Product Invariant Piecewise Polynomial Approximations of Signals

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Abstract— **The Strang and Fix conditions relate the accuracy of a finite element method to its ability to reproduce polynomials. A similar condition is proved to exist for the approximation of the product on these finite elements. Piecewise polynomial approximations are studied further, including a constructive description of all related approximate product operators.**

I. Introduction

Some piecewise polynomial approximation results are presented as particular examples of finite elements approximations. Section II recalls the classical results by Strang and Fix [1]. Section III derives a similar result on the approximation of the product operator when functions are approximated by finite elements. Both results are essentially based of the local representation of functions by polynomials. To exploit this, section IV presents two classes of finite element approximations which produce piecewise polynomial functions. Section V uses the result of section II to prove that linear approximation operators on these piecewise polynomial finite elements should verify the same conditions as in section II. Finally, section VI uses section III to characterize product operators on these piecewise polynomial finite elements which approximate the natural product. Such products are defined by Hermite interpolation.

All of these results are used in [2] to build time/scale analysis of signals which may be used in a nonlinear framework.

II. Approximation by finite elements: the Strang and Fix conditions

The approximation conditions of Strang and Fix describe a large family of finite element methods.

Theorem 1 (Strang & Fix [1]) Let $K \in \mathbf{L^2}_{\mathrm{Loc}}(\mathbb{R}\times\mathbb{R})$ such that

$$K(t+1, s+1) = K(t,s) \text{ a.e.} \tag{1}$$

$$\exists M \text{ s.t. } K(t,s) = 0 \text{ if } |t-s| \geq M \tag{2}$$

and, for $\delta > 0$, define P_δ as

$$P_\delta f(t) = \frac{1}{\delta}\int_{\mathbb{R}} K\left(\frac{t}{\delta}, \frac{s}{\delta}\right) f(s)ds \tag{3}$$

Then

- There exists $C \geq 0$ such that, for any $f \in \mathbf{L^2}(\mathbb{R})$ and $\delta \leq 1$,

$$\|P_\delta f\|_{\mathbf{L^2}(\mathbb{R})} \leq C\|f\|_{\mathbf{L^2}(\mathbb{R})} \tag{4}$$

- The three following statements are equivalent:
  - For any $f \in \mathbf{H}^N(\mathbb{R})$,

$$\delta^{-N}\|P_\delta f - f\|_{\mathbf{L^2}(\mathbb{R})} \to 0 \text{ when } \delta \to 0 \tag{5}$$

  - For any $f \in \mathbf{H}^{N+1}(\mathbb{R})$ and $\delta \leq 1$,

$$\|P_\delta f - f\|_{\mathbf{L^2}(\mathbb{R})} \leq C\delta^{N+1}\left\|f^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})} \tag{6}$$

  - For any integer p, $0 \leq p \leq N$,

$$\int_{\mathbb{R}} K(t,s)s^p ds = t^p \tag{7}$$

A complete proof is available at [3]; a paper copy is available by request to the author.

The localization of P_δ yields a local approximation result which does not require the signal to have a finite energy on the whole real axis:

Corollary 1: Let K a kernel which satisfies the assumptions of theorem 1. Additionnaly, it is assumed that

$$\kappa \stackrel{\text{def}}{=} \sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} |K(t,s)|\, ds < +\infty. \tag{8}$$

Then, for any bounded function f,

$$|P_\delta f(t)| \leq \kappa \sup_{|s-t|\leq M\delta} |f(s)|. \tag{9}$$

Moreover, if K satisfies condition (7), then, for any f of class C^{N+1}

$$|P_\delta f(t) - f(t)| \leq C\delta^{N+1} \sup_{|s-t|\leq M\delta} |f^{(N+1)}(s)|$$

Proof: Condition (9) is a straightforward consequence of assumptions (2) and (8). Denote by $\mathcal{T}f$ the Taylor expansion of f at point t and degree N. Then

$$\begin{aligned} |P_\delta \mathcal{T}f(t) - P_\delta f(t)| &= \left|\frac{1}{\delta}\int_{t-M\delta}^{t+M\delta} K\left(\frac{t}{\delta}, \frac{s}{\delta}\right)[\mathcal{T}f(s) - f(s)]\right| \\ &\leq \frac{\kappa}{(N+1)!}(M\delta)^{N+1} \sup_{|s-t|\leq M\delta}\left|f^{(N+1)}(s)\right| \end{aligned}$$

By definition of $\mathcal{T}f$, $\mathcal{T}f(t) = f(t)$. Finally, condition (7) implies that $(P_\delta\mathcal{T}f)(t) = \mathcal{T}f(t)$. Hence $|P_\delta f(t) - f(t)| = |P_\delta f(t) - P_\delta\mathcal{T}f(t)|$ and the result is proved. ■

Further additional assuption: the kernel K will be now assumed to satisfy (8). Then $P_\delta f$ is defined for any locally bounded f. Observe that condition (8) is actually a localized condition; indeed, condition (1) implies that $\kappa = \sup_{t\in[0,1]}\int_{\mathbb{R}}|K(t,s)|\,ds$ and condition (2) further implies $\kappa = \sup_{t\in[0,1]}\int_{t-M}^{t+M}|K(t,s)|\,ds$.

III. Product approximation on finite elements

Theorem 1 relates the accuracy of the approximation operator P_δ to its ability to reproduce polynomials. Most often, the image space $\mathbf{I}_\delta$ of P_δ is not invariant by product. Projecting the product on the finite element space does not solve the problem because the resulting product operator is generally not associative, (see [2] for a discussion on this topic).

However, it is sometimes possible to retrieve product invariance by using an approximate product instead of the "natural". The following theorem gives necessary and sufficient conditions for a product operator on $\mathbf{I}_\delta$ to be an accurate approximation of the usual product, that is, as accurate as the operator P_δ itself.

The study is restricted to the product operators $*$ which satisfy the following assumptions:

- *Product invariance:*

$$\forall f, g \in \mathbf{I}_1, f * g \in \mathbf{I}_1 \tag{10}$$

- *Shift invariance:* there exists $L > 0$ such that, for any f, g in $\mathbf{I}_1$,

$$f(t-L) * g(t-L) = (f*g)(t-L) \tag{11}$$

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- *Localization and continuity:* there exists K and μ such that,

$$|f * g|(t) \leq K \sup_{|s-t|\leq\mu} |f(s)| \sup_{|s-t|\leq\mu} |g(s)| \tag{12}$$

A product operator is defined on the image $\mathbf{I}_\delta$ of P_δ by rescaling:

$$f\left(\frac{t}{\delta}\right) *_\delta g\left(\frac{t}{\delta}\right) = (f * g)\left(\frac{t}{\delta}\right) \tag{13}$$

Theorem 2 (Characterization of approximate products) A product operators which satisfies conditions (10), (11) and (12) is assumed to exist. Then the two following conditions are equivalent:

- There exist K such that, for any functions f and g of class C^{N+1} and any $\delta \leq 1$,

$$\begin{aligned} &|fg(t) - [(P_\delta f) *_\delta (P_\delta g)](t)| \\ &\leq K\delta^{N+1} \sup_{\substack{0\leq k\leq N+1 \\ 0\leq l\leq N+1 \\ k+l\geq N+1}} \sup_{|s-t|\leq(\mu+M)\delta} \left|f^{(k)}(s)\right| \left|g^{(l)}(s)\right| \end{aligned} \tag{14}$$

- the following consistency condition is satisfied

$$t^i * t^j = t^{i+j} \text{ if } i+j \leq N \tag{15}$$

that is, the usual product applies on polynomials when the multipliers and the result belong to the image space.

Proof: *Necessary condition:* observe that, if i and j are two integers such that $i+j \leq N$, then any pair of integers (k,l) such that $k+l \geq N+1$ must verify $k \geq i+1$ or $l \geq j+1$. Substituting $f(t) = t^i$ and $g(t) = t^j$ with $i+j \leq N$ in (14) and using condition (7) hence implies (15) since the product of the derivatives is always zero.
Sufficient condition: let f and g two functions of class C^{N+1}. Denote by $\mathcal{T}x$ the Taylor expansion of a function x at point t with degree N, $\mathcal{T}x(s) = \sum_{i=0}^{i=N} f^{(i)}(t)/i!(s-t)^i$. The left handside of (14) is decomposed as

$$\begin{aligned} &|fg(t) - [(P_\delta f) *_\delta (P_\delta g)](t)| \\ \leq\ &|fg(t) - \mathcal{T}(fg)(t)| && (16) \\ &+ |[P_\delta f *_\delta P_\delta(g - \mathcal{T}g)](t)| && (17) \\ &+ |[P_\delta(f - \mathcal{T}f) *_\delta P_\delta \mathcal{T}g](t)| && (18) \\ &+ |\mathcal{T}(fg)(t) - [P_\delta(\mathcal{T}f) *_\delta P_\delta(\mathcal{T}g)](t)| && (19) \end{aligned}$$

By definition of $\mathcal{T}$, (16) is zero. Observe that (12) and (13) imply

$$|(f *_\delta g)(t)| \leq K \sup_{|s-t|\leq\mu\delta} |f(s)| \sup_{|s-t|\leq\mu\delta} |g(s)|. \tag{20}$$

Corollary 1 and (20) imply that (17) is bounded by

$$\frac{K\kappa^2}{(N+1)!}(M\delta)^{N+1} \sup_{|s-t|\leq(\mu+M)\delta} |f(s)| \sup_{|s-t|\leq(\mu+M)\delta} |g^{(N+1)}(s)|.$$

Since $P_\delta \mathcal{T}g = \mathcal{T}g$, the same arguments prove that (18) is bounded by

$$\frac{K\kappa^2}{(N+1)!}(M\delta)^{N+1} \sup_{|s-t|\leq(\mu+M)\delta} |f^{(N+1)}(s)| \sup_{|s-t|\leq\mu\delta} |\mathcal{T}g(s)|$$

with

$$\sup_{|s-t|\leq\mu\delta} |\mathcal{T}g(s)| \leq \left(\sup_{|s-t|\leq\mu\delta} |g(s)| + \frac{(\mu\delta)^{N+1}}{(N+1)!} \sup_{|s-t|\leq\mu\delta} \left|g^{(N+1)}(s)\right|\right).$$

Finally, condition (7) implies $P_\delta \mathcal{T}f = \mathcal{T}f$ and $P_\delta \mathcal{T}g = \mathcal{T}g$; condition (15) implies that the difference $\mathcal{T}(fg) - \mathcal{T}f *_\delta \mathcal{T}g$ has the expression

$$\begin{aligned} [\mathcal{T}(fg) - \mathcal{T}f *_\delta \mathcal{T}g](t) &= \sum_{k+l\geq N+1} \frac{f^{(k)}(t)}{k!}\frac{g^{(l)}(t)}{l!} \\ &\quad \times \left((s-t)^i *_\delta (s-t)^j\right)(t) \end{aligned}$$

The product bound (20) implies

$$\left|\left((s-t)^k *_\delta (s-t)^l\right)(t)\right| \leq K(\mu\delta)^{k+l} \leq K(\mu\delta)^{N+1}$$

Hence there exists an nonnegative real c such that (19) is bounded by

$$c(\mu\delta)^{N+1} \sup_{k+l\geq N+1} \left|f^{(k)}(t)\right| \left|g^{(l)}(t)\right|$$

Combining the bounds on (17), (18) and (19) yields the bound (14). ■

IV. From finite elements to piecewise polynomial approximations

Theorems 1 and 2 characterize the approximation properties of linear and product operators by the way they operate on polynomials. Condition (7) has been studied for a wide class of kernels, without necessarily using polynomials explicitely. Finding a product operator on finite elements which satisfies (15) is another matter. This task is considerably simplified is the finite elements are themeselves polynomials. For this reason, kernels which produce piecewise polynomial approximations are presented here.

It is assumed thar the kernel K can be written as

$$K(t,s) = \sum_{n\in\mathbb{Z}} \phi(t-n)\phi^*(t-n)$$

This is the case if, for instance, the approximation is obtained by projection on a resolution space related to wavelets (see [4], [6] for a nice presentation of wavelets). This section presents two methods which derive a piecewise polynomial kernel $\tilde{K}$ with the structure

$$\tilde{K}(t,s) = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{l=N} (t-k)^l 1_{[k,k+1)}(t)\varphi_l(s-k) \tag{21}$$

from the functions ϕ and ϕ^*. In both methods, φ is a finite linear combination of some $\phi^*(t-k)$, $k \in \mathbb{Z}$. It can be thus extected that the filtering properties of ϕ^* are preserved in the new kernel.

The following proposition specializes the Strang and Fix theorem 1 to kernels which satisfy (21).

Proposition 1: Let $\tilde{K}$ defined by (21) and assume that φ is compactly supported with a finite energy. Then the approximation conditions (5) and (6) are satisfied if and only if

$$\sum_{l=0}^{l=N} t^l \int_{\mathbb{R}} s^p \varphi_l(s) ds = t^p \text{ if } 0 \leq p \leq N \tag{22}$$

Proof: $\tilde{K}$ is indeed in $\mathbf{L^2}_{\mathrm{Loc}}(\mathbb{R}\times\mathbb{R})$; equation (21) shows that condition (1) is satisfied; condition (2) is also satisfied since φ is compactly supported. The proposition is proved if (22) is proved to be equivalent to (7). It is obviously necessary. Assume now that it is satisfied. Then (7) is satisfied if $(t+k)^p = \sum t^l \int_{\mathbb{R}} (s+k)^p \varphi_l(s) ds$. This is verified by expanding the polynomials on both sides. ■

The piecewise polynomial presentations are now presented.

A. Polynomial identification on the scaling coefficients

This method is based on the following theorem, which proves that any sequence of $q+1$ real numbers can be interpreted as a sequence of scaling coefficients $\int Q(s)\phi^*(n-s)ds$ of a polynomial Q of degree smaller or equal to q. Using this polynomial identification procedure leads to a piecewise polynomial approximation operator which preserves polynomials.

Lemma 1 (Generalized Lagrange) Let $(x_0,\ldots,x_q)$ a sequence of $q+1$ real numbers. Then, for any $j\in\mathbb{N}$, there exists a unique polynomial Q of degree $\le q$ such that

$$x_k = < Q, \phi^*_{j,k} > \text{ for all } 0\le k\le q \tag{23}$$

This lemma is proved in appendix A.

Let M the matrix which transforms a sequence $(x_0,\ldots,x_q)$ into a polynomial of degree less than q. This linear transformation is extended to sequences of arbitrary length by using a shift invariant construction.

If f is an integrable signal, a piecewise polynomial representation of f,

$$\tilde{P}_\delta f(t) = p_k(t) \text{ if } t\in[k(q+1)\delta,(k+1)(q+1)\delta),$$

is defined by

$$\begin{aligned} p_{k,\delta}(t) &= \frac{1}{\delta}\sum_{j=0}^{j=q}\sum_{i=0}^{i=q} m_{j,i}\left(\frac{t}{\delta}\right)^j \\ &\quad \int_{\mathbb{R}} f(s)\phi^*\left(\frac{s}{\delta}-k(q+1)-i\right)ds \end{aligned}$$

The related kernel is

$$\begin{aligned} \tilde{K}_\delta(t,s) &= \sum_{k\in\mathbb{Z}}\sum_{j=0}^{j=q}\sum_{i=0}^{i=q} m_{j,i}\left(\frac{t}{\delta}-k(q+1)\right)^j \\ &\quad 1_{[k,k+1)}\left(\frac{t}{(q+1)\delta}\right)\phi^*\left(\frac{s}{\delta}-k(q+1)-i\right) \end{aligned}$$

This representation is an approximation:

Proposition 2: Define $\tilde{K}(t,s)=\tilde{K}_1(t,s)$ and $\tilde{P}_\delta$ using (3). Then $\tilde{P}_\delta$ satisfies the approximation equations (5) and (6) with the order q, independently of the approximation order related to ϕ.

Proof: Using the change of scales $\delta'=(q+1)\delta$ shows that $\tilde{K}_{\delta'}$ has the structure (21) with

$$\varphi_l(s)=\sum_{i=0}^{i=q}(q+1)^{l+1}m_{l,i}\phi^*((q+1)s-i)$$

The assumptions of proposition 1 are satisfied. Let us verify (22).

$$\begin{aligned} &\sum_{j=0}^{j=q} t^j\int_{\mathbb{R}} t^s\varphi_j(s)ds \\ =& \sum_{i,j=0}^{i,j=q} t^j(q+1)^{j+1}m_{j,i}\int_{\mathbb{R}} s^p\phi^*((q+1)s-i)ds \\ =& \sum_{i,j=0}^{i,j=q} t^j(q+1)^{j-p}m_{j,i}\int_{\mathbb{R}} s^p\phi^*(s-i)ds \\ =& \sum_{i,j=0}^{i,j=q} t^j(q+1)^{j-p}m_{j,i}a_{i,p} \\ =& \sum_{i,j=0}^{i,j=q} t^j(q+1)^{j-p}\delta_{j,p}=t^p \end{aligned}$$

■

Observe that the approximating operator, and hence its order of approximation, does not depend on the synthesis finite element ϕ. Moreover, any polynomial sequence $(p_k)_{k\in\mathbb{Z}}$ can be interpreted as an element of a resolution $\mathbf{V}_J$ using the matrix $a_{i,j}$ described in theorem 1 and taking $\delta=2^J$. Hence there is a one to one correspondence between polynomial sequences and the elements of a resolution space; and it is also a correspondence between two different approximation operators.

B. Lagrange interpolation on the approximation

The drawback of the previous method is that it multiplies the shift length δ by a factor $q+1$. This section presents an approximation method which keeps the shift length unchanged. To do so, it performs a Lagrange interpolation of degree N on the elements of the image of P_δ, at knots which are equally spaced with a distance δ/N; this interpolation is extended to the whole real axis using shifts of length δ. Here N is defined by the Strang and Fix conditions.

Let p_j the Lagrange interpolation of $\phi(t-j)$ at the points $t=\frac{i}{N}$, for $0\le i\le N$:

$$p_j\left(\frac{i}{N}\right)=\phi\left(\frac{i}{N}-j\right) \text{ for } 0\le i\le N \tag{24}$$

Only a finite number of p_j are non zero. Observe that $p_j\left(\frac{t}{\delta}\right)$ is the Lagrange interpolation of $\phi\left(\frac{t}{\delta}-j\right)$ at the points $t=\frac{i\delta}{N}$. If $g=\sum_{j\in\mathbb{Z}}c_j\phi(t/\delta-j)$ is an element of $\mathbf{I}_\delta$, then its Lagrange interpolation $\mathcal{L}_\delta g$ at the points $\frac{i\delta}{N}$, $0\le i\le N$ is $\mathcal{L}_\delta g$ with

$$\mathcal{L}_\delta g(t)=\sum_{j\in\mathbb{Z}}c_jp_j\left(\frac{t}{\delta}\right)$$

Similarly, the interpolation of g at the points $\frac{i\delta}{N}+k$, $0\le i\le N$ is $\mathcal{L}_{k,\delta}g$ with

$$\mathcal{L}_{k,\delta}g(t)=\sum_{j\in\mathbb{Z}}c_{j+k}p_j\left(\frac{t}{\delta}-k\right).$$

If $g=P_\delta f$, then

$$\sum_{k\in\mathbb{Z}}\mathcal{L}_{k,\delta}P_\delta f(t)=\sum_{j\in\mathbb{Z}}p_j\left(\frac{t}{\delta}-k\right)\int_{\mathbb{R}} f(s)\phi^*\left(\frac{s}{\delta}-j-k\right)ds$$

is a piecewise polynomial representation of f. The related kernel is

$$\tilde{K}_\delta(t,s)=\sum_{k,j\in\mathbb{Z}}p_j\left(\frac{t}{\delta}-k\right)1_{[k,k+1)}\left(\frac{t}{\delta}\right)\phi^*\left(\frac{s}{\delta}-j-k\right)$$

The previous representation is an approximation:

Proposition 3: Define $\tilde{K}(t,s)=\tilde{K}_1(t,s)$ and $\tilde{P}_\delta$ using (3). Then $\tilde{P}_\delta$ satisfies the approximation equations (5) and (6) with the order N related to K by (7).

Proof: Since p_j has a degree smaller or equal to N and since ϕ^* is compactly supported, there exist a family φ_l, $0\le l\le N$ such that

$$\sum_{j\in\mathbb{Z}}p_j(t)\phi^*(s-j)=\sum_{l=0}^{l=N}t^l\varphi_l(s)$$

Each φ_l has a finite energy since it is a linear combination of a finite number of $\phi^*(s-j)$. It is also compactly supported. With these notations, $\tilde{K}$ satisfies the assumptions of proposition

1. Let us check that it satisfies condition (22). Condition (24) implies that

$$\begin{aligned}\sum_{l=0}^{l=N}\left(\frac{i}{N}\right)^l\int_{\mathbb{R}} s^p\varphi_l(s)ds &= \sum_{j\in\mathbb{Z}} p_j\left(\frac{i}{N}\right)\int_{\mathbb{R}} s^p\phi^*(s-j)ds \\ &= \sum_{j\in\mathbb{Z}}\phi\left(\frac{i}{N}-j\right)\int_{\mathbb{R}} s^p\phi^*(s-j)ds \\ &= \left(\frac{i}{N}\right)^p \text{ for } 0\le i\le N\end{aligned}$$

because of condition (7). Hence, the polynomial $\sum_{l=0}^{l=N} t^l\int_{\mathbb{R}} s^p\varphi_l(s)ds$ coincides with t^p at the $N+1$ sample points i/N; it is necessarily equal to t^p. ■

V. Approximating linear operators for piecewise polynomials

Multiresolution approximations of signals are obtained by applying a linear operator to a fine approximation of a signal to get a cruder approximation described by a smaller number of parameters. This section characterizes the linear transformations on piecewise polynomial functions which also generate approximations.

A. *Representation of linear operators over piecewise polynomial approximations*

Let $S_{N,\delta}(t)$ the space of piecewise polynomial function with degree smaller or equal to N over the intervals $[k\delta,(k+1)\delta)$, $k\in\mathbb{Z}$, and $\mathcal{Q}$ an operator from $S_{N,\delta}(t)$ into itself. It is assumed that $\mathcal{Q}$ is localized and shift generated. More precisely, it is assumed that the image $\mathcal{Q}r$ of a piecewise polynomial function r

$$r(t)=\sum_{k\in\mathbb{Z}} 1_{[k,k+1)}(t)\sum_{l=0}^{l=N} r_{l,k}(t-k)^l$$

is written as

$$\mathcal{Q}r(t)=\sum_{k\in\mathbb{Z}} 1_{[k,k+1)}(t)\sum_{i=0}^{i=q-1}\sum_{j=i-M}^{j=i+M}\sum_{l=0}^{l=N} q_{l,i,j,m}r_{m,j+kq}(t-k)^l$$

$\mathcal{Q}$ commutes with shifts of length q. Observe that $\mathcal{Q}$ defines a unique linear operator Q on the space $S_N[t]$ of polynomial sequences $(r_k)_{k\in\mathbb{Z}}$ of degree smaller than N, with

$$(Qr)_k(t)=\sum_{i=0}^{i=q-1}\sum_{j=i-M}^{j=i+M}\sum_{l=0}^{l=N} q_{l,i,j,m}r_{m,j+kq}(t-k)^l \tag{25}$$

Consider now a kernel $\tilde{K}$ which has the structure (21), and its related approwimation operators $\tilde{P}_\delta$. Then the composition $\mathcal{Q}\tilde{P}_1$ defines a new kernel H with

$$\begin{aligned}H(t,s) &= \sum_{k\in\mathbb{Z}}\sum_{i=0}^{i=q-1}\sum_{j=i-M}^{j=i+M}\sum_{l=0}^{l=N} q_{l,i,j,m}p_m(s-j-kq) \\ &\quad (t-k)^l 1_{[k,k+1)}(t)\end{aligned}$$

Using (3), this kernel defines a new family of scaled operators. The following proposition gives a decomposition of such operators.

Proposition 4: Let $\mathcal{P}_\delta$ the operator defined by

$$\mathcal{P}_\delta f(t)=\frac{1}{\delta}\int_{\mathbb{R}} H\left(\frac{t}{\delta},\frac{s}{\delta}\right)f(s)ds$$

Let $\mathcal{F}_\delta$ the one to one operator between $S_N[t]$ and $S_{N,\delta}(t)$ defined by

$$(\mathcal{F}_\delta r)(t)=\sum_{k\in\mathbb{Z}} r_k\left(\frac{t}{\delta}-k\right)1_{[k,k+1)}\left(\frac{t}{\delta}\right).$$

Then

$$\mathcal{P}_\delta=\mathcal{F}_\delta Q\mathcal{F}_\delta^{-1}P_\delta$$

This proposition is proved in appendix B.

B. *Approximation condition*

The following theorem characterizes which scale independent operators $\mathcal{Q}$ generate piecewise polynomial approximation operators $\mathcal{P}_\delta$.

Theorem 3 (Approximation and polynomials) Let P_δ a family of piecewise polynomial approximation operators defined by (3) with an order N, and Q an localized shift invariant linear operator from $S_N[t]$ into $S_N[t]$, defined by (25). Define the operator $\mathcal{P}_\delta$ by

$$\mathcal{P}_\delta=\mathcal{F}_\delta Q\mathcal{F}_\delta^{-1}P_\delta$$

Then $\mathcal{P}_\delta$ satisfies the approximation property (6) of theorem 1 if and only if

$$Q\left[((t+k)^p)_{k\in\mathbb{Z}}\right]=((t+k)^p)_{k\in\mathbb{Z}} \text{ for } 0\le p\le N \tag{26}$$

Proof: Proposition 4 proves that $\mathcal{P}_\delta$ is obtained by using the rescaled kernel H. Hence it satisfies (6) if and only if $\mathcal{P}_1 t^p = t^p$ for $p\le N$. From the definition of $\mathcal{F}_\delta$, this amounts precisely to (26). ■

Corollary 2 (Images of polynomials) Assume that $\mathcal{P}_\delta$ satisfies (6). Then the image of Q includes the sequences

$$((t+k)^p)_{k\in\mathbb{Z}}$$

for $0\le p\le N$.
Building a system of successive approximations similar to multiresolution analysis thus implies that each approximation space includes the polynomials $((t+k)^p)_{k\in\mathbb{Z}}$. This is used in [2] to study successive approximation in product invariant spaces.

VI. Piecewise polynomial product approximations

As mentioned in section III, the image space $\mathbf{I}_\delta$ of P_δ is generally not invariant by product. If there exists an underlying succesive approximation scheme, as in multiresolution analysis, then its structure is broken by the product. Using an approximate product operator which satisfies the conditions of theorem 2 is a possibilty for retrieving product invariance on an approximation space. This section characterizes all approximate product operator on piecewise polynomial approximations; these products are determined by a Hermite interpolation.

A. *Characterization of approximate product operators for piecewise polynomial functions*

Theorem 2 is specialized to piecewise polynomial representations of signals.

Lemma 2 (Characterization of approximate products)
Assume that $\times$ is a product operator over $\mathbb{R}_N[t]$, and define the operator $*_\delta$ over $S_{N,\delta}(t)$ by

$$(f*_\delta g)(t)=\mathcal{F}_\delta\left((\mathcal{F}_\delta^{-1}f)_k\times(\mathcal{F}_\delta^{-1}g)_k\right)_{k\in\mathbb{Z}} \tag{27}$$

Then $*_\delta$ approximates the product on the image of P_δ like in condition (14) of theorem 2 if and only if $\times$ satisfies (15):

$$t^i\times t^j=t^{i+j}\in S_N[t] \text{ if } i+j\le N \tag{28}$$

Proof: the result is proved by proving that the assumptions of theorem 2 are verified by $*_\delta$ and that (15) holds if and only if $t^i \times t^j = t^{i+j}$ when $i + j \leq N$.

Let us first prove that $*_\delta$ is obtained by rescaling. Indeed,

$$\begin{aligned} f\left(\frac{t}{\delta}\right) *_\delta g\left(\frac{t}{\delta}\right) &= \mathcal{F}_\delta\left(\left(\mathcal{F}_\delta^{-1} f\left(\frac{t}{\delta}\right)\right) \times\times \left(\mathcal{F}_\delta^{-1} g\left(\frac{t}{\delta}\right)\right)\right) \\ &= \mathcal{F}_\delta(\mathcal{F}_1 f \times\times \mathcal{F}_1 g) \\ &= \mathcal{F}_1(\mathcal{F}_1 f \times\times \mathcal{F}_1 g)\left(\frac{t}{\delta}\right) \\ &= (f * g)\left(\frac{t}{\delta}\right) \end{aligned}$$

where $*$ denotes the operator $*_1$.

The first condition on $*$ states that it must be shift invariant. Denote by σ the shift operator on $S_N[t]$ defined by $(\sigma r)_k = r_{k-1}$, Σ_δ the shift operator on $S_{N,\delta}(t)$ defined by $(\Sigma_\delta f)(t) = f(t-\delta)$ and $\times\times$ the product operator on $S_N[t]$ defined by $(r \times\times s)_k = r_k \times s_k$. Then

$$\Sigma_\delta \mathcal{F}_\delta = \mathcal{F}_\delta \sigma \tag{29}$$

Condition (27) implies

$$\begin{aligned} (\Sigma_\delta f * \Sigma_\delta g) &= \mathcal{F}_\delta\left((\mathcal{F}_\delta^{-1}\Sigma_\delta f) \times\times (\mathcal{F}_\delta^{-1}\Sigma_\delta g)\right) \\ &= \mathcal{F}_\delta\left((\sigma\mathcal{F}_\delta^{-1} f) \times\times (\sigma\mathcal{F}_\delta^{-1} g)\right) \\ &= \mathcal{F}_\delta\sigma\left((\mathcal{F}_\delta^{-1} f) \times (\mathcal{F}_\delta^{-1} g)\right) \\ &= \Sigma_\delta\mathcal{F}_\delta\left((\mathcal{F}_\delta^{-1} f) \times (\mathcal{F}_\delta^{-1} g)\right) \\ &= \Sigma_\delta(f * g) \end{aligned}$$

which proves that the product is shift invariant.

The second condition is a condition of local continuity. Any product operator $\times$ over a finte dimensional algebra is continuous with $\|a \times b\| \leq C\|a\|\|b\|$. Hence $*$ is continuous over each finite dimensional algebra defined as the space of polynomial functions over the integer interval $[k, k+1)$. Since all norms are equivalent in finite dimensional spaces, $*$ is continuous for the sup norm. Finally, the definition of $*$ implies that, if the minimum integer intervals which includes the supports of two functions do not overlap, then their product is zero. Hence condition (12) is satisfied.

Therefore the assumptions of theorem 2 are satisfied. Hence the product approximation condition (14) is verified if and only if (15) is satisfied in $S_{N,1}(t)$. Observe that $\mathcal{F}_1^{-1} t^p = ((t-k)^p)_{k\in\mathbb{Z}}$. Applying $\mathcal{F}_1^{-1}$ to condition (15) and using (27) proves that (15) is verified if and only if $(t-k)^i \times (t-k)^j = t^{i+j}$ if $i + j \leq N$ in $\mathbb{R}_N[t]$. This is verified for any $k \in \mathbb{Z}$ if and only if it is true for $k = 0$; this proves that (14) is equivalent to (28). ∎

B. Constructive description of approximate product operators for piecewise polynomial functions

The following lemma characterizes the product operators on $R_N[t]$ which satisfy (15).

Lemma 3 (Hermite interpolation product) Let $\times$ an associative, commutative product over $\mathbb{R}_N[t]$ which satisfies (15), and define $T_{N+1} \in \mathbb{R}_N[t]$ by $T_{N+1} = t^N \times t$. Then, for any p and q in $\mathbb{R}_N[t]$, $p \times q$ is the Hermite interpolation of pq at the (possibly multiple) zeros of $t^{q+1} - T_{N+1}(t)$ in $\mathbb{C}$. Conversely, any such product is associative, commutative, and satisfies (15) in $\mathbb{R}_N[t]$. This lemma is proved in appendix C.

Assembling lemmas 2 and 3 proves the main theorem of this paper:

Theorem 4 (Product approximation using finite elements) Assume that $\times$ is a product operator over $\mathbb{R}_N[t]$, and define the operator $*_\delta$ over $S_{N,\delta}(t)$ by equation (27). Then $*_\delta$ approximates the product on the image of P_δ like in condition (14) of theorem 2 if and only if $\times$ is a Hermite interpolation of the product like in lemma 3.

C. Interpretation of the product

It is convenient to think of $\times$ as a sequence of simpler products at the interpolation points. Specifically, denote by $(z_1, \ldots, z_M)$ the set of interpolation points which defines $\times$ and $(\omega_1, \ldots, \omega_M)$ the corresponding orders of interpolation, and denote by $\mathcal{T}_\omega p$ the Taylor expansion of a polynomial p at order ω. Then, for any pair of polynomials (p, q) with expansions $(p_1, \ldots, p_N)$ and $(q_1, \ldots, q_N)$ at points $(z_1, \ldots, z_M)$, the corresponding product $p \times q$ has the expansions $(\mathcal{T}_{\omega_1}(p_1 q_1), \ldots, \mathcal{T}_{\omega_M}(p_M q_M))$ at the same points. Figure 1 shows how cubic polynomials can be parameterized as couples of linear expansions.

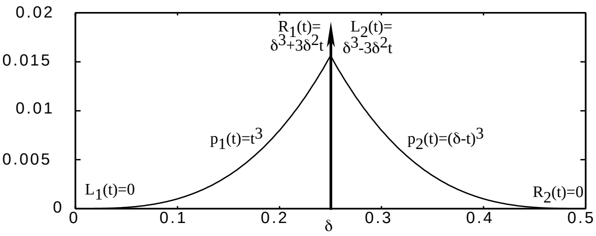

Fig. 1. A sequence $(p_1(t) = t^3, p_2(t) = (\delta - t)^3)$ of two cubic polynomials can be viewed as a sequence $((L_1(t) = 0, R_1(t) = \delta^3 + 3\delta^2 t), (L_2(t) = \delta^3 - 3\delta^2 t, R_2(t) = (\delta - t)^3)))$ of couples of polynomials with degree 1. L_i polynomials are Taylor expansions at the left point; R_i are expansions at the right point.

VII. Conclusion

A general product approximation theorem on finite elements has been proved in section III. To build such approximate product operators, section IV derives piecewise polynomial approximations from general finite element approximations. In this framework, all approximate product operators are defined by a Hermite interpolation on the usual product.

Appendix

I. Proof of lemma 1

Proof: Define the matrix $A = (a_{i,j})$ with $a_{k,i} =< t^i, \varphi_k^* >=< (t+k)^i, \varphi_0^* >$ and $B = (b_{i,j})$ with $b_{i,j} = \begin{pmatrix} j-1 \\ i-1 \end{pmatrix}$ if $i \leq j$ and $b_{i,j} = 0$ if $i > j$. B is invertible, and the matrix that changes the basis $(1, (t+k), \ldots (t+k)^q)$, $k \in \mathbb{Z}$, into $(1, t, \ldots t^q)$ is equal to B^k. If we denote by M the vector matrix of the $q+1$ first moments of φ^*, then we have:

$$A = \begin{bmatrix} M^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M^T \end{bmatrix} \begin{bmatrix} \mathrm{Id} \\ B \\ \vdots \\ B^q \end{bmatrix}$$

Denote by $\mathcal{B}$ the matrix which is obtained from B by replacing each of its element by the outer product of it with the identity matrix which has the same size as B. Then we have

$$\begin{bmatrix} \mathrm{Id} \\ B \\ \vdots \\ B^q \end{bmatrix} = \mathcal{B}^T \begin{bmatrix} \mathrm{Id} \\ B - \mathrm{Id} \\ \vdots \\ (B - \mathrm{Id})^q \end{bmatrix}$$

Observe that $(B - \mathrm{Id})^k$ has its first k columns equal to zero (as well as its last k rows). On the other hand, the product of the large M^T-diagonal matrix by $\mathcal{B}^T$ is in fact equal to $B^T M^T$, and hence:

$$A = B^T M^T \begin{bmatrix} \mathrm{Id} \\ B - \mathrm{Id} \\ \vdots \\ (B - \mathrm{Id})^q \end{bmatrix} \stackrel{\mathrm{def}}{=} B^T M^T B_-$$

Observe that the product of M^T with the large matrix B_- on the right end is upper triangular, with the first (and non zero) moment of ϕ^* on the diagonal. Since B is regular, A has full rank. ■

II. Proof of proposition 4

Proof: $P_\delta f$ is equal to

$$\begin{aligned} P_\delta f(t) &= \sum_{k\in\mathbb{Z}} \sum_{l=0}^{l=N} \left(\frac{t}{\delta} - k\right)^l 1_{[k,k+1)}\left(\frac{t}{\delta}\right) \\ &\quad \frac{1}{\delta}\int_{\mathbb{R}} p_l\left(\frac{s}{\delta} - k\right) f(s)ds \end{aligned}$$

Hence $\mathcal{F}_\delta^{-1} P_\delta f$ is the sequence of polynomials

$$r_k(t) = \sum_{l=0}^{l=N} (t-k)^l \frac{1}{\delta}\int_{\mathbb{R}} p_l\left(\frac{s}{\delta} - k\right) f(s)ds$$

and $Q\mathcal{F}_\delta^{-1} P_\delta f$ is the sequence $(\rho_k)_{k\in\mathbb{Z}}$ with

$$\begin{aligned} \rho_k(t) &= \sum_{i=0}^{i=q-1} \sum_{j=i-M}^{j=i+M} \sum_{l=0}^{l=N} q_{l,i,j,m}(t-k)^l \\ &\quad \frac{1}{\delta}\int_{\mathbb{R}} p_m\left(\frac{s}{\delta} - j + kq\right) f(s)ds \end{aligned}$$

Finally,

$$\begin{aligned} \mathcal{F}_\delta Q \mathcal{F}_\delta^{-1} P_\delta f(t) &= \sum_{k\in\mathbb{Z}} \sum_{i=0}^{i=q-1} \sum_{j=i-M}^{j=i+M} \sum_{l=0}^{l=N} q_{l,i,j,m} \\ &\quad \left(\frac{t}{\delta} - k\right)^l 1_{[k,k+1)}\left(\frac{t}{\delta}\right) \\ &\quad \frac{1}{\delta}\int_{\mathbb{R}} p_m\left(\frac{s}{\delta} - j + kq\right) f(s)ds \\ &= \mathcal{P}_\delta f(t) \end{aligned}$$

■

III. Proof of lemma 3

Proof: Let $T_{N+1} = t^N \times t$. There exists a unique family $(a_0, \ldots, a_N)$ such that

$$T_{N+1} = \sum_{i=0}^{i=N} a_i t^i \stackrel{\mathrm{def}}{=} Q(t) \tag{30}$$

Let $(z_0, \ldots, z_N)$ the complex roots of the (usual) polynomial $t^{N+1} - Q(t)$. Then T_{N+1} is the Hermite interpolation of t^{N+1} at these points. We are going to show that $\times$ is obtained by Hermite interpolation of the usual product at $(z_0, \ldots, z_N)$. To do so, we only have to prove it is true for monomials generated by $\times$.

The latter are defined recursively by $T_{N+p} = T_{N+p-1} \times t$. We assume that T_{N+p-1} is the interpolation of t^{N+p-1} at $(z_0, \ldots, z_N)$. There also exist a polynomial $q(t)$ of degree $< N$ and a real b such that $T_{N+p-1} = q + bt^N$. Then

$$T_{N+p} = qt + bT_{N+1}$$

By assumption, $q(z_i) + b(z_i)^N = z_i^{N+p-1}$; hence,

$$\begin{aligned} T_{N+p}(z_i) &= \left(z_i^{N+p-1} - b(z_i)^N\right) z_i + bT_{N+1}(z_i) \\ &= z_i^{N+p} - bz_i^{N+1} - bz_i^{N+1} = z_i^{N+p} \end{aligned}$$

which proves the interpolation result at the order 0. For higher orders k, the induction becomes:

$$\begin{aligned} T_{N+p}^{(k)}(z_i) &= (qt)^{(k)}(z_i) + bT_{N+1}^{(k)}(z_i) \\ &= \left(T_{N+p-1}t - bt^{N+1}\right)^{(k)}(z_i) + bT_{N+1}^{(k)}(z_i) \\ &= \left(T_{N+p-1}t\right)^{(k)}(z_i) \\ &= T_{N+p-1}^{(k)}(z_i)z_i + T_{N+p-1}^{(k-1)}(z_i) \\ &= \left(t^{N+p-1}\right)^{(k)}(z_i)z_i + \left(t^{N+p-1}\right)^{(k-1)}(z_i) \\ &= \left(t^{N+p}\right)^{(k)}(z_i) \end{aligned}$$

Conversely, one verifies that the Hermite interpolation is indeed associative and that, by definition, it preserves the polynomials of degree $\leq N$. ■

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