2015 European Control Conference (ECC)
July 15-17, 2015. Linz, Austria

Homogeneity applied to the controllability of a system of parabolic equations

Jean-Michel Coron, Sergio Guerrero, Philippe Martin and Lionel Rosier

***Abstract*— We consider a system of two parabolic equations with a forcing term present in one equation and a nonlinear coupling term of a particular type in the other one. We establish a local null controllability result, even if the linearized control system fails to be null controllable. The proof combines the return method, a Carleman inequality and an argument based on homogeneity. Our result can be applied to a reaction-diffusion system from Chemistry and to the Ginzburg-Landau equation.**

I. Introduction

The control of linear (or semilinear) parabolic equations is by now well understood since the introduction of Carleman inequalities by Fursikov and Imanuvilov in the nineties (see e.g. [1] and the references therein). The consideration of **systems** of parabolic equations is much more recent. The attention was paid in [2] to the null controllability of linear control systems with constant coefficients and internal control

$$w_t = (D\Delta + A)w + 1_\omega Bh, \qquad \text{in } (0,T)\times\Omega \tag{1}$$

$$w = 0, \qquad \text{in } (0,T)\times\partial\Omega \tag{2}$$

where $w : (0,T)\times\Omega \to \mathbb{R}^n$ is the state to be controlled, $h : (0,T)\times\omega \to \mathbb{R}^m$ is the control input, $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given matrices, $\Delta = \sum_{1\le i\le k} \partial^2/\partial x_i^2$ is the Laplacian operator, $\omega \subset \Omega \subset \mathbb{R}^k$ is a nonempty open set (the control region), and 1_ω is the characteristic function of ω, defined as

$$1_\omega(x) := \begin{cases} 1 & \text{if } x\in\omega, \\ 0 & \text{if } x\in\Omega\setminus\omega. \end{cases}$$

It was proved in [2] that the null controllability of (1)-(2) was equivalent to the **extended Kalman condition**

$$\text{rank } [L_p|B] = n \quad \forall p \ge 1,$$

where $[A|B] := [B, AB, A^2B, ..., A^{n-1}B]$ is the Kalman matrix and $L_p := -\lambda_p D + A$, $(\lambda_p)_{p\ge1}$ denoting the sequence of eigenvalues of the operator $-\Delta$ on Ω with Dirichlet boundary conditions. Thus, when $D = I$, the null controllability holds whenever rank $[A|B] = n$, i.e. when the finite-dimensional control system

$$\dot{w} = Aw + Bu$$

is itself controllable. The situation is more tricky for systems with variable coefficients or for boundary control. We refer the reader to [3] for a survey. A challenging problem is the issue whether the cascade system

$$u_t - \Delta u = 1_\omega h \qquad \text{in } (0,T)\times\Omega, \tag{3}$$

$$v_t - \Delta v = 1_{\mathcal{O}} u \qquad \text{in } (0,T)\times\Omega, \tag{4}$$

$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega \tag{5}$$

is null controllable when $\mathcal{O} \subset \Omega$ is **any** nonempty open set. If a positive answer is known when $\omega\cap\mathcal{O} \ne \emptyset$, the situation is less obvious when $\omega\cap\mathcal{O} = \emptyset$. In that case, a positive answer was given when $n = 1$ (see [4]) or when $n \ge 1$ and the **Geometric Control Condition** (required for the exact controllability of the wave equation) holds for both the open sets ω and $\mathcal{O}$ (see [5]).

Here, we are concerned with a system of two parabolic equations with only one control input in the first equation, and for which the linearized system fails to be null controllable. In [6], we proved the null controllability of the system

$$u_t - \Delta u = f(u,v) + 1_\omega h \qquad \text{in } (0,T)\times\Omega, \tag{6}$$

$$v_t - \Delta v = u^3 + Rv \qquad \text{in } (0,T)\times\Omega, \tag{7}$$

$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{8}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given function of class C^∞ with $f(0,0) = 0$ and $R \in \mathbb{R}$ is a given constant. Clearly, the linearized system around the trivial solution $u = v = 0$, namely

$$u_t - \Delta u = \frac{\partial f}{\partial u}(0,0)u + \frac{\partial f}{\partial v}(0,0)v + 1_\omega h \qquad \text{in } (0,T)\times\Omega,$$

$$v_t - \Delta v = Rv \qquad \text{in } (0,T)\times\Omega,$$

$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega,$$

is not null controllable, for the dynamics of v is not affected by h and u. To prove the null controllability of (6)-(8), we used the **return method**; that is, we linearized system (6)-(8) around a (smooth) nontrivial solution of (6)-(8). Next, we applied the classical approach based on Carleman estimates to prove first the null controllability of the linearized system, and next the null controllability of the original nonlinear system. We also noticed that the null controllability fails when the coupling term u^3 is replaced by u^2.

The aim of this paper is to generalize the above result to more general nonlinear systems, i.e. to systems with a nonlinear coupling term assuming a more general form than u^3. To simplify the exposition, we will assume that $k = 1$ in what follows, and take a system as simple as possible. The general case will be considered elsewhere.

J.-M. Coron and S. Guerrero are with the Université Pierre et Marie Curie -Paris 6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005 France

P. Martin and L. Rosier are with the Centre Automatique et Systèmes, MINES ParisTech, PSL Research University, 75006 Paris, France {coron,guerrero}@ann.jussieu.fr, {philippe.martin,lionel.rosier}@mines-paristech.fr

Let us consider a nonempty open set $\omega \subset \Omega := (0,1)$, and two maps $F : \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \mapsto F(u,v)$ and $G : \mathbb{R}^2 \to \mathbb{R}$, $(u,v) \mapsto G(u,v)$ of class C^∞. We assume that

$$F(0,0) = G(0,0) = 0, \tag{9}$$

and that

$$\frac{\partial^i G}{\partial u^i}(0,0) = 0, \quad \forall i \in \{0,...,2\}, \tag{10}$$

$$\frac{\partial^3 G}{\partial u^3}(0,0) \neq 0. \tag{11}$$

The control system we consider reads

$$u_t - u_{xx} + F(u,v) = h1_\omega \qquad \text{in } (0,T)\times\Omega, \tag{12}$$
$$v_t - v_{xx} + G(u,v) = 0 \qquad \text{in } (0,T)\times\Omega, \tag{13}$$
$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega. \tag{14}$$

Our aim is to establish the following (local) null controllability result.

Theorem 1: Let $T > 0$ and $p \in (3,+\infty)$. Then there exist $C_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $u^0, v^0 \in W_0^{1,\infty}(\Omega)$ with

$$||u^0||_{W^{1,\infty}(\Omega)} \leq \varepsilon_0,\ ||v^0||_{W^{1,\infty}(\Omega)} \leq \varepsilon_0, \tag{15}$$

there exists $h \in L^p((0,T)\times\Omega)$ satisfying

$$||h||_{L^p((0,T)\times\Omega)} \leq C_0\big(||u^0||_{W^{1,\infty}(\Omega)} + ||v^0||^{1/3}_{W^{1,\infty}(\Omega)}\big) \tag{16}$$

and such that the solution of the Cauchy problem

$$u_t - u_{xx} + F(u,v) = h1_\omega \qquad \text{in } (0,T)\times\Omega, \tag{17}$$
$$v_t - v_{xx} + G(u,v) = 0 \qquad \text{in } (0,T)\times\Omega, \tag{18}$$
$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{19}$$
$$u(0,.) = u^0,\ v(0,.) = v^0 \qquad \text{in } \Omega \tag{20}$$

satisfies

$$u(T,.) = v(T,.) = 0 \quad \text{in } \Omega. \tag{21}$$

We note that a similar result can be proved when the homogeneous Dirichlet boundary condition (19) is replaced by the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega. \tag{22}$$

Let us indicate two possible applications of Theorem 1.

A reaction-diffusion system describing a reversible chemical reaction [7], [8] takes the form

$$u_t = u_{xx} - ap(u^p - v^q) \qquad \text{in } (0,T)\times\Omega \tag{23}$$
$$v_t = v_{xx} + bp(u^p - v^q) \qquad \text{in } (0,T)\times\Omega \tag{24}$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } (0,T)\times\partial\Omega. \tag{25}$$

In (23)-(24), a and b denote some positive constants, and p and q are positive integers. The corresponding reversible chemical reaction is $pA \rightleftharpoons qB$. If we incorporate a forcing term $h1_\omega$ in (23), we obtain a system of the form (17)-(18) with the nonlinear terms F and G satisfying the conditions (9), (10) and (11) (if $p = 3$). We then infer from Theorem 1 that the complete system is locally null controllable. The same result is likely true when the exponent p in (23)-(24) is an odd integer, while the null controllability of the system clearly fails when p is even.

The Ginzburg-Landau equation

$$w_t = (1+i\alpha)w_{xx} + Rw - (1+i\beta)|w|^2 w + h1_\omega \tag{26}$$
$$w = 0 \qquad \text{on } (0,T)\times\partial\Omega \tag{27}$$

provides a simple model of turbulence. Here, $w = w(t,x) \in \mathbb{C}$ is the state function, $h = h(t,x) \in \mathbb{C}$ is the control input, and α, β, R are real constants. The null controllability of (26)-(27) was proved in [9], [10] with $h \in L^\infty((0,T)\times\omega,\mathbb{C})$. Assume now that the control h takes only **real** values. Writing $w = u + iv$ with u,v real valued, we obtain the system

$$u_t = u_{xx} - \alpha v_{xx} + Ru - (u^2+v^2)(u-\beta v) + h1_\omega, \tag{28}$$
$$v_t = \alpha u_{xx} + v_{xx} + Rv - (u^2+v^2)(\beta u + v), \tag{29}$$
$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega. \tag{30}$$

If $\alpha \neq 0$, then we can prove that the linearized system is null controllable (and the same is true for the nonlinear system). Assume now that $\alpha = 0$. The system becomes

$$u_t = u_{xx} + Ru - (u^2+v^2)(u-\beta v) + h1_\omega, \tag{31}$$
$$v_t = v_{xx} + Rv - (u^2+v^2)(\beta u + v), \tag{32}$$
$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega. \tag{33}$$

If $\beta \neq 0$, we can apply Theorem 1 to conclude that system (31)-(33) is locally null controllable. (Here, $G(u,v) = -Rv + (u^2+v^2)(\beta u + v)$ satisfies (10)-(11).) Finally, if $\beta = 0$, then the system fails to be null controllable. Indeed, if $v_0 \geq 0$ a.e. and $||R - (u^2+v^2)||_{L^\infty((0,T)\times\Omega)} \leq C$, then we have

$$v(t,.) \geq e^{t(\partial_x^2 - C)} v_0 > 0 \quad \text{in } \Omega.$$

The paper is outlined as follows. We sketch the proof of the null controllability of (6)-(8) in Section 2. In Section 3, we explain why the above method of proof needs to be strongly modified to deal with the general system (12)-(14). In particular, a key **homogeneity** argument will come into play. The proof of Theorem 1 is sketched in Section 3.

II. Sketch of the proof of the null controllability of (6)-(8)

In this section, we sketch the proof of the null controllability of the "toy" problem

$$u_t - u_{xx} = 1_\omega h \qquad \text{in } (0,T)\times\Omega, \tag{34}$$
$$v_t - v_{xx} = u^3 \qquad \text{in } (0,T)\times\Omega, \tag{35}$$
$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega \tag{36}$$

that was exposed in [6]. (We dropped the terms $f(u,v)$ and Rv for the sake of simplicity.) The proof of the null controllability rests on the **return method** introduced by Jean-Michel Coron for Euler equations of incompressible perfect fluids [11], [12]. Roughly speaking, we take the linearization along a smooth **nontrivial** trajectory $(\bar{u},\bar{v})$ such that

$$(\bar{u},\bar{v})_{|t=0} = (0,0) = (\bar{u},\bar{v})_{|t=T}.$$

There are three steps in the proof. In the first step, we construct the reference trajectory. In the second step, we prove the controllability of the linearized system along the reference trajectory. In the last step, we derive the null controllability of (34)-(36) by a fixed-point argument.

Step 1: Construction of the reference trajectory

We need the following proposition:

Proposition 1: Let $\rho > 0$. There exists a function $\bar{v} = \bar{v}(t,x)$, $\bar{v} \neq 0$ such that

$$\bar{v} \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x),\ \bar{v}(t,x) = 0 \text{ for } |t| \geq \rho \text{ or } |x| \geq \rho,$$
$$\bar{v}_t = \bar{v}_{xx} + \bar{u}^3 \qquad \text{with } \bar{u} \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x).$$

The corresponding control reads then $\bar{h} := \bar{u}_t - \bar{u}_{xx} \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x)$. Note that ρ can be chosen arbitrarily small. Using a translation, we can thus impose that

$$\text{supp }(\bar{h}) \subset (0,T) \times \omega.$$

Using a simple scaling, we see there is no loss in assuming $\rho = 1$. Note also that the main difficulty in the proof of Proposition 1 is the **smoothness** of $\bar{u} = (\bar{v}_t - \bar{v}_{xx})^{1/3}$. This requires to have "some control" on the behavior of $\bar{v}$ in a neighbourhood of the set $\{(t,x) :\ \bar{v}_t - \bar{v}_{xx} = 0\}$. The function $\bar{v}$ is taken of the form

$$\bar{v}(t,r) = \sum_{i=0}^{3} f_i(t) g_i(z)$$

where $r := |x|$, $z := r/\lambda(t)$, $\lambda(t) := \varepsilon(1-t^2)^2$, $f_0(t) = \exp(-(1-t^2)^{-1})1_{(-1,1)}(t)$, and $g_0 \in C_c^\infty(\mathbb{R})$, $f_i \in C_c^\infty(\mathbb{R})$, $1 \leq i \leq 3$, and $g_i \in C_c^\infty(\frac{1}{4}, \frac{3}{4})$, $1 \leq i \leq 3$, are conveniently chosen. Let $k(t,r) := \bar{v}_t - \bar{v}_{xx}$. The support of the function k is drawn in Figure 1.

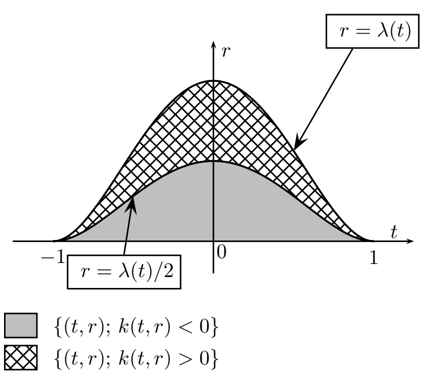


Fig. 1. Support of $k = \bar{v}_t - \bar{v}_{xx}$

Step 2: controllability of the linearized system

Let $(\bar{u},\bar{v})$ be the reference trajectory designed in Step 1, and let (u,v) be the solution of the control problem (S)

$$(\bar{S}) \begin{cases} \bar{u}_t - \bar{u}_{xx} = \bar{h}1_\omega, \\ \bar{v}_t - \bar{v}_{xx} = \bar{u}^3, \\ (\bar{u},\bar{v})_{|t=0} = (0,0), \\ (\bar{u},\bar{v})_{|t=T} = (0,0) \end{cases} \qquad (S) \begin{cases} u_t - u_{xx} = h1_\omega, \\ v_t - v_{xx} = u^3, \\ (u,v)_{|t=0} = (u_0,v_0), \\ (u,v)_{|t=T} = (0,0). \end{cases}$$

Then $w = (w_1,w_2) := (u-\bar{u}, v-\bar{v})$ satisfies

$$\begin{cases} w_{1,t} - w_{1,xx} = (h-\bar{h})1_\omega, \\ w_{2,t} - w_{2,xx} = a(w_1,\bar{u})w_1, \\ (w_1,w_2)_{|t=0} = (u_0,v_0), \qquad (w_1,w_2)_{|t=T} = (0,0), \end{cases}$$

where

$$a(z_1,z_2) := (3\bar{z}_2^2 + 3z_1z_2 + z_1^2).$$

Note that $a(0,\bar{u}) = 3\bar{u}^2 > const > 0$ somewhere, say for $t_1 \leq t \leq t_2$ and $x \in \overline{\omega_0} \subset \omega$. Thus $a(z_1,\bar{u}) > const > 0$ on $[t_1,t_2] \times \overline{\omega_0}$ for $||z_1||_{L^\infty((0,T)\times\Omega)} \ll 1$.

We impose that $\hat{h} := h - \bar{h}$ be null for $t < t_1$ and for $t > t_2$. We aim to drive the system to 0 in the time interval $[t_1,t_2]$.

We can assume that $[0,T] = [t_1,t_2]$, $a(z_1,\bar{u}) > const > 0$ for $0 \leq t \leq T$, $x \in \overline{\omega_0}$, and $||z_1||_{L^\infty((0,T)\times\Omega)} < \varepsilon$.

The adjoint system to

$$\begin{aligned} w_{1,t} - w_{1,xx} &= \hat{h}1_\omega && \text{in } (0,T)\times\Omega, \\ w_{2,t} - w_{2,xx} &= a(z_1,\bar{u})w_1 && \text{in } (0,T)\times\Omega, \\ w_1 = w_2 &= 0 && \text{on } (0,T)\times\partial\Omega, \\ (w_1,w_2)_{|t=0} &= (u_0,v_0) && \text{in } \Omega \end{aligned}$$

reads

$$\begin{aligned} -\varphi_{1,t} - \varphi_{1,xx} &= a(z_1,\bar{u})\varphi_2 && \text{in } (0,T)\times\Omega, \\ -\varphi_{2,t} - \varphi_{2,xx} &= 0 && \text{in } (0,T)\times\Omega, \\ \varphi_1 = \varphi_2 &= 0 && \text{on } (0,T)\times\partial\Omega, \\ (\varphi_1,\varphi_2)_{|t=T} &= (\varphi_{1,T},\varphi_{2,T}) && \text{in } \Omega. \end{aligned}$$

The observability inequality to be proved for $\varphi := (\varphi_1,\varphi_2)$ reads:

$$\int_\Omega |\varphi(0,x)|^2 dx \leq C \iint_{(0,T)\times\omega_0} |\varphi_1|^2 dxdt.$$

To establish it, we need the following

Proposition 2: (Global Carleman estimate for the heat equation [1]) There exist some constants $s_0 > 0$, $C > 0$ such that for every $q \in L^2(0,T,H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T,L^2(\Omega))$ and all $s \geq s_0$, we have

$$\begin{aligned} &\iint_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}\big((s\eta)^3|q|^2 + s\eta|q_x|^2 \\ &\qquad + (s\eta)^{-1}(|q_{xx}|^2 + |q_t|^2)\big) \\ &\leq C\Big(\iint_{(0,T)\times\Omega} e^{-s\rho(x)\eta(t)}|q_t + q_{xx}|^2 \\ &\qquad + \iint_{(0,T)\times\omega_0} e^{-s\rho(x)\eta(t)}(s\eta)^3|q|^2\Big) \end{aligned}$$

for some smooth function $\rho(x) \geq C > 0$ satisfying

$$\rho_x(x) \neq 0 \ \text{ on } \ \Omega\setminus\omega_0 \ \text{ and } \ \partial\rho/\partial\nu \geq 0 \ \text{ on } \ \partial\Omega,$$

and for some function $\eta \in C^\infty((0,T),(0,+\infty))$ such that (see [13])

$$\eta(t) = \begin{cases} t^{-1} & \text{if } t \leq T/3, \\ (T-t)^{-1} & \text{if } t \geq 2T/3. \end{cases} \qquad (37)$$

Recall that $a(z_1,\bar{u}) \geq const > 0$ on $[0,T]\times\overline{\omega_0}$. Letting $q = \varphi_i$, $i = 1,2$ with

$$-\varphi_{1,t} - \varphi_{1,xx} = a(z_1,\bar{u})\varphi_2$$
$$-\varphi_{2,t} - \varphi_{2,xx} = 0$$

yields for $\varphi=(\varphi_1,\varphi_2)$ and s large enough

$$\begin{aligned}\iint_{(0,T)\times\Omega} & e^{-s\rho(x)\eta(t)}\big((s\eta)^3|\varphi|^2+s\eta|\varphi_x|^2 \\ & \qquad +(s\eta)^{-1}(|\varphi_{xx}|^2+|\varphi_t|^2)\big) \\ \leq C\iint_{(0,T)\times\omega_0} & e^{-s\rho(x)\eta(t)}(s\eta)^3(|\varphi_1|^2+|\varphi_2|^2).\end{aligned}$$

Using local energy estimates, we can drop the observation of φ_2 in the last integral term to obtain

$$\begin{aligned}\iint_{(0,T)\times\Omega} & e^{-s\rho(x)\eta(t)}\big((s\eta)^3|\varphi|^2+s\eta|\varphi_x|^2 \\ & \qquad +(s\eta)^{-1}(|\varphi_{xx}|^2+|\varphi_t|^2)\big) \\ \leq C\iint_{(0,T)\times\omega_0} & e^{-s\rho(x)\eta(t)}(s\eta)^7|\varphi_1|^2.\end{aligned}$$

This yields the desired observability inequality

$$||\varphi(0,.)||^2_{L^2(\Omega)}\leq C\iint_{(0,T)\times\omega_0} e^{-s\rho(x)\eta(t)}(s\eta)^7|\varphi_1|^2$$

to prove the null controllability of the linearized system.

Step 3: Fixed-point argument

For $||z||_{L^\infty((0,T)\times\Omega)}\ll 1$ and $(u_0,v_0)\in L^2(\Omega)^2$, the control h driving the system to $(0,0)$ satisfies

$$||h||_{L^\infty((0,T)\times\Omega)}\leq C||(u_0,v_0)||_{L^2(\Omega)^2}.$$

It can be shown that for $(u_0,v_0)\in W_0^{1,\infty}(\Omega)^2$, the map $z\mapsto w$ is continuous and compact in the space $L^\infty((0,T)\times\Omega)$. An application of Kakutani theorem yields the existence of a fixed-point, which completes the proof.

III. Proof of Theorem 1

The main difficulty in the above approach is the construction of a **nontrivial, compactly supported, reference trajectory** for

$$\bar v_t-\bar v_{xx}+G(\bar u,\bar v)=0.$$

One can treat the case $G=G_0:=c_1u^{2k+1}+c_2v$, $k\in\mathbb{N}$, but there is no hope to do that for an arbitrary function G fulfilling $\frac{\partial^i G}{\partial u^i}(0,0)=0$ for $i\leq 2k$ and $\frac{\partial^{2k+1}G}{\partial u^{2k+1}}(0,0)\neq 0$.

For $G=G_0$, we observe that

$$\bar v=O(\bar u^{2k+1})\qquad \text{as}\quad t\to 0^+ \text{ or } t\to T^-.$$

We are led to assign different weights to u et v: we expand $G(u,v)$ with respect to the **dilation**

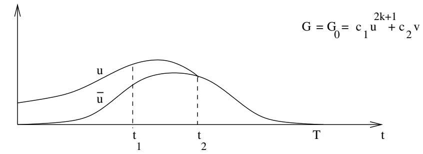
$$d_\varepsilon(u,v):=(\varepsilon u,\varepsilon^{2k+1}v).$$

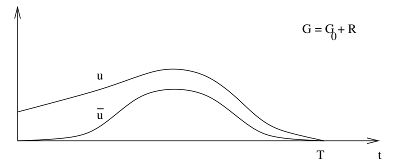
In the general case, the consideration of the Taylor expansion of G

$$G(u,v)=G_0(u,v)+R(u,v)$$

leads to keep the reference trajectory associated with G_0, and to consider the term $R(u,v)$ as a **disturbance** to be balanced on the time interval $(0,T)$.

The two approaches are sketched as follows:





Thus, to balance the (small) term $R(u,v)$, the control has to be applied up to the time $t=T$. On the other hand, in order to limit the control effort, it is desirable to have the support of $\bar u$ and its magnitude as large as possible. For simplicity, we still assume that $k=1$.

Let $\mathcal{X}$ and $\mathcal{Y}$ denote some Banach spaces that will be defined later on. Consider the map

$$\begin{aligned}\mathcal{A}:\ \mathbb{R}\times W_0^{1,\infty}(\Omega)^2\times\mathcal{X} &\to \mathcal{Y}\\ (\varepsilon,U^0,V^0,(U,V,H)) &\mapsto (\mathcal{A}_1,\mathcal{A}_2,U(0)-U^0,V(0)-V^0)\end{aligned}$$

defined by

$$\mathcal{A}_1=\begin{cases}\frac{1}{\varepsilon}(u_t-u_{xx}+F(u,v)-h1_\omega) & \text{if } \varepsilon\neq 0,\\ U_t-U_{xx}+f_0U-H1_\omega & \text{if } \varepsilon=0,\end{cases}$$

$$\mathcal{A}_2=\begin{cases}\frac{1}{\varepsilon^3}(v_t-v_{xx}+G(u,v)) & \text{if } \varepsilon\neq 0,\\ V_t-V_{xx}+\frac{1}{3}g_0((U+\bar u)^3-\bar u^3)+g_1V & \\ & \text{if } \varepsilon=0,\end{cases}$$

with

$$u:=\varepsilon(\bar u+U),\quad v:=\varepsilon^3(\bar v+V),\quad h:=\varepsilon(\bar h+H)$$

and $f_0:=\frac{\partial F}{\partial u}(0,0)$, $g_0:=\frac{1}{2}\frac{\partial^3 G}{\partial u^3}(0,0)$, $g_1:=\frac{\partial G}{\partial v}(0,0)$.

Denoting by

$$\mathcal{L}=\frac{\partial\mathcal{A}}{\partial(U,V,H)}(0,0,0,(0,0,0))$$

it holds

$$\begin{aligned}\mathcal{L}(U,V,H) = {} & (U_t-U_{xx}+f_0U-H1_\omega,\\ & V_t-V_{xx}+g_0\bar u^2U+g_1V,U(0),V(0)).\end{aligned}\tag{38}$$

To complete the proof with a version of the **implicit function theorem**, one has to show that the map $\mathcal{L}$ is **onto**.

We notice that we can pick $\lambda(t)=\varepsilon t(T-t)$, $0<t<T$, (instead of $\lambda(t)=\varepsilon(T^2-t^2)^2$, $-T<t<T$) to construct the reference trajectory satisfying

$$\begin{aligned}\bar u_t-\bar u_{xx}+f_0\bar u &= \bar h1_\omega,\\ \bar v_t-\bar v_{xx}+\frac{g_0}{3}\bar u^3+g_1\bar v &= 0.\end{aligned}$$

On the other hand, we can as well replace f_0 by $f_0^\mu\in C_c^\infty(\mathbb{R})$ for any $\mu>0$. Finally, by a simple translation we can assume that supp $\bar u$ $\cup$ supp $\bar v\subset[0,T]\times\{|x-x_0|\leq\rho\}$, with

$$\bar u(0,.)=\bar v(0,.)=\bar u(T,.)=\bar v(T,.)=0.$$

Furthermore, we can impose the following for $t \in (0,T)$, $x \in (0,1)$

$$\begin{array}{rcl} \bar{u}(t,x) & \geq & c_1 \eta^{\frac{2}{3}}(t) e^{-\frac{\mu}{3}\eta(t)} \text{ if } |x-x_0| < \lambda(t)/4 \\ |\partial_t^l \partial_x^k \bar{u}(t,x)| & \leq & c_2 \eta^{2l+k+\frac{2}{3}} e^{-\frac{\mu}{3}\eta(t)}, \quad l,k = 0,1,\dots \\ |\partial_t^l \partial_x^k \bar{v}(t,x)| & \leq & c_3 \eta^{2l+k} e^{-\mu\eta(t)} \quad l,k = 0,1,\dots \end{array}$$

for some positive constants c_1, c_2, c_3. The support of $(\bar{u}, \bar{v})$ now contains a cone as $t \to 0^+$ or for $t \to T^-$, since

$$\{|x-x_0| < \lambda(t)/4\} \supset \{|x-x_0| < c|t-T|\}$$

for $c > 0$ small enough. It turns that one can still derive a Carleman estimate for a heat equation with an internal observation on a cone-shaped domain.

Proposition 3: (Observation on a cone) There exist some constants $s_0 > 0$, $C > 0$ such that for all $z \in L^2(0,T,H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T,L^2(\Omega))$ and all $s \geq s_0$, we have

$$\begin{aligned} &\iint_{(0,T)\times\Omega} e^{-s\rho(t,x)\eta(t)} \big((s\eta)^3|z|^2 + s\eta|z_x|^2 \\ &\qquad + (s\eta)^{-1}(|z_{xx}|^2 + |z_t|^2)\big) \\ &\leq C\bigg(\iint_{(0,T)\times\Omega} e^{-s\rho(t,x)\eta(t)} |z_t + z_{xx}|^2 \\ &\qquad + \iint_{|x-x_0|<\lambda(t)/4} e^{-s\rho(t,x)\eta(t)} s^3\eta^4|z|^2\bigg) \end{aligned}$$

for a smooth function $\rho(t,x) \geq const > 0$ satisfying

- $|\rho_x(t,x)| = 1$ for $|x-x_0| \geq \lambda(t)$,
- $\rho^* - \varepsilon \leq \rho(t,x) \leq \rho^*$ for all (t,x) (with $0 < \varepsilon \ll \rho^*$),
- $\partial\rho/\partial\nu \geq 0$ in $(0,T)\times\partial\Omega$,

and $\eta \in C^\infty((0,T),(0,+\infty))$ is as in (37).

We pick

$$\mu := s(\rho^* + \varepsilon)$$

in the definition of the reference trajectory $(\bar{u}, \bar{v})$. Using local energy estimates, we obtain for any solution $\varphi = (\varphi_1, \varphi_2)$ of

$$\begin{aligned} &-\varphi_{1,t} - \varphi_{1,xx} + f_0\varphi_1 + g_0\bar{u}^2\varphi_2 = h_1 \\ &-\varphi_{2,t} - \varphi_{2,xx} + g_1\varphi_2 = h_2 \\ &\varphi_1 = \varphi_2 = 0 \text{ in } (0,T)\times\partial\Omega \end{aligned}$$

the following estimate for all $s \geq s_0$

$$\begin{aligned} &||\varphi_1(0)||^2_{L^2(\Omega)} + ||\varphi_2(0)||^2_{L^2(\Omega)} \\ &+ \iint_{(0,T)\times\Omega} e^{-s\rho\eta} \big((s\eta)^3|\varphi_1|^2 + s\eta|\varphi_{1,x}|^2 \\ &\qquad + (s\eta)^{-1}(|\varphi_{1,xx}|^2 + |\varphi_{1,t}|^2)\big) \\ &+ \iint_{(0,T)\times\Omega} e^{-3s\rho\eta} \big((s\eta)^3|\varphi_2|^2 + s\eta|\varphi_{2,x}|^2 \\ &\qquad + (s\eta)^{-1}(|\varphi_{2,xx}|^2 + |\varphi_{2,t}|^2)\big) \\ &\leq C\bigg(\iint_{|x-x_0|<\lambda(t)/2} e^{(-3s\rho+2\mu)\eta}(s\eta)^7\eta^2|\varphi_1|^2 \\ &\qquad + \iint_{(0,T)\times\Omega} e^{-3s\rho\eta}(e^{2\mu\eta}s^3\eta^5|h_1|^2 + |h_2|^2)\bigg). \end{aligned}$$

Next, proceeding as in [1], [14], we "simplify" the weights so that they are no longer singular at $t = 0$. Introduce the function

$$l(t) := \begin{cases} \eta(\frac{2T}{3}) = \frac{3}{T} & \text{if } 0 < t < \frac{2T}{3}, \\ \eta(t) = (T-t)^{-1} & \text{if } \frac{2T}{3} < t < T. \end{cases}$$

We obtain (fixing some $s \geq s_0$)

$$\begin{aligned} &||\varphi_1(0)||^2_{L^2(\Omega)} + ||\varphi_2(0)||^2_{L^2(\Omega)} \\ &+ \iint_{(0,T)\times\Omega} e^{-s\rho l} \big(l^3|\varphi_1|^2 + l|\varphi_{1,x}|^2 \\ &\qquad + l^{-1}(|\varphi_{1,xx}|^2 + |\varphi_{1,t}|^2)\big) \\ &+ \iint_{(0,T)\times\Omega} e^{-3s\rho l} \big(l^3|\varphi_2|^2 + l|\varphi_{2,x}|^2 \\ &\qquad + l^{-1}(|\varphi_{2,xx}|^2 + |\varphi_{2,t}|^2)\big) \\ &\leq C\bigg(\iint_{(0,T)\times\omega} e^{(-3s\rho+2\mu)l} l^9|\varphi_1|^2 \\ &\qquad + \iint_{(0,T)\times\Omega} e^{-3s\rho l}(e^{2\mu l}l^5|h_1|^2 + |h_2|^2)\bigg). \end{aligned}$$

We are then in a position to state a null controllability result for a system with forcing terms. Consider the system

$$w_{1,t} - w_{1,xx} + f_0 w_1 = k1_\omega + F_1 \tag{39}$$

$$w_{2,t} - w_{2,xx} + g_0\bar{u}^2 w_1 + g_1 w_2 = F_2 \tag{40}$$

$$w_1 = w_2 = 0 \qquad \text{on } (0,T)\times\partial\Omega \tag{41}$$

$$(w_1, w_2)_{|t=0} = (w_1^0, w_2^0). \tag{42}$$

Let

$$\rho_* := \rho^* - \varepsilon \leq \min_{(t,x)\in[0,T]\times[0,1]} \rho(t,x).$$

Then the following holds.

Proposition 4: For any $(w_1^0, w_2^0) \in W_0^{1,\infty}(\Omega)^2$ and for any $F = (F_1, F_2)$ with $e^{s\rho l} l^{-3/2} F_1 \in L^2((0,T)\times\Omega)$ and $e^{3s\rho l} l^{-3/2} F_2 \in L^2((0,T)\times\Omega)$, there is a pair (w,k) solving (39)-(42) together with $(w_1, w_2)_{|t=T} = (0,0)$, and such that

$$\begin{aligned} &e^{(3s\rho-2\mu)l} l^{-5/2} w_1 \in L^2((0,T)\times\Omega), \\ &e^{3s\rho l} w_2 \in L^2((0,T)\times\Omega), \\ &e^{(3s\rho-2\mu)l} l^{-9/2} k \in L^2((0,T)\times\Omega). \end{aligned}$$

Furthermore, for $0 < \gamma < (\rho^* - 5\varepsilon)/(\rho^* - \varepsilon)$, we have

$$\begin{aligned} &e^{\gamma s\rho_* l} l^{-3/2} w_1 \in L^\infty((0,T)\times\Omega), \\ &e^{3\gamma s\rho_* l} l^{-3/2} w_2 \in L^\infty((0,T)\times\Omega). \end{aligned}$$

We set

$$Q := (0,T)\times\Omega,$$

and we let $\mathcal{X}$ denote the set of functions $(U,V,H) : Q \to \mathbb{R}^3$ such that

$$\begin{array}{rcl} e^{(3s\rho-2\mu)l} l^{-5/2} U & \in & L^2(Q) \\ e^{3s\rho l} V & \in & L^2(Q) \\ e^{(3s\rho-2\mu)l} l^{-9/2} H & \in & L^2(Q) \\ e^{s\rho l} l^{-3/2}(U_t - U_{xx} + f_0 U - H1_\omega) & \in & L^2(Q) \\ e^{3s\rho l} l^{-3/2}(V_t - V_{xx} + g_0\bar{u}^2 U + g_1 V) & \in & L^2(Q) \\ (U,V) & \in & C([0,T], H_0^1(\Omega)^2) \\ U(0,.) \in W_0^{1,\infty}(\Omega),\ V(0,.) & \in & W_0^{1,\infty}(\Omega). \end{array}$$

The vector space $\mathcal{X}$ is a Banach space when equipped with the norm

$$\begin{aligned}
||(U,V,H)||_{\mathcal{X}} &:= ||e^{(3s\rho-2\mu)l} l^{-5/2} U||_{L^2(Q))} \\
&+||e^{3s\rho l} V||_{L^2(Q)} \\
&+||e^{(3s\rho-2\mu)l} l^{-9/2} H||_{L^2(Q)} \\
&+||e^{s\rho l} l^{-3/2}(U_t - U_{xx} + f_0 U - H 1_\omega)||_{L^2(Q)} \\
&+||e^{3s\rho l} l^{-3/2}(V_t - V_{xx} + g_0 \bar{u}^2 U + g_1 V)||_{L^2(Q)} \\
&+||(U,V)||_{L^\infty(0,T,H_0^1(\Omega)^2)} \\
&+||U(0,.)||_{W^{1,\infty}(\Omega)} + ||V(0,.)||_{W^{1,\infty}(\Omega)}.
\end{aligned}$$

Let $\mathcal{Y}$ be the set of $(y_1,y_2,y_3,y_4) \in L^2(Q)^2 \times W_0^{1,\infty}(\Omega)^2$ such that

$$e^{s\rho l} l^{-3/2} y_1 \in L^2(Q) \text{ and } e^{3s\rho l} l^{-3/2} y_2 \in L^2(Q).$$

It is a Banach space when equipped with the norm

$$\begin{aligned}
&||(y_1,y_2,y_3,y_4)||_{\mathcal{Y}} \\
&\quad := ||e^{s\rho l} l^{-3/2} y_1||_{L^2(Q)} + ||e^{3s\rho l} l^{-3/2} y_2||_{L^2(Q)} \\
&\quad\quad +||y_3||_{W^{1,\infty}(\Omega)} + ||y_4||_{W^{1,\infty}(\Omega)}.
\end{aligned}$$

One can see that the map $\mathcal{A}: \mathbb{R} \times W_0^{1,\infty}(\Omega)^2 \times \mathcal{X} \to \mathcal{Y}$ is well defined and of class C^1, with $\mathcal{L} = \partial\mathcal{A}/\partial(U,V,H)(0)$ given by (38). Furthermore, using Proposition 4, we have that $\mathcal{L}$ is onto. The following inverse mapping theorem is needed (see [15]):

Theorem 2: Let E_1 and E_2 be two Banach spaces and let $G: E_1 \to E_2$ be a map of class C^1. Assume that $x_0 \in E_1$ is such that $G'(x_0): E_1 \to E_2$ is onto. Then there exists $\delta > 0$ such that for every $y \in E_2$ with $||y - G(x_0)||_{E_2} < \delta$, there exists $x \in E_1$ such that $G(x) = y$.

Corollary 1: Let B_1, B_2, and B_3 be three Banach spaces, and let $F: B_1 \times B_2 \to B_3$ be a map of class C^1. Let $(x_0,y_0) \in B_1 \times B_2$ be such that $\frac{\partial F}{\partial y}(x_0,y_0): B_2 \to B_3$ is onto, and let $z_0 = F(x_0,y_0)$. Then there exists $\delta > 0$ such that for all $(x,z) \in B_1 \times B_3$ with $||x-x_0||_{B_1} + ||z-z_0||_{B_3} < \delta$, there exists $y \in B_2$ such that $F(x,y) = z$.

Corollary 1 follows at once from Theorem 2 by letting $E_1 = B_1 \times B_2$, $E_2 = B_1 \times B_3$, and $G(x,y) = (x, F(x,y))$ for $(x,y) \in E_1$. Indeed, G is C^1 with

$$G'(x_0,y_0)(u,v) = (u, \frac{\partial F}{\partial x}(x_0,y_0)u + \frac{\partial F}{\partial y}(x_0,y_0)v).$$

It is clear that $G'(x_0,y_0)$ is onto. Therefore, there exists $\delta > 0$ such that for $||x-x_0||_{B_1} + ||z-z_0||_{B_3} < \delta$, one may find $y \in B_2$ such that $z = F(x,y)$.

To conclude the proof of Theorem 1, it is sufficient to apply Corollary 1 to the spaces $B_1 := \mathbb{R} \times W_0^{1,\infty}(\Omega)^2$, $B_2 := \mathcal{X}$, $B_3 := \mathcal{Y}$, and the map $F := \mathcal{A}$ (taking $(x_0,y_0,z_0) = (0,0,0)$). Then there exists some $\varepsilon_1 > 0$ such that for $(\varepsilon, U^0, V^0) \in B_1$ with

$$|\varepsilon| \le \varepsilon_1, \quad ||U_0||_{W^{1,\infty}(\Omega)} \le \varepsilon_1, \quad ||V_0||_{W^{1,\infty}(\Omega)} \le \varepsilon_1,$$

one may find $(U,V,H) \in \mathcal{X}$ such that $\mathcal{A}(\varepsilon, U^0, V^0, (U,V,H)) = (0,0,0,0)$. The conclusion of Theorem 1 holds for $(u^0,v^0) \in W_0^{1,\infty}(\Omega)^2$ with

$$||u^0||_{W^{1,\infty}(\Omega)} \le \frac{\varepsilon_1^2}{2}, \quad ||v^0||_{W^{1,\infty}(\Omega)} \le \frac{\varepsilon_1^4}{2}$$

Indeed, the result is trivial when $u^0 = v^0 = 0$. Otherwise, we let

$$\varepsilon := \left(\frac{||u^0||^6_{W^{1,\infty}(\Omega)}}{\varepsilon_1^6} + \frac{||v^0||^2_{W^{1,\infty}(\Omega)}}{\varepsilon_1^2} \right)^{\frac{1}{6}}, \; U^0 := \frac{u^0}{\varepsilon}, \; V^0 := \frac{v^0}{\varepsilon^3}.$$

Then $0 < \varepsilon \le \varepsilon_1$, $||U^0||_{W^{1,\infty}(\Omega)} \le \varepsilon_1$, and $||V^0||_{W^{1,\infty}(\Omega)} \le \varepsilon_1$. The conclusion follows with $u := \varepsilon(\bar{u} + U)$, $v := \varepsilon^3(\bar{v} + V)$, and $h := \varepsilon(\bar{h} + H)$.

ACKNOWLEDGMENTS

JMC and LR acknowledge the support of the ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

REFERENCES

[1] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, ser. Lecture Notes Series. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996, vol. 34.

[2] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos, "Controllability for a class of reaction-diffusion systems: the generalized Kalman's condition," *C. R. Math. Acad. Sci. Paris*, vol. 345, no. 10, pp. 543–548, 2007.

[3] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa, "Recent results on the controllability of linear coupled parabolic problems: a survey," *Math. Control Relat. Fields*, vol. 1, no. 3, pp. 267–306, 2011.

[4] L. Rosier and L. de Teresa, "Exact controllability of a cascade system of conservative equations," *C. R. Math. Acad. Sci. Paris*, vol. 349, no. 5-6, pp. 291–296, 2011.

[5] F. Alabau-Boussouira and M. Léautaud, "Indirect controllability of locally coupled wave-type systems and applications," *J. Math. Pures Appl. (9)*, vol. 99, no. 5, pp. 544–576, 2013.

[6] J.-M. Coron, S. Guerrero, and L. Rosier, "Null controllability of a parabolic system with a cubic coupling term," *SIAM J. Control Optim.*, vol. 48, no. 8, pp. 5629–5653, 2010.

[7] D. Bothe and D. Hilhorst, "A reaction-diffusion system with fast reversible reaction," *J. Math. Anal. Appl.*, vol. 286, no. 1, pp. 125–135, 2003.

[8] P. Érdi and J. Tóth, *Mathematical models of chemical reactions*, ser. Nonlinear Science: Theory and Applications. Princeton University Press, Princeton, NJ, 1989, theory and applications of deterministic and stochastic models.

[9] L. Rosier and B.-Y. Zhang, "Null controllability of the complex Ginzburg-Landau equation," *Ann. Inst. H. Poincaré Anal. Non Linéaire*, vol. 26, no. 2, pp. 649–673, 2009.

[10] X. Fu, "Null controllability for the parabolic equation with a complex principal part," *J. Funct. Anal.*, vol. 257, no. 5, pp. 1333–1354, 2009.

[11] J.-M. Coron, "On the controllability of 2-D incompressible perfect fluids," *J. Math. Pures Appl. (9)*, vol. 75, no. 2, pp. 155–188, 1996.

[12] ——, *Control and nonlinearity*, ser. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007, vol. 136.

[13] F. W. Chaves-Silva, L. Rosier, and E. Zuazua, "Null controllability of a system of viscoelasticity with a moving control," *J. Math. Pures Appl. (9)*, vol. 101, no. 2, pp. 198–222, 2014.

[14] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel, "Local exact controllability of the Navier-Stokes system," *J. Math. Pures Appl. (9)*, vol. 83, no. 12, pp. 1501–1542, 2004.

[15] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin, *Optimal control*, ser. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1987, translated from the Russian by V. M. Volosov.