

Table of Contents

Advances in the Theory of Control, Signals and Systems with Physical Modeling

Flatness Characterization: Two Approaches	1
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Flatness Characterization: Two Approaches

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Abstract. We survey two approaches to flatness necessary and sufficient conditions and compare them on examples.

1 Introduction

In this survey we consider underdetermined implicit systems of the form

$$F(x, \dot{x}) = 0 \tag{1}$$

with $x \in X$, X being an infinitely differentiable manifold of dimension n , whose tangent bundle is denoted by TX , and $F : TX \rightarrow \mathbb{R}^{n-m}$ regular in the sense that $\text{rk} \frac{\partial F}{\partial \dot{x}} = n - m$ in a suitable open dense subset of TX . Differential flatness, or more shortly, flatness was introduced in 1992 [20,11]. In the setting of implicit control systems it may be roughly described as follows: there exists a smooth mapping $x = \varphi(y, \dot{y}, \dots, y^{(r)})$ with $y = (y_1, \dots, y_m)^T$ of dimension m , $r = (r_1, \dots, r_m)^T \in \mathbb{N}^m$, such that

$$F(\varphi(y, \dot{y}, \dots, y^{(r)}), \dot{\varphi}(y, \dot{y}, \dots, y^{(r+1)})) \equiv 0 \tag{2}$$

with φ invertible in the sense that there exists a locally defined smooth mapping ψ and a multi-index s such that $y = \psi(x, \dot{x}, \dots, x^{(s)})$.

The vector y is called a *flat output*.

This concept has inspired an important literature. See [10,21,19,26,27,31] for surveys on flatness and its applications. Various formalisms have been introduced: finite dimensional differential geometric approaches [4,14,30], [32,28], differential algebra and related approaches [12,3,15], infinite dimensional differential geometry of jets and prolongations [13,33,19,6,7,23], [22,24], which is adopted here. The interested reader may refer to [1,13,16], [19,23,34] for more details.

The first part of the paper recalls the mathematical setting. In Section 3 the approach introduced in [19,2] for the characterization of differentially flat systems is recalled. Then, in Section 4, we introduce a novel characterization using the so-called Generalized Euler-Lagrange Operator. We conclude the paper with examples.

2 Implicit control systems on manifolds of jets of infinite order

Given an infinitely differentiable manifold X of dimension n , we denote its tangent space at $x \in X$ by $T_x X$, and its tangent bundle by TX . Let F be a meromorphic function from TX to \mathbb{R}^{n-m} . We consider an underdetermined implicit system of the form (1) regular in the sense that $\text{rk} \frac{\partial F}{\partial \bar{x}} = n - m$ in a suitable open dense subset of TX .

Following [17,18], we consider the infinite dimensional manifold \mathfrak{X} defined by $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}_\infty^n \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \dots$, made of an infinite (but countable) number of copies of \mathbb{R}^n , with the global infinite set of coordinates³ $\bar{x} = (x, \dot{x}, \dots, x^{(k)}, \dots)$, endowed with the product topology.

Recall that, in this topology, a function φ from \mathfrak{X} to \mathbb{R} is *continuous* (resp. *differentiable*) if φ depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables. C^∞ or analytic or meromorphic functions from \mathfrak{X} to \mathbb{R} are then defined as in the usual finite dimensional case since they only depend on a finite number of variables. We endow \mathfrak{X} with the so-called trivial Cartan field ([16,34]) $\tau_{\mathfrak{X}} = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}$. We

also denote by $L_{\tau_{\mathfrak{X}}} \gamma = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial \gamma}{\partial x_i^{(j)}} = \frac{d\gamma}{dt}$ the Lie derivative of a differentiable function γ along $\tau_{\mathfrak{X}}$ and $L_{\tau_{\mathfrak{X}}}^k \gamma$ its k th iterate. Since $\frac{d}{dt} x_i^{(j)} \stackrel{\text{def}}{=} \dot{x}_i^{(j)} = x_i^{(j+1)}$, the Cartan field acts on coordinates as a shift to the right. \mathfrak{X} is thus called *manifold of jets of infinite order*.

A *regular implicit control system* is defined as a triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ with $\mathfrak{X} = X \times \mathbb{R}_\infty^n$, $\tau_{\mathfrak{X}}$ its associated trivial Cartan field, and F meromorphic from TX to \mathbb{R}^{n-m} satisfying $\text{rk} \frac{\partial F}{\partial \bar{x}} = n - m$ in a suitable open subset of TX .

We next consider the cotangent space $T_{\bar{x}}^* \mathfrak{X}$ with $dx_i^{(j)}$, $i = 1, \dots, n$, $j \geq 0$ as basis, dual to the $\frac{\partial}{\partial x_i^{(j)}}$'s. 1-forms on \mathfrak{X} are then defined in the usual way. The set of 1-forms is noted $\Lambda^1(\mathfrak{X})$. We also denote by $\Lambda^p(\mathfrak{X})$ the module of all the p -forms on \mathfrak{X} .

2.1 Flatness

We recall the following definitions and result [17,18,19]:

Given two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$, with $\mathfrak{X} = X \times \mathbb{R}_\infty^n$, $\dim X = n$ and $\text{rk} \frac{\partial F}{\partial \bar{x}} = n - m$, and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, with $\mathfrak{Y} = Y \times \mathbb{R}_\infty^p$, $\dim Y = p$, $\tau_{\mathfrak{Y}}$ its trivial Cartan field, and $\text{rk} \frac{\partial G}{\partial \bar{y}} = p - q$, we set $\mathfrak{X}_0 = \{\bar{x} \in \mathfrak{X} | L_{\tau_{\mathfrak{X}}}^k F(\bar{x}) = 0, \forall k \geq 0\}$ and $\mathfrak{Y}_0 = \{\bar{y} \in \mathfrak{Y} | L_{\tau_{\mathfrak{Y}}}^k G(\bar{y}) = 0, \forall k \geq 0\}$. They are endowed with the topologies and differentiable structures induced by \mathfrak{X} and \mathfrak{Y} respectively.

Definition 1 *The control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ are said locally Lie-Bäcklund equivalent (or shortly L-B equivalent) in a neighbourhood $\mathcal{X}_0 \times \mathcal{Y}_0$ of the pair $(\bar{x}_0, \bar{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$ if and only if*

³ From now on, \bar{x}, \bar{y}, \dots stand for the sequences of jets of infinite order of x, y, \dots

- (i) there exists a one-to-one meromorphic mapping $\Phi = (\varphi, \dot{\varphi}, \dots)$ from \mathcal{Y}_0 to \mathcal{X}_0 satisfying $\Phi(\bar{y}_0) = \bar{x}_0$ and such that $\Phi_*\tau_{\mathcal{Y}} = \tau_{\mathcal{X}}$;
- (ii) there exists Ψ one-to-one and meromorphic from \mathcal{X}_0 to \mathcal{Y}_0 , with $\Psi = (\psi, \dot{\psi}, \dots)$, such that $\Psi(\bar{x}_0) = \bar{y}_0$ and $\Psi_*\tau_{\mathcal{X}} = \tau_{\mathcal{Y}}$.
- The mappings Φ and Ψ are called mutually inverse Lie-Bäcklund isomorphisms at (\bar{x}_0, \bar{y}_0) .

Definition 2 The implicit system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat in a neighborhood of $(\bar{x}_0, \bar{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}_{\infty}^m$ if and only if it is locally L-B equivalent around (\bar{x}_0, \bar{y}_0) to the trivial implicit system $(\mathbb{R}_{\infty}^m, \tau_{\mathbb{R}_{\infty}^m}, 0)$. In this case, the mutually inverse L-B isomorphisms Φ and Ψ are called inverse trivializations.

Theorem 1 The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat at $(\bar{x}_0, \bar{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}_{\infty}^m$ if and only if there exists a local meromorphic invertible mapping Φ from \mathbb{R}_{∞}^m to \mathfrak{X}_0 , with meromorphic inverse, satisfying $\Phi(\bar{y}_0) = \bar{x}_0$, and such that⁴

$$\Phi^* dF = 0. \quad (3)$$

3 Necessary and Sufficient Conditions: Generalized Moving Frame Structure Equations

3.1 Algebraic characterization of the differential of a trivialization

Consider the following matrix, polynomial with respect to the differential operator $\frac{d}{dt}$ (we use indifferently $\frac{d}{dt}$ for $L_{\tau_{\mathfrak{X}}}$ or $L_{\tau_{\mathbb{R}_{\infty}^m}}$, the context being unambiguous):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt}, \quad P(\varphi) = \sum_{j \geq 0} \frac{\partial \varphi}{\partial y^{(j)}} \frac{d^j}{dt^j} \quad (4)$$

with $P(F)$ (resp. $P(\varphi)$) of size $(n - m) \times n$ (resp. $n \times m$). Equation (3) reads:

$$\Phi^* dF = P(F)P(\varphi)dy = 0. \quad (5)$$

Clearly, the entries of the matrices in (4) are polynomials in the differential operator $\frac{d}{dt}$ with meromorphic coefficients from \mathfrak{X} to \mathbb{R} . We denote by \mathfrak{K} the field of meromorphic functions from \mathfrak{X} to \mathbb{R} and by $\mathfrak{K}[\frac{d}{dt}]$ the (non-commutative) principal ideal ring of polynomials in $\frac{d}{dt}$ with coefficients in \mathfrak{K} . For $r, s \in \mathbb{N}$, let us denote by $\mathcal{M}_{r,s}[\frac{d}{dt}]$ the module of $r \times s$ matrices over $\mathfrak{K}[\frac{d}{dt}]$ (see e.g. [8]). Matrices whose inverse belong to $\mathcal{M}_{r,r}[\frac{d}{dt}]$ are called *unimodular matrices*. They form a multiplicative group denoted by $\mathcal{U}_r[\frac{d}{dt}]$.

⁴ Note that if Φ is a meromorphic mapping from \mathcal{Y} to \mathfrak{X} , the (backward) image by Φ of a 1-form is defined in the same way as in the finite dimensional context.

Every matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ admits a *Smith decomposition* (or diagonal reduction)

$$VMU = (\Delta, 0_{r,s-r}) \text{ if } r \leq s, \text{ and } \begin{pmatrix} \Delta \\ 0_{r-s,s} \end{pmatrix} \text{ if } s \leq r \quad (6)$$

with $V \in \mathcal{U}_r[\frac{d}{dt}]$ and $U \in \mathcal{U}_s[\frac{d}{dt}]$ and Δ diagonal (see e.g. [8]). U and V are indeed non unique. We say that $U \in \mathbf{R} - \mathbf{Smith}(M)$ and $V \in \mathbf{L} - \mathbf{Smith}(M)$.

A matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ is said *hyper-regular* if and only if its Smith decomposition leads to $\Delta = I$. An interpretation of this property in terms of controllability in the sense of [9], may be found in [18].

From now on, we assume that $P(F)$ is hyper-regular in a neighborhood of \bar{x}_0 . In place of (5), we first solve the matrix equation:

$$P(F)\Theta = 0 \quad (7)$$

where $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ is not supposed to be of the form $P(\varphi)$. It may be verified that matrices $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (7) have the structure

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \quad (8)$$

with $U \in \mathbf{R} - \mathbf{Smith}(P(F))$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ arbitrary. Clearly Θ is itself hyper-regular and admits the Smith decomposition

$$Q\Theta Z = QU \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} WZ = Q\hat{U}R = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} \quad (9)$$

with $Q \in \mathcal{U}_n[\frac{d}{dt}]$, $Z \in \mathcal{U}_m[\frac{d}{dt}]$, $R = WZ$ and $\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$.

3.2 Integrability

We denote by ω the m -dimensional vector 1-form defined by

$$\omega(\bar{x}) = \begin{pmatrix} \omega_1(\bar{x}) \\ \vdots \\ \omega_m(\bar{x}) \end{pmatrix} = (I_m, 0_{m,n-m}) Q(\bar{x}) dx|_{\mathcal{X}_0} \quad (10)$$

with Q given by (9), the restriction to \mathcal{X}_0 meaning that $\bar{x} \in \mathcal{X}_0$ satisfies $L_{\tau_{\bar{x}}}^k F = 0$ for all k and that the $dx_j^{(k)}$ are such that $dL_{\tau_{\bar{x}}}^k F = 0$ in \mathcal{X}_0 for all k . Since Q is hyper-regular, the forms $\omega_1, \dots, \omega_m$ are independent by construction.

Theorem 2 *A necessary and sufficient condition for system (1) to be locally flat around (\bar{x}_0, \bar{y}_0) is that there exist $U \in \mathbf{R} - \mathbf{Smith}(P(F))$, $Q \in \mathbf{L} - \mathbf{Smith}(\hat{U})$, with \hat{U} given by (9) and a matrix $M \in \mathcal{U}_m[\frac{d}{dt}]$ such that $d(M\tau) = 0$.*

We denote by $(\Lambda^p(\mathfrak{X}))^m$ the space of m -dimensional vector p -forms on \mathfrak{X} , by $(\Lambda(\mathfrak{X}))^m$ the space of m -dimensional vector forms of arbitrary degree on \mathfrak{X} , and by $\mathcal{L}_q((\Lambda(\mathfrak{X}))^m) = \bigcup_{p \geq 1} \mathcal{L}((\Lambda^p(\mathfrak{X}))^m, (\Lambda^{p+q}(\mathfrak{X}))^m)$ the space of linear operators from $(\Lambda^p(\mathfrak{X}))^m$ to $(\Lambda^{p+q}(\mathfrak{X}))^m$ for all $p \geq 1$, where $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ denotes the set of linear mappings from a given space \mathcal{P} to a given space \mathcal{Q} .

In order to develop the expression $d(\mu\kappa)$ for $\mu \in \mathcal{L}_q((\Lambda(\mathfrak{X}))^m)$ and for all $\kappa \in (\Lambda^p(\mathfrak{X}))^m$ and all $p \geq 1$, we define the operator \mathfrak{d} by:

$$\mathfrak{d}(\mu)\kappa = d(\mu\kappa) - (-1)^q \mu d\kappa. \quad (11)$$

Note that (11) uniquely defines $\mathfrak{d}(\mu)$ as an element of $\mathcal{L}_{q+1}((\Lambda(\mathfrak{X}))^m)$.

Theorem 3 *The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat iff there locally exists $\mu \in \mathcal{L}_1((\Lambda(\mathfrak{X}))^m)$, and a matrix $M \in \mathcal{U}_m[\frac{d}{dt}]$ such that*

$$d\omega = \mu\omega, \quad \mathfrak{d}(\mu) = \mu^2, \quad \mathfrak{d}(M) = -M\mu. \quad (12)$$

with the notation $\mu^2 = \mu\mu$ and where ω is defined by (10). In addition, if (12) holds true, a flat output y is obtained by integration of $dy = M\omega$.

Remark 1 *Note that the two first conditions of (12) are comparable to conditions (A) and (B) of [6,7]. However, the last condition of (12) is different from condition (C) of [6,7] and is easier to check.*

Note also that conditions (12) may be seen as a generalization in the framework of manifolds of jets of infinite order of Cartan's well-known moving frame structure equations (see e.g. [5]).

3.3 A Sequential Procedure

We start with $P(F)$ hyper-regular and compute the vector 1-form ω defined by (10).

1. We identify the operator μ such that $d\omega = \mu\omega$ componentwise. It is proven in [19] that such μ always exists.
2. Among the possible μ 's, only those satisfying $\mathfrak{d}(\mu) = \mu^2$ are kept. It is shown in [19] that such μ always exists.
3. We then identify M such that $\mathfrak{d}(M) = -M\mu$ componentwise.
4. If, among such M 's, there is a unimodular one, the system is flat and a flat output is obtained by integration of $dy = M\omega$. Otherwise the system is not flat.

More details and examples may be found in [18,19].

4 Necessary and Sufficient Conditions using the Generalized Euler-Lagrange Operator

Another way of analysing (3) consists in characterizing the change of coordinates corresponding to the mapping Φ in (3). More precisely (3) reads

$$\sum_{j=1}^m \sum_{k=0}^{r_j} \left(\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_j^{(k)}} \right) dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k)}} dy_j^{(k+1)} \right) = 0 \quad (13)$$

Since the one forms $dy_1, \dots, dy_1^{(r_1)}, \dots, dy_m, \dots, dy_m^{(r_m)}$ are independent by assumption, (13) yields, for every $j = 1, \dots, m$,

$$\begin{cases} \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_j^{(k)}} \right) + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k-1)}} = 0, \quad \forall k = 1, \dots, r_j \\ \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_j} \right) = 0 \end{cases} \quad (14)$$

The Generalized Euler-Lagrange operator \mathcal{E}_F associated to F is defined by

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) \quad (15)$$

In the case $n - m = 1$, it is well-known that the curves that extremize the cost function $J = \int_0^T F(x, \dot{x}) dt$ are those satisfying the Euler-Lagrange equation $\mathcal{E}_F = 0$, which justifies our terminology. Using (15) and elementary calculus, (14) yields:

Theorem 4 *A necessary and sufficient condition for (1) to be differentially flat is that there exist (r_1, \dots, r_m) with $\sum_{i=1}^m r_i + m \geq n$ and a solution φ of the following triangular system of PDEs in an open dense subset of \mathfrak{X}*

$$\begin{cases} \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(l)}} = \sum_{k=0}^{r_j-l-1} (-1)^{k+1} \frac{d^k}{dt^k} \left(\mathcal{E}_F \frac{\partial \varphi}{\partial y_j^{(l+k+1)}} \right), \quad \forall l = 0, \dots, r_j - 1, \\ 0 = \sum_{k=0}^{r_j} (-1)^k \frac{d^k}{dt^k} \left(\mathcal{E}_F \frac{\partial \varphi}{\partial y_j^{(k)}} \right) \end{cases} \quad (16)$$

satisfying $d\varphi_1 \wedge \dots \wedge d\varphi_n \neq 0$.

Remark 2 *If there exists a coordinate transformation φ that satisfies the conditions of Theorem 4 with given r_1, \dots, r_m , meaning that the system is flat, then $g_j = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial y_j^{(r_j)}} \frac{\partial}{\partial \dot{x}_i}$, if non zero, defines a ruled direction [32,25,19].*

5 Examples

5.1 An Academic Example: Generalized Moving Frame Approach

We consider the 3-dimensional system with 2 inputs:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = \sin \left(\frac{u_1}{u_2} \right) \quad (17)$$

or, in implicit form:

$$F(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) \triangleq \dot{x}_3 - \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right) = 0. \quad (18)$$

It is readily seen that $P(F) = \left[-\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\dot{x}_2^{-1}\frac{d}{dt} \middle| \dot{x}_1 \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\dot{x}_2^{-2}\frac{d}{dt} \middle| \frac{d}{dt} \right]$ and that $VP(F)U = (1 \ 0 \ 0)$ with

$$V = 1, \quad U = \left(\begin{array}{c|c} \frac{\dot{x}_1}{a\dot{x}_2} & 1 + \frac{\dot{x}_1}{a(\dot{x}_2)^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} \\ \frac{1}{a} & \frac{1}{a\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} \\ 0 & 0 \end{array} \middle| \begin{array}{c} \frac{\dot{x}_1}{a\dot{x}_2} \frac{d}{dt} \\ -\frac{1}{a} \frac{d}{dt} \\ 1 \end{array} \right) \quad (19)$$

where $a = -\frac{1}{\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \left(\frac{\dot{x}_1\dot{x}_2 - \dot{x}_1\dot{x}_2}{(\dot{x}_2)^2}\right)$. Then, $Q\hat{U}R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is computed with

$$Q = \left(\begin{array}{c|c} 1 & -\frac{\dot{x}_1}{\dot{x}_2} \\ 0 & 0 \\ -\frac{1}{a\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} & \frac{\dot{x}_1}{a(\dot{x}_2)^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} \end{array} \middle| \begin{array}{c} 0 \\ 1 \\ \frac{1}{a} \frac{d}{dt} \end{array} \right), \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (20)$$

So, $(\omega_1 \ \omega_2)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q dx = \left(dx_1 - \frac{\dot{x}_1}{\dot{x}_2} dx_2 \ \ dx_3 \right)^T$ and $d\omega = \left(\frac{1}{\sqrt{1-(\dot{x}_3)^2}} dx_2 \wedge dx_3 \ 0 \right)^T$. According to section 3.3, step 1,

$$\mu = \begin{pmatrix} 0 & \left(-\frac{\dot{x}_3}{(1-(\dot{x}_3)^2)^{\frac{3}{2}}} dx_2 \wedge d\dot{x}_3 + \eta d\dot{x}_3 \right) \wedge \frac{d}{dt} \\ 0 & 0 \end{pmatrix}. \quad (21)$$

Step 2 yields $\eta = \frac{x_2\dot{x}_3}{(1-\dot{x}_3)^{\frac{3}{2}}} + \sigma(\dot{x}_3)$. For step 3 we set $M = \begin{pmatrix} 1 & m_{12} \frac{d}{dt} \\ 0 & 1 \end{pmatrix}$

which yields $m_{12} = -\left(\frac{x_2}{\sqrt{1-(\dot{x}_3)^2}} + \sigma_1(\dot{x}_3)\right)$ with σ_1 a primitive of σ .

Thus, $d(M\omega) = 0$ and setting $(dy_1 \ dy_2)^T = M\omega$, one obtains

$$y_1 = x_1 - \frac{\dot{x}_1}{\dot{x}_2} x_2 + \sigma_2(\dot{x}_3), \quad y_2 = x_3 \quad (22)$$

where $\sigma_2(\dot{x}_3)$ is an arbitrary meromorphic function (a primitive of σ_1). By inversion of (22) we get

$$\begin{aligned} x_1 &= y_1 - \arcsin(\dot{y}_2) \frac{\sqrt{1-(\dot{y}_2)^2}}{\dot{y}_2} (\dot{y}_1 - \sigma_1(\dot{y}_2)\dot{y}_2) - \sigma_2(\dot{y}_2) \\ x_2 &= -\frac{\sqrt{1-(\dot{y}_2)^2}}{\dot{y}_2} (\dot{y}_1 - \sigma_1(\dot{y}_2)\dot{y}_2) \\ x_3 &= y_2 \end{aligned} \quad (23)$$

5.2 Academic Example: Euler-Lagrange Operator

We consider once more the example (18). We have

$$\frac{\partial F}{\partial \dot{x}} = \left(-\dot{x}_2^{-1} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), \dot{x}_1 \dot{x}_2^{-2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), 1 \right), \quad \mathcal{E}_F = (\eta_1, \eta_2, 0) \quad (24)$$

with $\eta_1 = -\frac{\ddot{x}_2}{\dot{x}_2^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) - \frac{\dot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$ and
 $\eta_2 = -\frac{\dot{x}_1 \dot{x}_2 - 2\dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) + \frac{\dot{x}_1(\dot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2)}{\dot{x}_2^4} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$.

The first two equations of (16), with $r_1 = r_2 = 2$, read

$$-\frac{1}{\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \left(\frac{\partial \varphi_1}{\partial \dot{y}_j} - \frac{\dot{x}_1}{\dot{x}_2} \frac{\partial \varphi_2}{\partial \dot{y}_j} \right) + \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0, \quad j = 1, 2 \quad (25)$$

If we assume that $\frac{\partial \varphi_3}{\partial \dot{y}_j} = \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0$, $j = 1, 2$ and introduce the variable

$$\psi = \frac{\dot{x}_1}{\dot{x}_2} \quad (26)$$

with $\frac{\partial}{\partial \dot{y}} \psi = 0$ we obtain from (25)

$$\frac{\partial \varphi_1}{\partial \dot{y}_j} - \psi \frac{\partial \varphi_2}{\partial \dot{y}_j} = \frac{\partial}{\partial \dot{y}_j} (\varphi_1 - \psi \varphi_2) = 0, \quad j = 1, 2$$

Setting $\kappa(y, \dot{y}) = \varphi_1 - \psi \varphi_2$, we get

$$\dot{\kappa} = \dot{\varphi}_1 - \psi \dot{\varphi}_2 - \dot{\psi} \varphi_2 = -\dot{\psi} \varphi_2 \quad (27)$$

Using the definition of κ and (27) we obtain:

$$\varphi_1 = \kappa - \frac{\dot{\kappa} \sqrt{1 - \dot{\varphi}_3}}{\dot{\varphi}_3} \arcsin(\dot{\varphi}_3), \quad \varphi_2 = -\frac{\dot{\kappa}}{\dot{\varphi}_3} \sqrt{1 - \dot{\varphi}_3}, \quad \varphi_3 = \varphi_3(y) \quad (28)$$

Choosing $\varphi_3 = y_2$, $\kappa = y_1$, we arrive at the invertible transformation

$$x_1 = \varphi_1 = y_1 - \frac{\dot{y}_1}{\dot{y}_2} \sqrt{1 - \dot{y}_2^2} \arcsin(\dot{y}_2), \quad x_2 = \varphi_2 = -\frac{\dot{y}_1}{\dot{y}_2} \sqrt{1 - \dot{y}_2^2},$$

with $x_3 = \varphi_3 = y_2$, which gives the same formula as (23) with $\sigma_1 = \sigma_2 = 0$. Hence (y_1, y_2) is indeed a flat output, which implies that the remaining equations of (16) are satisfied.

5.3 An Example Proposed by P. Rouchon

Consider the implicit control system

$$F(x, \dot{x}) = \dot{x}_1 \dot{x}_3 - (\dot{x}_2)^2 = 0. \quad (29)$$

We thus have $\frac{\partial F}{\partial x} = (0 \ 0 \ 0)$, $\frac{\partial F}{\partial \dot{x}} = (\dot{x}_3 \ -2\dot{x}_2 \ \dot{x}_1)$ and

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) = -\frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) = (-\ddot{x}_3 \ 2\ddot{x}_2 \ -\ddot{x}_1).$$

The lowest possible choice of (r_1, r_2) in Theorem 4 is $r_1 = r_2 = 1$. However, there is no solution of (16) for these values, and we choose $r_1 = r_2 = 2$. The two first equations of (16) read

$$\dot{\varphi}_3 \frac{\partial \varphi_1}{\partial \ddot{y}_j} - 2\dot{\varphi}_2 \frac{\partial \varphi_2}{\partial \ddot{y}_j} + \dot{\varphi}_1 \frac{\partial \varphi_3}{\partial \ddot{y}_j} = 0, \quad j = 1, 2 \quad (30)$$

We divide (30) by $\dot{\varphi}_3$ to obtain

$$\frac{\partial \varphi_1}{\partial \ddot{y}_j} - 2\psi \frac{\partial \varphi_2}{\partial \ddot{y}_j} + \psi^2 \frac{\partial \varphi_3}{\partial \ddot{y}_j} = 0, \quad j = 1, 2 \quad (31)$$

where, taking account of the system equation (29),

$$\psi = \frac{\dot{\varphi}_2}{\dot{\varphi}_3} = \sqrt{\frac{\dot{\varphi}_1}{\dot{\varphi}_3}}. \quad (32)$$

If we assume that ψ doesn't depend on \dot{y}_1 and \dot{y}_2 , equation (31) reads $\frac{\partial}{\partial \ddot{y}_j} (\varphi_1 - 2\psi\varphi_2 + \psi^2\varphi_3) = 0$, for $j = 1, 2$. In other words, there exists a function κ satisfying $\frac{\partial \kappa}{\partial \ddot{y}_j} = 0$ for $j = 1, 2$, such that

$$\varphi_1 - 2\psi\varphi_2 + \psi^2\varphi_3 = \kappa \quad (33)$$

Differentiating the latter relation with respect to t , and taking into account the relation $\dot{\varphi}_1 - 2\psi\dot{\varphi}_2 + \psi^2\dot{\varphi}_3 = 0$ obtained from (29) and (32), we get

$$\varphi_2 - \psi\varphi_3 = -\frac{\dot{\kappa}}{2\dot{\psi}}. \quad (34)$$

We again differentiate the latter relation with respect to t to obtain

$$\varphi_3 = \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \quad (35)$$

thanks to $\dot{\varphi}_2 - \psi\dot{\varphi}_3 = 0$ from (32). Thus, solving the system (33)–(35), we immediately obtain

$$\begin{aligned} \varphi_1 &= \kappa - \psi \frac{\dot{\kappa}}{\dot{\psi}} + \psi^2 \left(\frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \right) \\ \varphi_2 &= -\frac{\dot{\kappa}}{2\dot{\psi}} + \psi \left(\frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \right) \\ \varphi_3 &= \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \end{aligned} \quad (36)$$

where κ and ψ are arbitrary functions of $y_1, y_2, \dot{y}_1, \dot{y}_2$.

Note that choosing $\kappa = y_1$ and $\psi = y_2$ yields, after inversion of (36) with (32):

$$y_1 = x_1 - 2x_2 \frac{\dot{x}_2}{\dot{x}_3} + x_3 \frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = \frac{\dot{x}_2}{\dot{x}_3},$$

which is similar to the solution obtained by F. Ollivier⁵.

Similarly, the solution of K. Schlacher and M. Schöberl [29] may be

⁵ personal communication

recovered by posing $\kappa = y_1 - y_2 \frac{\dot{y}_1}{y_2}$ and $\psi = \frac{\dot{y}_1}{2y_2}$ which, again after inversion of (36) with (32), yields:

$$y_1 = x_1 - x_3 \frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = x_2 - x_3 \frac{\dot{x}_2}{\dot{x}_3}.$$

6 Conclusion

In this survey we presented two dual approaches to flatness necessary and sufficient conditions, one based on the integration of 1-forms and the second based on the integration of a set of PDEs involving a generalized Euler-Lagrange operator. Their complexity is compared on examples.

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