Wavelets and Differentiation

François Chaplais*

October 29, 1999

Abstract

The approximation conditions of Strang and Fix are first recalled. An elementary result on the differentiation of a finite elements approximation is proved, followed by a result by Lemarié on the differentiation of a wavelet decomposition. Daubechies' spline example is detailed.

1 The Strang and Fix conditions

1.1 Main result

The following theorem relates the approximating properties of a discrete shift invariant operator to its ability to reproduce polynomials.

Theorem 1 (Fix-Strang [1]) *Let* $K \in \mathbf{L^2}_{Loc}(\mathbb{R} \times \mathbb{R})$ *such that*

$$K(t+1, s+1) = K(t, s) \ a.e. \tag{1}$$

$$\exists M \ s.t. \ K(t, s) = 0 \ if \ |t - s| \geq M \tag{2}$$

and, for $\delta > 0$*, define* P_δ *as*

$$P_\delta f(t) = \frac{1}{\delta} \int_{\mathbb{R}} K\left(\frac{t}{\delta}, \frac{s}{\delta}\right) f(s) ds \tag{3}$$

Then the two following statements are equivalent:

*Centre Automatique et Systèmes, École Nationale Supérieure des Mines de Paris, 35 rue Saint Honoré, 77305 Fontainebleau Cedex FRANCE, e-mail: chaplais@cas.ensmp.fr, http://cas.ensmp.fr/~chaplais/

- *For any $f \in \mathbf{H}^{N+1}(\mathbb{R})$ and $\delta \leq 1$,*

$$\|P_\delta f - f\|_{\mathbf{L^2}(\mathbb{R})} \leq C\delta^{N+1} \left\|f^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})} \tag{4}$$

- *For any integer p, $0 \leq p \leq N$,*

$$\int_{\mathbb{R}} K(t,s)s^p ds = t^p \tag{5}$$

A proof of this theorem is available at [2].

1.2 Example

Building an operator K which satisfies (1) is rather straightforward: assume that $k(t,s)$ is an L^2 compactly supported function. Then

$$K(t,s) = \sum_{i\in\mathbb{Z}} k(t-i, s-i) \tag{6}$$

is locally L^2 and satisfies (1) and (2).

Wavelet bases are constructed using *scaling functions*. Related approximation results are obtained by setting

$$k(t,s) = \phi(t)\phi^*(s)$$

where ϕ and ϕ^* are conjugare scaling functions. The seed kernel k is then separable.

2 Differentiation of shift generated approximations

The kernel K is now assumed to have the structure

$$K(t,s) = \sum_{i\in\mathbb{Z}} k(t-i, s-i)$$

It is assumed that k is differentiable with respect to t, that it is compactly supported and that its derivative has a finite energy. Then

$$k_1(t,s) = \sum_{j=0}^{j=+\infty} \frac{\partial k}{\partial t}(t-j, s) \tag{7}$$

is well defined because the sum is finite at every point. Hence

$$\tilde{k}(t,s) = \int_s^{s+1} k_1(t,\sigma)d\sigma \tag{8}$$

is also defined because k_1 is compactly supported and has a finite energy with respect to s. Define $\tilde{K}$ as

$$\tilde{K}(t,s) = \sum_{i\in\mathbb{Z}} \tilde{k}(t-i,s-i) \tag{9}$$

and $\tilde{P}_\delta$ by

$$\left(\tilde{P}_\delta x\right)(t) = \frac{1}{\delta}\int \tilde{K}(t,s)x(s)ds$$

Theorem 2 *$\tilde{K}$ has locally a finite energy. If K satisfies (5) for $p=0$, then $\tilde{K}$ satisfies (2) for some $M \geq 0$. The following commutation formula holds:*

$$\frac{d}{dt}P_\delta x = \tilde{P}_\delta \frac{dx}{dt} \tag{10}$$

and, if K satisfies (5), then $\tilde{K}$ satisfies

$$\int_{\mathbb{R}} \tilde{K}(t,s)s^p ds = t^p \text{ if } 0 \leq p \leq N-1 \tag{11}$$

Proof. Since k is compactly supported with a finite energy,

$$\tilde{k}_1(t,s) = \int_s^{s+1} \frac{\partial k}{\partial t}(t,s)ds$$

is compactly supported with a finite energy. Hence

$$\tilde{k}(t,s) = \sum_{i\in\mathbb{Z}}\sum_{j=0}^{j=+\infty} \tilde{k}_1(t-i-j,s-i)$$

has locally a finite energy because the summations are locally finite. Moreover, if the support of k is in $[a,b]\times[\alpha,\beta]$, then $\tilde{k}(t,s)$ is non zero only if $t-b-j \leq i \leq t-a-j$ and $s-\beta \leq i \leq s-\alpha$, which is impossible if $s-t > \beta - a$. Hence

$$\tilde{K}(t,s) = 0 \text{ if } s-t > \beta - a \tag{12}$$

On the other hand, the Strang and Fix conditions for $p=0$ imply

$$\begin{aligned} 1 &= \sum_{j\in\mathbb{Z}} \int k(t-j,s-j)ds \\ &= \sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} \int_s^{s+1} k(t-j,\sigma+i-j)d\sigma \\ &= \sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} \int_s^{s+1} k(t-i-j,\sigma-j)d\sigma \end{aligned}$$

which implies

$$0=\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\int_s^{s+1}\frac{\partial k}{\partial t}(t-i-j,\sigma-j)d\sigma \tag{13}$$

For some values of $t-s$, the summation i is restricted to $i\geq 0$. Indeed, the integral is non zero only if $s-\beta\leq j\leq s+1-\alpha$. On the other hand, $\partial k/\partial t(t-i-j,s-j)$ is non zero only if $a+j\leq t-i\leq b+j$. There exists an index j which satisfies the previous conditions with $s-\beta\leq j\leq s+1-\alpha$ only if $a+s-\beta\leq t-i\leq b+s+1-\alpha$. If $t-s\geq b+1-\alpha$, this implies $i\geq 0$, and hence

$$\begin{aligned}0 &= \sum_{j\in\mathbb{Z}}\sum_{i=0}^{i=+\infty}\int_s^{s+1}\frac{\partial k}{\partial t}(t-i-j,\sigma-j)d\sigma \\ &= \tilde{K}(t,s) \text{ if } t-s\geq b+1-\alpha\end{aligned} \tag{14}$$

which proves that

$$\tilde{K}(t,s)=0 \text{ if } |t-s|>\max(|\beta-a|,|b+1-\alpha|)$$

Since

Observe that the commutation formula holds if and only if its is valid for $\delta=1$. In this case

$$\frac{d}{dt}\int K(t,s)x(s)ds=\sum_{i\in\mathbb{Z}}\int\frac{\partial k}{\partial t}(t-i,s-i)x(s)ds \tag{15}$$

Equation (7) implies

$$\frac{\partial k}{\partial t}(t,s)=k_1(t,s)-k_1(t-1,s) \tag{16}$$

Substituting (16) into (15) yields

$$\begin{aligned}\frac{d}{dt}\int K(t,s)x(s)ds &= \sum_{i\in\mathbb{Z}}\int(k_1(t-i,s-i)-k_1(t-i-1,s-i))x(s)ds \\ &= \sum_{i\in\mathbb{Z}}\int(k_1(t-i,s-i)-k_1(t-i,s-i+1))x(s)ds\end{aligned} \tag{17}$$

Using (9) in (17) yields

$$\begin{aligned}\frac{d}{dt}\int K(t,s)x(s)ds &= -\sum_{i\in\mathbb{Z}}\int\frac{\partial\tilde{k}}{\partial s}(t-i,s-i)x(s)ds \\ &= \sum_{i\in\mathbb{Z}}\int\tilde{k}(t-i,s-i)\frac{dx}{ds}(s)ds \\ &= \int\tilde{K}(t,s)\frac{dx}{ds}(s)ds\end{aligned}$$

Finally, one can verify that (11) is a direct consequence of the commutation formula (10) applied to (5). □

3 Differentiation of wavelet decompositions

3.1 Wavelet decomposition

Theorem 2 is extended to the decomposition of a signal on a wavelet basis. A signal x is represented as

$$x(t) = \frac{1}{\delta}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\phi\left(\frac{t}{\delta}-k\right)\phi^*\left(\frac{s}{\delta}-k\right)x(s)ds \tag{18}$$

$$+ \sum_{j=0}^{j=-\infty}\frac{1}{2^j\delta}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\psi\left(\frac{t}{2^j\delta}-k\right)\psi^*\left(\frac{s}{2^j\delta}-k\right)x(s)ds \tag{19}$$

where ϕ and ϕ^* are conjugate scaling functions and ψ and ψ^* the related wavelets.

The first part (18) of the decomposition is a particular case of the approximation (3) with (6) with

$$k(t,s) = \phi(t)\phi^*(s). \tag{20}$$

Scaling functions verify a scaling equation:

$$\phi\left(\frac{t}{2}\right) = \sum_{k\in\mathbb{Z}} h_k\phi(t-k) \tag{21}$$

or, in the Fourier domain:

$$\hat{\phi}(2\omega) = \frac{1}{2}\hat{h}(\omega)\hat{\phi}(\omega)$$

where the Fourier transform of the sampled data filter h is given by

$$\hat{h}(\omega) = \sum_{k=-\infty}^{k=+\infty} h_k e^{-ik\omega} \tag{22}$$

h is called the *synthesis low pass filter*. The Fourier transform of ϕ is

$$\hat{\phi}(\omega) = \frac{1}{2}\prod_{j=1}^{j=+\infty}\hat{h}\left(2^{-k}\omega\right)\hat{\phi}(0).$$

A similar equation exists for ϕ^* with a filter h^* which is called the *analysis low pass filters*. Wavelets also satisfy a scaling equation:

$$\psi\left(\frac{t}{2}\right) = \sum_{k\in\mathbb{Z}} g_k\phi(t-k) \tag{23}$$

where g is generally taken as

$$g_k = (-1)^{k-1} h^*_{1-k} \tag{24}$$

and g^* as

$$g^*_k = (-1)^{k-1} h_{1-k}. \tag{25}$$

g is the *synthesis high pass filter* and g^* is the *analysis high pass filter* (actualling, they are band pass filters). The decomposition (18,19) is performed by a cascade of FIR filters whose coefficients are those of h_{-k} and g_{-k}; the reconstruction of the signal is performed using FIR filters whose coefficients are those of h^*_k and g^*_k. This algorithm is given by theorem 4 of section 3.5.

3.2 Main result

Lemarié [3] has extended theorem 2 and has given the related transformations on the low pass filters h and h^*:

Theorem 3 (Lemarié) *Let ϕ and ϕ^* two conjugate scaling functions such that ϕ is C^ϵ for some $\epsilon > 0$. Then there exist two conjugate scaling functions $\tilde{\phi}$ and $\tilde{\phi}^*$ such that:*

$$\phi'(t) = \tilde{\phi}(t) - \tilde{\phi}(t-1) \text{ and } \phi^*(t+1) - \phi^*(t) = \tilde{\phi}^{*\prime}(t) \tag{26}$$

Moreover, the trigonometric polynomials $\tilde{h}$ and $\tilde{h}^*$ and the biorthogonal wavelets $\tilde{\psi}$ and $\tilde{\psi}^*$ related to $\tilde{\phi}$ and $\tilde{\phi}^*$, the projectors $\tilde{P}_\delta$ related to $\tilde{k}(t,s) = \tilde{\phi}(t)\tilde{\phi}^*(t)$ and the projector $\tilde{Q}_\delta = Id - \tilde{P}_\delta$ satisfy

$$\begin{aligned} \hat{\tilde{h}}(\omega) &= \frac{2}{1+e^{-i\omega}} \hat{h}(\omega) & (27) \\ \widehat{\tilde{h}^*}(\omega) &= \frac{1+e^{i\omega}}{2} \widehat{h^*}(\omega) & (28) \end{aligned}$$

$$\tilde{\psi}(t) = \frac{1}{4}\psi'(t) \text{ and } \tilde{\psi}^{*\prime}(t) = -4\psi^*(t) \tag{29}$$

$$\frac{d}{dt} \circ P_\delta = \tilde{P}_\delta \circ \frac{d}{dt} \text{ (commutation formula)} \tag{30}$$

$$\frac{d}{dt} \circ Q_\delta = \tilde{Q}_\delta \circ \frac{d}{dt}. \tag{31}$$

Equation (26) proves that the new scaling functions $\tilde{\phi}$ and $\tilde{\phi}^*$ define a new kernel seed $\tilde{k}$ which is related to k by (7) and (8); hence the commutation formula (30) is a consequence of theorem 2.

3.3 Filter computations

3.3.1 Low pass filters

Proposition 1 *The low pass filters $\tilde{h}$ and $\tilde{h}^*$ satisfy*

$$\tilde{h}_k = 2 \sum_{p=0}^{p=+\infty} (h_{k-2p} - h_{k-2p-1}) \tag{32}$$

and

$$\tilde{h}^*_k = \frac{h^*_k + h^*_{k+1}}{2} \tag{33}$$

Proof. Equations (27) and (28) lead to the time representation of the filters $\tilde{h}$ and $\tilde{h}^*$. Using (22) in (27) yields

$$\tilde{h}_k + \tilde{h}_{k-1} = 2h_k$$

which implies

$$\tilde{h}_k - \tilde{h}_{k-2} = 2(h_k - h_{k-1})$$

and hence

$$\tilde{h}_k = 2 \sum_{p=0}^{p=+\infty} (h_{k-2p} - h_{k-2p-1})$$

The filter $\tilde{h}$ has a finite impulse response if ϕ has a compact support since the length of $\tilde{h}$ is equal to 1 plus the length of the support of $\tilde{\phi}$, which is itself finite.

Using (22) in (28) yields

$$\tilde{h}^*_k = \frac{h^*_k + h^*_{k+1}}{2}$$

□

3.3.2 High pass filters

Proposition 2 *The high pass filters $\tilde{g}$ and $\tilde{g}^*$ satisfy*

$$\tilde{g}_k = \frac{g^*_k - g^*_{k-1}}{2} \tag{34}$$

and

$$\tilde{g}^*_k = \sum_{p=0}^{p=+\infty} (g^*_{k+2p} + g^*_{k+2p+1}) \tag{35}$$

Proof. Equations (24) and (33) gives the expression of $\tilde{g}$:

$$\begin{aligned}\tilde{g}_k &= (-1)^{k-1}\frac{h^*_{1-k}+h^*_{1-k+1}}{2}\\ &= \frac{g^*_k - g^*_{k-1}}{2}\end{aligned}$$

Equations (25) and (32) gives the expression of $\tilde{g}^*$:

$$\begin{aligned}\tilde{g}^*_k &= (-1)^{k-1}2\sum_{p=0}^{p=+\infty}(h_{1-k-2p}-h_{1-k-2p-1})\\ &= 2\sum_{p=0}^{p=+\infty}(g^*_{k+2p}+g^*_{k+2p+1})\end{aligned}$$

□

Propositions 1 and 2 provide an explict computation of the “derivative” analysis and reconstruction filters from the original filters.

3.4 Coefficient computations

Proposition 3 *If a differentiable signal x is decomposed as*

$$x(t) = \frac{1}{\delta}\sum_{k\in\mathbb{Z}} c_k\phi\left(\frac{t}{\delta}-k\right) + \sum_{j=0}^{j=-\infty}\frac{1}{2^j\delta}\sum_{k\in\mathbb{Z}} d_{j,k}\psi\left(\frac{t}{2^j\delta}-k\right)$$

then

$$\frac{dx}{dt} = \frac{1}{\delta}\sum_{k\in\mathbb{Z}}\frac{c_k-c_{k-1}}{\delta}\tilde{\phi}\left(\frac{t}{\delta}-k\right) + \sum_{j=0}^{j=-\infty}\frac{1}{2^j\delta}\sum_{k\in\mathbb{Z}}\frac{4d_{j,k}}{2^j\delta}\tilde{\psi}\left(\frac{t}{2^j\delta}-k\right) \qquad (36)$$

Proof. The derivative of a signal x can be decomposed on the biortogonal system $\tilde{\phi}, \tilde{\phi}^*, \tilde{\psi}, \tilde{\psi}^*$.

If

$$P_\delta x = \sum_{k\in\mathbb{Z}} c_k\phi\left(\frac{t}{\delta}-k\right)$$

then

$$\tilde{P}_\delta\frac{dx}{dt} = \sum_{k\in\mathbb{Z}}\gamma_k\tilde{\phi}\left(\frac{t}{\delta}-k\right)$$

with

$$\begin{aligned}
\gamma_k &= \frac{1}{\delta}\int_{\mathbb{R}} \tilde{\phi}^*\left(\frac{t}{\delta}-k\right)\frac{dx}{ds}ds \\
&= -\frac{1}{\delta^2}\int_{\mathbb{R}} \frac{d\tilde{\phi}^*}{ds}\left(\frac{t}{\delta}-k\right)x(s)ds \\
&= \frac{1}{\delta^2}\int_{\mathbb{R}} \left(\phi^*\left(\frac{s}{\delta}-k\right)-\phi^*\left(\frac{s}{\delta}-k+1\right)\right)x(s)ds \\
&= \frac{c_k-c_{k-1}}{\delta}
\end{aligned}$$

Similarly, if

$$Q_\delta x = \sum_{k\in\mathbb{Z}} d_k\psi\left(\frac{t}{\delta}-k\right)$$

then

$$\tilde{Q}_\delta\frac{dx}{dt} = \sum_{k\in\mathbb{Z}} \delta_k\tilde{\psi}\left(\frac{t}{\delta}-k\right)$$

$$\begin{aligned}
\gamma_k &= \frac{1}{\delta}\int_{\mathbb{R}} \tilde{\psi}^*\left(\frac{t}{\delta}-k\right)\frac{dx}{ds}ds \\
&= -\frac{1}{\delta^2}\int_{\mathbb{R}} \frac{d\tilde{\psi}^*}{ds}\left(\frac{t}{\delta}-k\right)x(s)ds \\
&= \frac{4}{\delta^2}\int_{\mathbb{R}} \psi^*\left(\frac{s}{\delta}-k\right)x(s)ds \\
&= \frac{4d_k}{\delta}
\end{aligned}$$

$\square$

Together with propositions 1 and 2, proposition 3 provides an explicit scheme to differentiate a signal given its wavelet decomposition.

3.5 Discrete computations

A discrete signal $x[k]$ is given with a sampling interval δ. This discrete data $x[n]$ is assimilated to the scaling coefficients of a continuous time signal x, e.g.

$$x[n] = \frac{1}{\delta}\int_{\mathbb{R}} x(s)\phi^*\left(\frac{s}{\delta}-n\right)ds \tag{37}$$

Given a discrete signal x, denote by $\bar{x}$ the signal defined by $\bar{x}[n]=x[-n]$ and by $\check{x}[n]$ the signal defined by

$$\check{x}[n] = \begin{cases} x[p] & \text{if } n=2p \\ 0 & \text{if } n=2p+1 \end{cases}.$$

The scaling and wavelets coefficients of x are denoted by

$$\begin{aligned} c_j[n] &= \frac{1}{2^j\delta}\int_{\mathbb{R}} x(s)\phi^*\left(\frac{s}{2^j\delta}-n\right)ds \\ d_j[n] &= \frac{1}{2^j\delta}\int_{\mathbb{R}} x(s)\psi^*\left(\frac{s}{2^j\delta}-n\right)ds \end{aligned}$$

The following theorem [4, 5] computes the scaling and wavelet coefficients recursively along the scales:

Theorem 4 (Mallat) *At the decomposition*

$$c_{j+1}[p] = \sum_{n=-\infty}^{+\infty} h^*[n-2p]\,c_j[n] = c_j \star \bar{h}^*[2p], \tag{38}$$

$$d_{j+1}[p] = \sum_{n=-\infty}^{+\infty} g^*[n-2p]\,d_j[n] = c_j \star \bar{g}^*[2p]. \tag{39}$$

At the reconstruction,

$$\begin{aligned} c_j[p] &= \sum_{n=-\infty}^{+\infty} h[p-2n]\,c_{j+1}[n] + \sum_{n=-\infty}^{+\infty} g[p-2n]\,d_{j+1}[n] \\ &= \check{c}_{j+1}\star h[n] + \check{d}_{j+1}\star g[n]. \end{aligned} \tag{40}$$

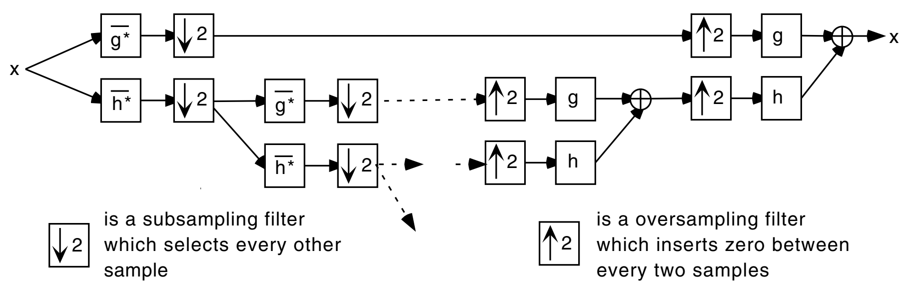

Figure 1: The decomposition and recontruction in a wavelet basis is implemented by a cascade of filter banks.

Important signal features, notably local regularity, can be described using wavelet coefficients. Manipulating these coefficients can perform denoising or local regularization. These manipulations typically occur between the analysis and reconstruction cascades, e.g. in the middle of figure 1.

The derivative of x is computed using the same decomposition, suitable coefficients manipulations as described in proposition 3 and reconstruction using the filters $\tilde{h}$ and $\tilde{g}$, as illustrated by figure 2. Denoising and/or local regularization can be performed just before the coefficients manipulation which performs the differentiation.

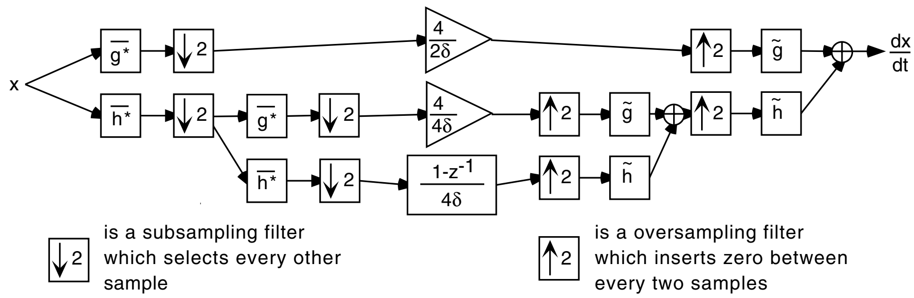

Figure 2: The derivative is reconstructed from finite differences on the scaling coefficients and scaled wavelet coefficients using the synthesis filters given by propositions 1 and 2. This example uses two cascades.

3.6 The spline case

Faure [6] has proved that the differentiation scheme of Lemarié transforms a spline biorthogonal basis into another one.

Spline biorthogonal bases are defined by their synthesis low pass filters:

$$\hat{h}(m,\omega) = \left(\cos\frac{\omega}{2}\right)^m \quad \text{if } m = 2\tilde{l} \tag{41}$$

$$\hat{h}(m,\omega) = e^{-i\frac{\omega}{2}}\left(\cos\frac{\omega}{2}\right)^m \quad \text{if } m = 2\tilde{l}+1 \tag{42}$$

and their analysis low pass filters

$$\widehat{h^*}(m,n,\omega) = \left(\cos\frac{\omega}{2}\right)^n \sum_{i=0}^{l+\tilde{l}-1} \begin{pmatrix} l+\tilde{l}-1+i \\ i \end{pmatrix} \left(\sin^2\frac{\omega}{2}\right)^i \quad \text{if } n = 2l \text{ and } m = 2\tilde{l} \tag{43}$$

$$\widehat{h^*}(m,n,\omega) = e^{-i\frac{\omega}{2}}\left(\cos\frac{\omega}{2}\right)^n \sum_{i=0}^{l+\tilde{l}} \begin{pmatrix} l+\tilde{l}+i \\ i \end{pmatrix} \left(\sin^2\frac{\omega}{2}\right)^i \quad \text{if } n = 2l+1 \text{ and } m = 2\tilde{l} \tag{44}$$

It can be verified that the synthesis scaling function ϕ is a B-spline. Hence signal approximations are splines of the same order. The scaling equation (23) implies that the detail signals are also splines.

Proposition 4 *Assume that h and h^* are two low pass filters defined by (41) and (43) (resp. (42) and (44)). Then the corresponding "derivative" filters $\tilde{h}$ and $\tilde{h}^*$ defined by (32) and (33) satisfy*

$$\left.\begin{aligned} \hat{\tilde{h}}(\omega) &= e^{i\omega}\hat{h}(m-1,\omega) \\ \widehat{\tilde{h}^*}(\omega) &= e^{i\omega}\widehat{h^*}(m-1,n+1,\omega) \end{aligned}\right\} \text{ if } m = 2\tilde{l} \text{ and } n = 2l \tag{45}$$

and

$$\left.\begin{array}{rcl}\widehat{\tilde{h}}(\omega) & = & \hat{h}(m-1,\omega) \\ \widehat{\tilde{h}^*}(\omega) & = & \widehat{h^*}(m-1,n+1,\omega)\end{array}\right\} \text{ if } m = 2\tilde{l}+1 \text{ and } n = 2l+1 \qquad (46)$$

Hence $\tilde{h}$ *and* $\tilde{h}^*$ *can be computed using (41) and (43) (resp. (42) and (44)) with a suitable change of parameters* m *and* n *and a possible shift.*

Proof. If $m = 2\tilde{l}$ then

$$\begin{aligned}\widehat{\tilde{h}}(\omega) &= \frac{2}{1+e^{-i\omega}}\left(\cos\frac{\omega}{2}\right)^m \\ &= \frac{2}{1+e^{-i\omega}}e^{i\tilde{l}\omega}\left(\frac{1+e^{-i\omega}}{2}\right)^{2\tilde{l}} \\ &= e^{i\tilde{l}\omega}\left(\frac{1+e^{-i\omega}}{2}\right)^{2\tilde{l}-1} \\ &= e^{i\frac{\omega}{2}}\left(\cos\frac{\omega}{2}\right)^{2\tilde{l}-1} \\ &= e^{i\omega}\hat{h}(m-1,\omega)\end{aligned}$$

If $m = 2\tilde{l}+1$ then

$$\begin{aligned}\widehat{\tilde{h}}(\omega) &= \frac{2}{1+e^{-i\omega}}e^{-i\frac{\omega}{2}}\left(\cos\frac{\omega}{2}\right)^m \\ &= \frac{2}{1+e^{-i\omega}}e^{i\tilde{l}\omega}\left(\frac{1+e^{-i\omega}}{2}\right)^{2\tilde{l}+1} \\ &= e^{i\tilde{l}\omega}\left(\frac{1+e^{-i\omega}}{2}\right)^{2\tilde{l}} \\ &= \hat{h}(m-1,\omega)\end{aligned}$$

On the analysis side, if $m = 2\tilde{l}$ and $n = 2l$ then

$$\begin{aligned}\widehat{\tilde{h}^*}(\omega) &= \frac{1+e^{i\omega}}{2}\left(\cos\frac{\omega}{2}\right)^n \sum_{i=0}^{l+\tilde{l}-1}\left(\begin{array}{c} l+\tilde{l}-1+i \\ i\end{array}\right)\left(\sin^2\frac{\omega}{2}\right)^i \\ &= e^{i\frac{\omega}{2}}\left(\cos\frac{\omega}{2}\right)^{n+1} \sum_{i=0}^{l+(\tilde{l}-1)}\left(\begin{array}{c} l+(\tilde{l}-1)+i \\ i\end{array}\right)\left(\sin^2\frac{\omega}{2}\right)^i \\ &= e^{i\omega}\widehat{h^*}(m-1,n+1,\omega)\end{aligned}$$

If $m = 2\tilde{l} + 1$ and $n = 2l + 1$ then

$$\begin{aligned}
\widehat{\tilde{h}^*}(\omega) &= \frac{1+e^{i\omega}}{2} e^{-i\frac{\omega}{2}} \left(\cos\frac{\omega}{2}\right)^n \sum_{i=0}^{l+\tilde{l}} \begin{pmatrix} l+\tilde{l}+i \\ i \end{pmatrix} \left(\sin^2\frac{\omega}{2}\right)^i \\
&= \left(\cos\frac{\omega}{2}\right)^{n+1} \sum_{i=0}^{(l+1)+\tilde{l}-1} \begin{pmatrix} (l+1)+\tilde{l}-1+i \\ i \end{pmatrix} \left(\sin^2\frac{\omega}{2}\right)^i \\
&= \widehat{h^*}(m-1, n+1, \omega)
\end{aligned}$$

$\square$

References

[1] G. Strang and G. Fix, "A Fourier analysis of the finite element variational method," *Construct. Aspects of Funct. Anal.*, pp. 796–830, 1971.

[2] F. Chaplais, "The strang and fix conditions." available at `http://cas.ensmp.fr/ chaplais/FTP/Mathematical_Notes/Strang_and_Fix.pdf`.

[3] P. Lemarié-Rieusset, "Analyses multi-résolutions non orthogonales, commutation entre projecteurs et dérivation et ondelettes vecteurs à divergence non nulle," *Revista Matemática Iberoamericana*, vol. 8, no. 2, pp. 221–237, 1992.

[4] S. Mallat, "An efficient image representation for multiscale analysis," in *Proc. of Machine Vision Conference*, (Lake Taho), February 1987.

[5] S. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation," *IEEE Trans. Patt. Recog. and Mach. Intell.*, vol. 11, pp. 674–693, July 1989.

[6] S. Faure, "Implémentation sous simulink de la dérivation de signaux à partir des coefficients d'ondelettes." Internal Report, 1999.