow$ (often)

Nonlinear system locally controllable around ψ_1 in time T.

Proof : Apply the inverse mapping theorem on $\Theta_T : u \mapsto \psi(T)$

Rk : The assumption « $\int_0^1 \mu(x)\sin(\pi x)\sin(k\pi x)dx \neq 0, \forall k$ » is necessary to control the linearized system. Otherwise, power series expansions and Coron's return method may be successful and T_{min} may be > 0. [KB(05,08), KB-Coron(06), Coron(06)]

Main difficulty of the proof

$$i\partial_t\psi = -\partial_x^2\psi - u(t)\mu(x)\psi, \quad \psi(t,0)=\psi(t,1)=0, \quad \psi(0,x)=\sqrt{2}\sin(\pi x).$$

$$\Theta_T : u \mapsto \psi(T)$$

Easy to prove :

- $\Theta_T : L^2(0,T) \to H^2$ is C^1,
- there exists $d\Theta_T(0)^{-1} : H^3 \to L^2(0,T)$.

Thus, there is an a priori loss of regularity.

- **First solution :** Nash-Moser theorem [KB(05)]
- **New solution :** regularizing effect **[KB-Laurent(10)]**

Rk : Every Riemannian manifold can be isometrically embedded in some $\mathbb{R}^k$, [Nash(56), Moser(60'), Günther(89,91)]

Step 1 : controllability of the linearized system

$$\begin{cases} i\partial_t \Psi(t,x) = -\partial_x^2 \Psi(t,x) - v(t)\mu(x)\psi_1(t,x), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0. \end{cases}$$

$$\Psi(T,x) = \sum_{k=1}^{\infty} \frac{i}{\langle \mu\varphi_1, \varphi_k \rangle} \int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} dt e^{-i\lambda_k T} \varphi_k(x)$$

For a given target Ψ_f, the equality $\Psi(T) = \Psi_f$ is equivalent to the trigonometric moment problem,

$$\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} dt = \frac{-i\langle \Psi_f, \varphi_k \rangle e^{i\lambda_k T}}{\langle \mu\varphi_1, \varphi_k \rangle}, \forall k \in \mathbb{N}^*.$$

Answer : $\forall T > 0, \forall \Psi_f \in H^3_{(0)}(0,1), \exists v \in L^2(0,T)$ solution [Ingham(36), Beurling, Haraux(89)]

Step 2 : Regularizing effect

$$i\partial_t\psi = -\partial_x^2\psi - u(t)\mu(x)\psi, \quad \psi(t,0)=\psi(t,1)=0, \quad \psi(0)=\varphi_1.$$

Goal : $u \in L^2(0,T) \quad \Rightarrow \quad \psi \in C^0([0,T], H^3_{(0)})$

$$H^3_{(0)} := \{\xi \in H^3(0,1); \xi = \xi'' = 0 \text{ at } x=0,1\}.$$

Classical strategy : fixed point ? $\quad C^0([0,T], H^3_{(0)}) \to ???$, $\quad \psi \mapsto \xi$

$$\text{where } \xi(t) := e^{-i\Delta t}\varphi_1 + i\int_0^t e^{-i\Delta(t-s)}[\ u(s)\mu\psi(s)\ ]ds, \forall t \in [0,T].$$

- $e^{-i\Delta\tau} : H^3_{(0)} \to H^3_{(0)}, \forall\tau$
- $\varphi \mapsto \mu\varphi$ does not preserve $H^3_{(0)}$: $(\mu\varphi)'' = 2\mu'\varphi'$ at $x = 0, 1$.

Proof of the regularizing effect

Lemma : If $f \in L^2((0,T), H^3 \cap H^1_0)$ then the map $F : t \mapsto \int_0^t e^{i\Delta s} f(s) ds$ belongs to $C^0([0,T], H^3_{(0)})$.

Qu : $$\|F(t)\|^2_{H^3_{(0)}} = \sum_{k=1}^{\infty} \left| k^3 \int_0^t e^{i\lambda_k s} \left\langle f(s), \sin(k\pi x) \right\rangle ds \right|^2 \to 0, \quad [t \to 0]$$

$$\left\langle f(s), \sin(k\pi x) \right\rangle = \frac{1}{(k\pi)^3} \left[(-1)^k \partial_x^2 f(s,1) - \partial_x^2 f(s,0) - \left\langle \partial_x^3 f(s), \cos(k\pi x) \right\rangle \right]$$

$$\begin{aligned} \|F(t)\|_{H^3_{(0)}} &\leqslant \sum_{\alpha \in \{0,1\}} \left\| \int_0^t e^{i\lambda_k s} \partial_x^2 f(s,\alpha) dx \right\|_{l^2} \\ &\quad + \left\| \int_0^t e^{i\lambda_k s} \left\langle \partial_x^3 f(s), \cos(k\pi x) \right\rangle ds \right\|_{l^2} \\ &\leqslant \mathcal{C} \sum_{\alpha \in \{0,1\}} \|\partial_x^2 f(.,\alpha)\|_{L^2(0,t)} + \|\partial_x^3 f\|_{L^1((0,t),L^2)} \end{aligned}$$

Conclusion on the 1D local exact controllability

This result enters the classical framework of local controllability results proved with fixed point methods.

Other results with the same proof (see [KB-Laurent(10)])

- generalization to H^5/H^1_0, H^7/H^2_0, H^9/H^3_0, etc
- on the 3D ball with radial data
- nonlinear Schrödinger equation
- nonlinear wave equation

Generalization of the 1D strategy in 2D

$$\begin{cases} i\partial_t \psi = -\Delta\psi - u(t)\mu(x)\psi, & x \in \Omega \subset \mathbb{R}^N, N = 2,3, \\ \psi(t,x) = 0, & x \in \partial\Omega. \end{cases}$$

2 points to understand :

- in which spaces the linearized system is controllable ?
- is the NL system well posed in these spaces ? (C^1)

Qu : Solve $\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} dt = d_k(\Psi_f), \forall k \in \mathbb{N}^*$

$\Leftrightarrow$ Gap : $\exists N > 0, \quad \lambda_{k+N} - \lambda_k \geqslant \frac{2\pi N}{T}, \quad \forall k \in \mathbb{N}^*$

[Castro-Zuazua(96), Jaffard-Tucsnak-Zuazua(97), Baiocchi-Komornik-Loreti(02)]

Weyl : Card$\{\lambda_k \in [0,t]\} = dt + o(t)$ $\rightarrow$ open problem.

Proof on a toy model and conjectures

$$\partial_t^2 w = \partial_x^2 w + u(t)\mu(x)w, \qquad \partial_x w(t,0) = \partial_x w(t,1) = 0$$

Theorem[KB(10)] : **Local exact controllability in** $H^3 \times H^2$**, around** $(w \equiv 1, u \equiv 0)$**, with controls in** $L^2(0,T)$:

- $T > 2$: yes
- $T = 2$: yes up to codimension one $\left(w - \int_0^1 w(x)dx, \partial_t w\right)$
- $T < 2$: no, reachable set $\subset$ non flat submanifold of $H^3 \times H^2$.

Key point of the proof :

- $T > 2$: uniform gap $> \frac{2\pi}{T}$
- $T < 2$: $(e^{ik\pi t})_{k\in\mathbb{Z}}$ contains a RB of $L^2(0,T)$ [Horvath-Joo(90)]
  second order term $\rightarrow$ non flat

Conjecture for 2D-Schrödinger equations : idem with $T_{min} = 2\pi/d$

And with continuous spectrum ?

Toy model : $\partial_t M(t,\omega) = [u(t)e_1 + v(t)e_2 + \omega e_3] \wedge M(t,\omega), \omega \in (a,b)$

$\mathcal{A}M := \omega e_3 \wedge M(\omega) \quad \rightarrow \quad \mathrm{Sp}(\mathcal{A}) = -i(\omega_*, \omega^*) \bigcup i(\omega_*, \omega^*)$

Results[KB-Coron-Rouchon(09), KB-Rouchon-Silva(10)] :

- **No exact controllability with a priori bounded controls in** $L^2(0,T)$: *non flat submanifold argument*
- **But we recover controllability with unbounded controls :**
  - global approximate controllability in $H^1(\omega_*, \omega^*)$: *non commutativity* [Li-Khaneja(06)] *+ variationnal method*
  - local exact controllability to e_3 with $T = +\infty$ and explicit unbounded controls : *Fourier series*
  - weak feedback stabilization of $\pm e_3$: *periodic impulse train structure to reduce dispersion.*

Conjectures for Schrödinger equations : idem. New technics ?

Conclusion and open problems about exact controllability

$$i\partial_t\psi(t,x) = \Big[-\Delta + V(x) - u(t)\mu(x) \Big]\psi(t,x), x \in \Omega$$

Well understood :

- exact controllability in 1D, Ω bounded, $V = 0$: simple proof

In progress :

- exact controllability in multi-D, Toy model $\Rightarrow$ conjectures
- with continuous spectrum, Toy model $\Rightarrow$ conjectures

Problems still open in 1D :

- characterization of T_{min} [in progress]
- simultaneous controllability/stabilization of N Schrödinger equations
- problematic linearized systems :

Ex : $\partial_t^2 w = \partial_x^2 w + u(t)\mu(x)w, \qquad w(t,0) = w(t,\pi) = 0$
around $(w = \sin(Lt)\sin(Lx), u \equiv 0)$:
$\forall T > 0$, $\{\sin(Lt)e^{ikt}; k \in \mathbb{Z}\}$ does not satisfy the RB prop in $L^2(0,T)$.

3rd part

Feedback stabilization

Feedback stabilization : state of the art

Well understood :

1. approximate stabilization in L^2
2. weak stabilization in H^2

Not well understood :

1. strong stabilization

Approximate stabilization in L^2

$$i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi$$

$$\mathcal{L}(\psi) = 1 - |\langle \psi, \varphi_1 \rangle|^2 - (1-\epsilon) \sum_{k=2}^{N} |\langle \psi, \varphi_k \rangle|^2$$

$$\frac{d\mathcal{L}}{dt} = -2u(t)\Im\Big(\sum_{k=1}^{N} a_k \langle \mu\psi, \varphi_k \rangle \langle \varphi_k, \psi \rangle \Big)$$

This Lyapunov function encodes two tasks :

- it prevents the L^2-mass lost through the high-energy eigenstates,
- it privileges the increase of the population in the first eigenstate.

[KB-Mirrahimi(09)] : semi-global approximate stabilization
$\limsup_{t\to+\infty} \text{dist}_{L^2}[\psi(t), \mathcal{C}_1] \leqslant \epsilon.$

Rk : Also valid with continuous spectrum [Mirrahimi(09)]

Weak stabilization in H^2

$$\left\{ \begin{array}{l} i\partial_t \psi = [-\Delta + V(x)]\psi + u(t)\mu(x)\psi, \quad x \in D \subset \mathbb{R}^N \\ \psi_{\partial D} = 0 \end{array} \right.$$

$$\mathcal{L}(\psi) := \alpha \|(-\Delta + V)P\psi\|^2 + 1 - |\langle \psi, e_1 \rangle|^2$$

$$u(\psi) := -\delta \Im \Big[\langle \alpha(-\Delta + V)P(\mu\psi), (-\Delta + V)P\psi \rangle - \langle \mu\psi, e_1 \rangle \langle e_1, \psi \rangle \Big]$$

[KB-Nersesyan(10)] : $\psi(t) \to \mathcal{C}_1$ weakly in H^2 (semi-global).

Key points :

1. H^2 strictly more regular than the L^2-constraint $\to$ invariant set OK
2. $u(\psi)$ well defined for $\psi \in H^{3/2}$

Rk : Idem for Bloch equation in H^1 (L^∞-constraint).

About strong stabilization

Difficulty : prove the compactness of the trajectories of the closed loop system in the functional space considered. [Coron-d'Andea Novel(98)]

[Couchouron(11)] : $w_{tt} + Aw + u(t)Bw = 0$
Goal : $(w, w_t) \to (0, 0)$ strongly in $H^1 \times L^2$
$\mathcal{L} := \int_0^1 w(t,x)^2 + w_x(t,x)^2 dx \quad \to \quad u(w, w_t) := \langle w_t, Bw(t) \rangle$
OK when B is diagonal wrt A...

Open pb : When B couples the different modes ? bidiagonal structure ?

With a polarizability term

$$i\partial_t\psi = -\partial_x^2\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi$$

[Morancey(11)] : explicit approximate controllability in H^s, $s < 2$

Key points :

1. periodic feedback $u(t,\psi) = \alpha(\psi) + \sin(t/\epsilon)\beta(\psi)$ as in [Coron-Grigoriu-Lefter-Turinici(09)]
2. weak H^2-stabilization of the average system as in [KB-Nersesyan(10)]
3. distance between the averaged and the true system : finite horizon only.

4th part

Approximate controllability

Approximate controllability : state of the art

Well understood : with discrete spectrum

- **For particular models** [Adami-Boscain(05)] : adiabatic theory + intersection of eigenvalues
- **In L^2, 1D, with $T=+\infty$** [KB-Mirrahimi(09)] : feedback approximate stabilization
- **In L^2, $T<+\infty$** [Boscain&al(09)] : geometric methods
- **In H^s, $s>0$, $T=+\infty$** [Nersesyan(09), KB-Nersesyan(10)] : feedback weak stabilization $\Rightarrow$ 1D global exact control
- **Trapped ion model** [Ervedoza-Puel(09)] : reduced model

Not well understood : with continuous spectrum

- Approximate controllability with $T=+\infty$ [Mirrahimi(09)]
- For a toy-model : Bloch equation [KB-Coron-Rouchon(09), KB-Rouchon-Silva(10)]

In L^2, with Galerkin approach

$$\frac{d\psi}{dt} = A\psi + u(t)B\psi$$

[Chambrion-Mason-Sigalotti-Boscain(09)] :

- A, B skew adjoint (possibly unbounded) on an Hilbert H
- discrete spectrum, $(\lambda_{n+1} - \lambda_n)_{n\in\mathbb{N}^*}$ $\mathbb{Q}$-linearly indepdt
- $\langle B\varphi_n, \varphi_{n+1}\rangle \neq 0, \forall n$

$\Rightarrow$ approximate controllability in L^2 with piecewise constant controls.

Conclusion of the talk

$$i\partial_t\psi = [-\Delta + V(x) - u(t)\mu(x)]\psi, \qquad x \in \Omega$$

- **Exact controllability :**
  OK in 1D, Ω bounded, $V = 0$
  In progress in 2D ($T_{min} > 0$)
  Probably false in 3D or with continuous spectrum
- **Feedback stabilization :**
  OK for approximate or weak stabilization
  Open for strong one
- **Approximate controllability :**
  OK in H^s on Ω multi-D bounded,
  OK in L^2 on Ω multi-D, with discrete spectrum
  Quite open with continuous spectrum