RECURSIVE FRAME INVERSE COMPUTATION USING WAVELETS ON THE REAL LINE

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Abstract. A scale and time recursive algorithm which computes the wavelet frame inverse of a signal is proposed. After some time, the algorithm coincides with the algorithme à trous.

1. Introduction

1.1. Problem motivation and statement. Wavelet processing provides tools for denoising [4] piecewise regular signals and estimating their regularity [6, 2]. In the control context, it can be used to process the controller inputs. For instance, measures can be denoised and their regularity may be computed before their are differentiated for observation purposes [3]. To do so, the wavelet transform must be computed online, as well as the signal reconstruction.

The simplest method computes the coefficients of the signal in a wavelet basis using a cascade of Finite Impulse Response (FIR) filters, performs the appropriate computations on them and reconstruct the related signal. Unfortunately, the amplitude of the wavelet coefficients largely depends of the location of the transient pattern. For instance, the wavelet coefficient related to a jump in a piecewise constant signal may have an amplitude which varies with a factor 2, depending on the jump's location.

This can be avoided by computing a shift invariant wavelet transform, whose amplitude is not related to the pattern location. It is a redundant representation; this implies that a typical wavelet processing such as thresholding may produce data which is not itself a wavelet transform. To recover a signal in the time domain, a least squares problem is solved to yield the signal whose transform is closest. This a *frame inverse* computation. This paper shows how to to compute the frame inverse signal recursively in time and scale. It is applied to the online denoising a a real life signal.

1.2. Assumptions and notations. All functions and discrete time signals will be assumed to be bounded and compactly supported. Filters are also assumed to have a Finite Impulse Response. If x is a discrete sequence, $x_{j\uparrow}[k]$ is defined by $x_{j\uparrow}[k] = x[p]$ if $k = 2^j p$ and $x_{j\uparrow}[k] = 0$ in the other case.

1.3. Dyadic wavelet transforms. Let h and g a pair of conjugate mirror filters. If J is an integer, a discrete signal a_0 can be recursively decomposed into a "regular" approximation a_J and a sequence d_j, $j = 1, \dots, J$ of detail signals using the following cascade of filter banks:

$$a_{j+1}[p] = \sum_{n\in\mathbb{Z}} h[n-2p]a_j[n] \tag{1}$$

$$d_{j+1}[p] = \sum_{n\in\mathbb{Z}} g[n-2p]a_j[n] \tag{2}$$

Conversely, a_0 can be recovered from the previous signals using the following recursion

$$a_j[p] = \sum_{n\in\mathbb{Z}} h[p-2n]a_{j+1}[n] + \sum_{n\in\mathbb{Z}} g[p-2n]d_{j+1}[n] \tag{3}$$

This is the discrete expression of the decomposition and reconstruction of a signal in an orthogonal wavelet basis [7]. The decomposition does not commute with shifts on the signal a_0. It is actually a subsampling of the following transform which is itself shift invariant:

$$\tilde{a}_{j+1}[p] = \sum_{n\in\mathbb{Z}} h_{j\uparrow}[n-p]\tilde{a}_j[n] \tag{4}$$

$$\tilde{d}_{j+1}[p] = \sum_{n\in\mathbb{Z}} g_{j\uparrow}[n-p]\tilde{a}_j[n] \tag{5}$$

and $\tilde{a}$, $\tilde{d}$ are subsamples of a and d:

$$a_j[n] = \tilde{a}_j[2^j n] \text{ and } d_j[n] = \tilde{d}_j[2^j n]. \tag{6}$$

The signal can be reconstructed by the *algorithme à trous*[5]:

$$(7)\quad \tilde{a}_j[p] = \frac{1}{2}\left[\sum_{n\in\mathbb{Z}} h_{j\uparrow}[p-n]\tilde{a}_{j+1}[n] + \sum_{n\in\mathbb{Z}} g_{j\uparrow}[p-n]\tilde{d}_{j+1}[n]\right]$$

The transform is redundant, e.g. not all sequences $(\tilde{a}_J[n])$ and $(d_j[n])$ are signal transforms. Various operations that are performed on the transform, such as denoising by wavelet threshold or local regularization, may produce sequences which are not signal images. An approximate signal is obtained by solving a least squares problem.

2. Recursive frame inverse on the real line

Here a sequence α_J and a family of sequences δ_j, $j = 1, \dots, J$ are given for all sample times $n \in \mathbb{Z}$. The reconstruction problem is to find a signal a_0 such that its dyadic wavelet transform at the scale J, defined by equations (4,5) is closest to the α_J and δ_j:

$$(8)\quad \min_{a_0}\left[\frac{1}{2^J}\sum_{k\in\mathbb{Z}}(a_J[k]-\alpha_J[k])^2 + \sum_{j=1}^{j=J}\frac{1}{2^j}\sum_{k\in\mathbb{Z}}(d_j[k]-\delta_j[k])^2\right]$$

Theorem 1. *The signal $\tilde{a}_0$ generated by the algorithme à trous (7) with $\tilde{a}_J = \alpha_J$ and $\tilde{d}_j = \delta_j$ is a solution to problem (8).*

Proof. A direct computational proof can be found in [1]. A more general one can be drawn from the results in chapter 3 of [2]. In both proofs, it is proven that the optimal estimation $\tilde{a}_0$ is the average of the 2^J reconstructions computed with (3) where the data α_J and δ_j has been shifted by i, $0 \le i < 2^J$, and the resulting signals are subsampled by a scale 2^j. The second part of the proof shows that this average is computed recursively by the algorithme à trous. □

3. Recursive frame inverse on the interval

The data is now only given on a finite interval $[K_-, K_+]$. The lower scale (e.g., 1) in problem (8) is allowed to vary with value j in order to find a scale recursive solution. At scale j, the problem is to find a signal $a_{j,0}$ such that its dyadic wavelet transform $(a_{j,J}, d_{j,i})_{j\le i\le J}$ from scale j to scale J is closest to the α_J and δ_j on the interval $[K_-, K_+]$:

$$(9)\quad \min_{a_{j,0}}\left[\frac{1}{2^J}\sum_{k=K_-}^{k=K_+}(a_{j,J}[k]-\alpha_J[k])^2 + \sum_{i=j+1}^{i=J}\frac{1}{2^i}\sum_{k=K_-}^{k=K_+}(d_{j,i}[k]-\delta_i[k])^2\right]$$

Since the filters have a finite impulse response, $a_{j,0}$ is not unique. Actually, we shall see that the optimal value $a_0[k]$ is unique for k in a given set of indices, and completely indeterminate for the other indices.

Remark: observe that the minimum cost is zero if the data is the exact wavelet transform of a signal. The solution of problem (9) hence provides an exact reconstruction when the data is in the wavelet transform's image, at indices where the optimum is uniquely defined.

3.1. Boundary and inner sets.

Since we use wavelets on the real axis to estimate a signal on an interval, boundary effects are to be expected. Let $[p_-, p_+]$ (resp. $[q_-, q_+]$) the support of h (resp. g). We will use the following adjacent[1] intervals at the scale j: for $j = J$, $\Gamma_J = [K^-, K^+], \Gamma_J^- = \emptyset, \Gamma_J^+ = \emptyset$; for $j < J$, $\Gamma_j^- = [\phi_j^-, \gamma_j^- - 1]$, $\Gamma_j = [\gamma_j^-, \gamma_j^+]$ and $\Gamma_j^+ = [\gamma_j^+ + 1, \phi_j^+]$ with $\gamma_j^- = K_- + (2^{J-1} - 2^j)p_+ + 2^{J-1}\max(q_+, p_+)$, $\gamma_j^+ = K_+ + (2^{J-1} - 2^j)p_- + 2^{J-1}\min(q_-, p_-)$, $\phi_j^- = K_- + (2^{J-1} - 2^j)p_- + 2^{J-1}\min(q_-, p_-)$ and $\phi_j^+ = K_+ + (2^{J-1} - 2^j)p_+ + 2^{J-1}\max(q_+, p_+)$.

3.2. Boundary matrices.

Let $\Phi_j^- = \Gamma_j^- - K_-$ and $\Phi_j^+ = \Gamma_j^+ - K_+$; they do not depend on K_- and K_+. The symmetric matrices which computes the cost on the boundary

[1] It is assumed that $K_+ - K_- \ge 2^{p-1}(p_+ - p_-) + 2^p(\max(p_+, q_+) - \min(p_-, q_-))$

sets are computed recursively with

(10)

$$M_j^\sigma(n,p) = \frac{1}{2^{j-1}} \sum_{k\in[0,\sigma\infty]} g_j[l-k]g_j[J-k] + \frac{1}{2^{j-1}} \sum_{k\in[\beta_j^\sigma,\sigma\infty]} h_j[l-k]h_j[J-k] + \sum_{m,n\in\Phi_j^\sigma} M_{j+1}^\sigma(m,n)h_j[l-m]h_j[J-n]$$

with $\sigma = +$ or $\sigma = -$ and $\beta_j^\sigma = (2^{J-1} - 2^j)p_\sigma + 2^{J-1}\max(q_\sigma, p_\sigma)$.

3.3. **Inner estimate.** An intermediate signal $\tilde{a}_j$ is defined by

(11)

$$\tilde{a}_j[p] = \frac{1}{2} \sum_{k\in[K_-,K_+]} g_j[p-k]d_{j+1}[k] + \frac{1}{2} \sum_{k\in\Gamma_{j+1}} h_j[p-k]a_{j+1}[k] + 2^{j-1} \sum_{m,k\in\Gamma_{j+1}^+} M_{j+1}^+(m-K_+,k-K_+)h_j[p-m]a_{j+1}[k] + 2^{j-1} \sum_{m,k\in\Gamma_{j+1}^-} M_{j+1}^-(m-K_-,k-K_-)h_j[p-m]a_{j+1}[k]$$

3.4. **Optimal signals.**

Theorem 2. *At the scale j the optimal signals $\check{a}_j$ satisfy*

(12)

$$\check{a}_j[n] = \begin{cases} \frac{1}{2^j}(M_j^-)^{-1}\tilde{a}_j[n] & \text{if } n \in \Gamma_j^- \\ \tilde{a}_j[n] & \text{if } n \in \Gamma_j \\ \frac{1}{2^j}(M_j^+)^{-1}\tilde{a}_j[n] & \text{if } n \in \Gamma_j^+ \end{cases}$$

and their values for the other indices are indeterminate.

Proof. The result is basically a left inverse computation. However some care has to be taken in order to preserve the recursivity of the solution with respect to time and scale. The detailed proof in [1] uses two possible reconstructions of the signal at the scale $j-1$ using the data at scale j. They are used to operate two changes of variables in the expression of the cost to simplify it and find the optimum. □

Remark observe that, if $n \in \Gamma_j$, then $\check{a}_j[n]$ is computed with the algorithme à trous and is identical to the optimal estimate of theorem 1.

4. Application

By setting $j = 0$ and incrementing K_+ in problem (9), signal estimates can be recovered online from a processed wavelet transform. Typically, a dyadic shift invariant wavelet transform has been performed on noisy data using equations (1,2). Wavelet coefficients whose magnitude is below a given threshold are set to zero to denoise the signal as in [4]. In general, the result is not a wavelet transform and a related signal has to be estimated. This is done by using the algorithm of theorem 2.

Figures 1 and 2 shows a measure from an actual plant which is controlled to produce piecewise constant outputs, and its denoising using a wavelet threshold and a reconstruction with the algorithme à trous. The signal has a length of 2424. The wavelet transform has been computed with Daubechies wavelets with two vanishing moments and over 7 scales. The threshold corresponds to an estimated variance of 9 ($\sigma = 3$). Figure 3 shows the restored signal when the reconstruction delay is shortened. In figure 4, the various signal estimates are then differentiated in the flat zone (differentiable regions can also be detected using the wavelet transform). In practice, the only acceptable derivative comes from the algorithme à trous.

5. Conclusion

A recursive algorithm which estimates a signal in the least squares sense from pervaded wavelet data is detailed. It has been tested on real life signals; these tests indicate that the price to pay for a good estimate may be some important delays.

References

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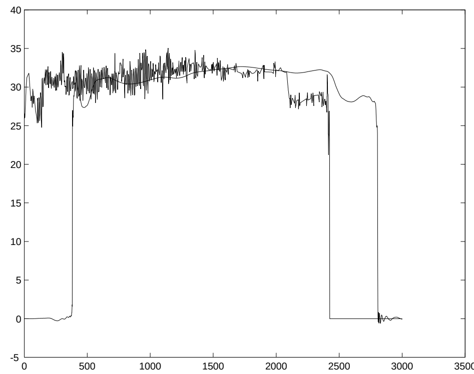

FIGURE 1. The original signal and its reconstruction with the algorithme à trous. The delay is given by $delay = (2^{scales} - 1) * (Filterlength - 1)$. Its value is 381 sampling intervals here.

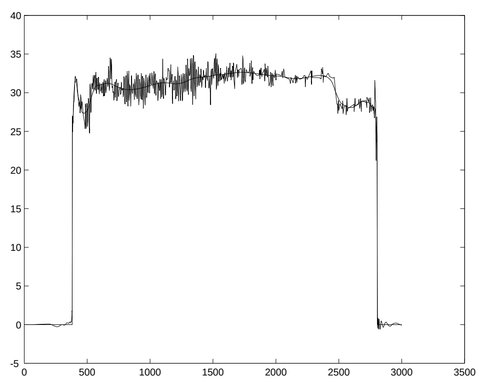

FIGURE 2. The delay is compensated to superpose the original and filtered signals. The jumps are well preserved.

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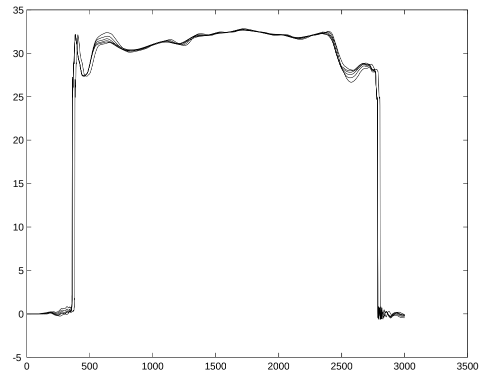

FIGURE 3. Here the signal is restored without waiting for the steady state estimate. The least squares algorithm uses the boundary matrices. The delays for the reconstructions are respectively 361, 341, 321, 301, 281 and 261. Gaining 120 sampling intervals implies a significant corruption of the restored signal

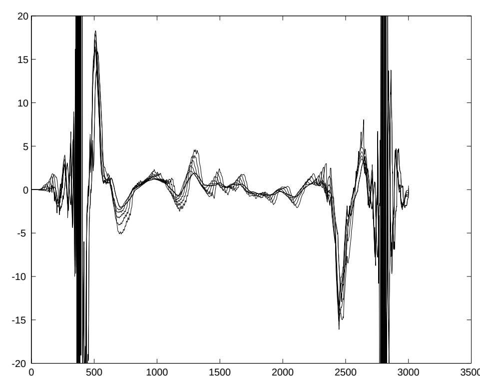

FIGURE 4. The steady state estimate and the shorter delayed estimates are differentiated. The discontinuities produces large peaks which are out of range. All derivatives differ, and the steady sate estimate seems to be the best choice

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