

A weak version of the small-gain theorem

Alessandro Astolfi and Laurent Praly

Abstract— A weak version of the small-gain theorem is derived. Connections with the classical *linear and nonlinear small-gain conditions* are established. The necessity of the weak small-gain conditions is discussed.

I. INTRODUCTION

Small-gain theorems have been widely used to establish stability properties of nonlinear interconnected systems. It is possible to provide several versions of the small-gain theorem, depending of the input-output property that is used to quantify the input-output behavior of the interconnected subsystems. Possible selections include the L_2 -gain, yielding an L_2 small-gain theorem [11], [10] (which generalizes to the nonlinear setting the linear H_∞ small-gain theorem [4]), and the property of Input-to-State Stability (ISS), which leads to the derivation of nonlinear small-gain theorems, such as the one in [7]. Other versions of the small-gain theorem have been developed in [5], [1], [6], in which interconnections of possibly non-ISS subsystems have been considered. Finally, small-gain theorems for large scale interconnected systems and for systems interconnected by means of communication channels have recently been developed in [3].

The purpose of this paper is to develop a weak version of the small-gain theorem, in the spirit of the Matrosov theorem derived in [2]. As a matter of fact, the paper partly extends, to a class of interconnected systems, the results therein which provide a weak version of Matrosov theorem. Note, however, that the results in [2] are somewhat stronger, since under some stability assumptions it is possible to establish strong convergence claims.

A. Astolfi is with the Dept. of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK and DISP, University of Roma “Tor Vergata”, Via del Politecnico 1, 00133 Rome, Italy, E-mail: a.astolfi@ic.ac.uk. This work is partly supported by the EPSRC Programme Grant “Control For Energy and Sustainability” EP/G066477 and by the MIUR under PRIN Project “Advanced Methods for Feedback Control of Uncertain Nonlinear Systems”.

L. Praly is with MINES ParisTech, CAS, Mathématiques et Systèmes, 35, rue St-Honoré, 77305 Fontainebleau CEDEX, France, E-mail: Laurent.Praly@mines-paristech.fr.

We consider a nonlinear system described by equations of the form

$$\dot{x} = f(x), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, and f is locally Lipschitz continuous. In addition, without loss of generality, we assume that $x = 0$ is an equilibrium of the system.

The ISS small-gain theorem, see [7], allows to establish asymptotic stability of the equilibrium of the system (1) when there exist

- two C^1 functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $V_1 + V_2$ is positive definite and radially unbounded,
- two class K_∞ functions, $\alpha_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and two continuous functions $\beta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

satisfying, along the solutions of system (1), the differential inequalities

$$\begin{aligned} \dot{V}_1 &\leq -\alpha_1(V_1) + \beta_1(V_2), \\ \dot{V}_2 &\leq -\alpha_2(V_2) + \beta_2(V_1), \end{aligned} \quad (2)$$

and the small-gain condition

$$\beta_2 \circ \alpha_1^{-1} \circ \beta_1 \circ \alpha_2^{-1} < Id, \quad (3)$$

where Id is the identity map.

The problem that we address in this paper is to study what happens *relaxing* the inequalities (2). This relaxation can be carried out in various directions. In particular, we are interested in the case in which the argument of the functions α_i and β_j are not the functions V_k , but some other functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ so that, along the solutions of system (1), we have

$$\begin{aligned} \dot{V}_1 &\leq -\alpha_1(h_1(x)) + \beta_1(h_2(x)), \\ \dot{V}_2 &\leq -\alpha_2(h_2(x)) + \beta_2(h_1(x)). \end{aligned} \quad (4)$$

Remark 1: Under additional assumptions on the functions V_i the inequalities (4) may be exploited to establish boundedness of all solutions of the system (1).

Remark 2: In the considered set up, borrowing from LaSalle invariance principle, and from the classical small-gain theorem, one may be tempted to conjecture

that the ω -limit set of the solutions of the system (1) is contained in the largest invariant set such that

$$\begin{aligned} 0 &= -\alpha_1(h_1(x)) + \beta_1(h_2(x)), \\ 0 &= -\alpha_2(h_2(x)) + \beta_2(h_1(x)). \end{aligned} \quad (5)$$

This, unfortunately, is not true in general.

Remark 3: The small-gain condition (3) and the inequalities (2) imply that the equations

$$\begin{aligned} 0 &= -\alpha_1(V_1(x)) + \beta_1(V_2(x)), \\ 0 &= -\alpha_2(V_2(x)) + \beta_2(V_1(x)), \end{aligned}$$

have the unique solution $x = 0$, *i.e.* that the system (1) has a unique equilibrium. This is, however, not implied by the inequalities (5).

Remark 4: The differential inequalities in [2] are a special case of the inequalities (4), obtained by setting β_1 to zero. This selection yields a triangular structure of the inequalities which, exploiting properties of asymptotically autonomous vector fields [8], dictates very specific properties for the ω -limit set of the solutions of the underlying system. In particular the ω -limit set is a chain recurrent set. This property is however lost in the current scenario, since there is no *driving inequality*.

The paper is organized as follows. In Section II a preliminary lemma, which generalizes the result in [2] and introduces a new small-gain condition, is stated. Section III discusses the new small-gain condition, establishes connections with the classical, nonlinear, small-gain condition, and clarifies the necessity of the new small-gain property. Section IV provides the main result of the paper, namely a weak version of the small-gain theorem. Finally, Section V contains a simple example and Section VI contains a few concluding remarks and observations.

II. A PRELIMINARY RESULT

This section contains a preliminary result which is instrumental to establish the weak small-gain theorem formulated in Section IV.

Lemma 1: Let $i = 1, 2$. Let $a_i : \mathbb{R}_+ \rightarrow [-\bar{a}, \bar{a}]$, be bounded absolutely continuous functions and $b_i : \mathbb{R}_+ \rightarrow [0, \bar{b}]$ be bounded, piecewise continuous, functions.

Assume there exist continuous positive definite functions $\alpha_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous functions $\beta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which are zero at zero, and a real number ε in $]0, 1[$ such that the following hold.

1) The differential inequalities

$$\begin{aligned} \dot{a}_1(t) &\leq -\alpha_1(b_1(t)) + \beta_1(b_2(t)), \\ \dot{a}_2(t) &\leq -\alpha_2(b_2(t)) + \beta_2(b_1(t)) \end{aligned} \quad (6)$$

hold for almost all t in \mathbb{R}_+ .

2) The small-gain like condition

$$\beta_1(b_2)\beta_2(b_1) \leq (1 - \varepsilon)\alpha_2(b_2)\alpha_1(b_1) \quad (7)$$

holds for all (b_1, b_2) in $[0, \bar{b}]^2$.

Then

$$\liminf_{t \rightarrow +\infty} [b_1(t) + b_2(t)] = 0. \quad (8)$$

III. THE SMALL-GAIN CONDITION (7)

In this section we study the condition (7) and we relate this condition with the classical nonlinear small-gain condition.

To start with, we observe that, if there exist real numbers ψ_1 and ψ_2 such that

$$\psi_1 = \sup_{b_1 \in]0, \bar{b}] } \frac{\beta_2(b_1)}{\alpha_1(b_1)}, \quad \psi_2 = \sup_{b_2 \in]0, \bar{b}] } \frac{\beta_1(b_2)}{\alpha_2(b_2)},$$

then the condition

$$\psi_1 \psi_2 \leq (1 - \varepsilon) \quad (9)$$

implies condition (7). The converse statement is also true. Namely, if condition (7) holds then the numbers ψ_1 and ψ_2 exist and satisfy condition (9).

This property justifies the terminology "linear small-gain condition" for condition (7).

We are now ready to relate the condition (7) to the classical nonlinear small-gain condition. To this end, and to simplify the discussion, assume that the functions β_i and α_i are defined on \mathbb{R}_+ and that the functions α_i are invertible. Assume also that \bar{b} , in (7), is infinity. Then, from the theory of interconnected nonlinear systems we would expect that stability properties be related to the nonlinear small-gain condition (4), namely

$$\beta_2 \circ \alpha_1^{-1} \circ \beta_1 \circ \alpha_2^{-1}(s) < s \quad \forall s > 0. \quad (10)$$

Lemma 2: Condition (7) implies, but it is not implied by, condition (10).

While necessity of the small-gain condition (7) is difficult to establish, we now show that violation of the non-strict inequality yields the existence of functions a_i and b_i such that the convergence result of Lemma 1 does not hold.

Lemma 3: Assume there exist strictly positive real numbers b_{1a} , b_{2b} and b_{2c} such that

$$\frac{\beta_1(b_{2b})\beta_2(b_{1a})}{\alpha_2(b_{2b})\alpha_1(b_{1a})} > 1 \quad \frac{\beta_1(b_{2c})\beta_2(b_{1a})}{\alpha_2(b_{2c})\alpha_1(b_{1a})} < 1. \quad (11)$$

Then there exist functions a_i and b_i such that the convergence result in Lemma 1 does not hold.

To illustrate the result in Lemma 3 consider the differential inequalities

$$\dot{a}_1 \leq -a_1 + b_2^2, \quad \dot{a}_2 \leq -b_2 + \gamma\sqrt{a_1},$$

with $\gamma > 0$. Note that the linear small-gain condition is violated, while the nonlinear one holds for $\gamma < 1$.

Let k be in $]0, 1[$ and

$$b_2(t) = \sqrt{1 - k \cos(t)}.$$

Then

$$a_1(t) = 1 - \frac{k}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right)$$

is a solution of the first inequality and

$$\liminf_{t \rightarrow +\infty} [b_1(t) + b_2(t)] > 0.$$

To conclude, it remains to establish that we can find a bounded absolutely continuous function a_2 which satisfies the second differential inequality. To this end note that, for all k in $]0, 1[$,

$$\rho(k) = \frac{\int_0^{2\pi} \sqrt{1 - \frac{k}{\sqrt{2}} \cos\left(t - \frac{\pi}{4}\right)} dt}{\int_0^{2\pi} \sqrt{1 - k \cos(t)} dt} \geq 1.$$

As a result, for all γ in $]1/\rho(k), 1[$,

$$\lim_{t \rightarrow +\infty} \int_0^t [-b_2(s) + \gamma\sqrt{a_1(s)}] ds = +\infty,$$

which implies that a function a_2 does exist¹.

IV. A WEAK SMALL-GAIN THEOREM

In this section we state the main result of the paper, namely a weak version of the small-gain theorem.

Theorem 1: Consider the nonlinear, time-invariant, system (1). Suppose there exist continuous functions $\beta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which are zero at zero, C^1 functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$, continuous functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$, continuous positive definite functions $\alpha_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

¹For example $a_2(t) = \text{sat}\left(\int_0^t [-b_2(s) + \gamma\sqrt{a_1(s)}] ds\right)$.

such that the conditions (4) hold. Suppose in addition that we have

$$\beta_1(b_2) \beta_2(b_1) \leq (1 - \varepsilon) \alpha_1(b_1) \alpha_2(b_2), \quad (12)$$

for some $\varepsilon > 0$ and all non-negative b_1 and b_2 .

Then, for any bounded solutions of system (1),

$$\liminf_{t \rightarrow +\infty} [h_1(x(t)) + h_2(x(t))] = 0. \quad (13)$$

Moreover, if the largest invariant set \mathcal{N} contained in the set

$$\{x \in \mathbb{R}^n : h_1(x) = h_2(x) = 0\},$$

is stable, then

$$\lim_{t \rightarrow +\infty} h_1(x(t)) + h_2(x(t)) = 0. \quad (14)$$

Remark 5: As explained in Section III it is not possible, in general, to obtain stronger convergence results, for example asymptotic convergence to zero of $h_1(x(t)) + h_2(x(t))$, nor to relax the linear small-gain condition (12).

Remark 6: The last point in Theorem 1 rephrases a well-known fact, see for instance [9, Lemma I.4].

V. AN ILLUSTRATIVE EXAMPLE

In this section we illustrate some of the ideas and results established by means of a simple example.

Consider the system

$$\begin{aligned} \dot{x}_1 &= (x_{1+}^2 + x_3^2) x_2, \\ \dot{x}_2 &= -(x_{1+}^2 + x_3^2) x_1 \\ \dot{x}_3 &= -x_3^3 + x_{1+}^{9/2}. \end{aligned} \quad (15)$$

Note that all solutions are bounded, since

$$\overline{x_{1+}^2 + x_2^2} = 0,$$

and the x_3 sub-system is ISS. Therefore, for any solution, there exists a constant c such that $x_{1+}^2 + x_2^2$ and x_3^2 are bounded by c^2 . In what follows we assume that these bounds hold. To apply the small-gain theorem let

$$V_1(x) = x_2, \quad V_2(x) = \frac{k}{2} x_3^2.$$

Then, Young's inequality yields

$$\begin{aligned} \dot{V}_1 &= -(x_{1+}^2 + x_3^2) x_{1+} + (x_{1+}^2 + x_3^2) |x_{1-}| \\ &\leq -x_{1+}^3 + c x_3^2 \\ \dot{V}_2 &= -k x_3^4 + k x_3 x_{1+}^{9/2} \\ &\leq -\left(k - \frac{k^4 \ell^8}{4}\right) x_3^4 + \frac{3}{4} \left(\frac{k}{\ell^2}\right)^{3/4} x_{1+}^6, \end{aligned}$$

which motivate the choice

$$\begin{aligned} h_1(x) &= x_{1+}^3, & h_2(x) &= x_3^2, \\ \alpha_1(s) &= s, & \beta_1(s) &= cs, \\ \alpha_2(s) &= \left(k - \frac{k^4 \ell^2}{4}\right) s^2, & \beta_2(s) &= \frac{3}{4} \left(\frac{k}{\sqrt{\ell}}\right)^{3/4} s^2. \end{aligned}$$

The linear small-gain condition does not hold for such functions, although the nonlinear one does hold selecting

$$k < \left(\frac{16}{27c^6}\right)^{1/4}, \quad \ell = \frac{(27c^6)^{1/8}}{k}.$$

Note that the conclusion of Theorem 1 would be, in this case,

$$\liminf_{t \rightarrow +\infty} [x_{1+}(t)^3 + x_3(t)^2] = 0. \quad (16)$$

As we shall prove this is not the case if $x_3(0)$ is non-zero. To this end, note that, at any equilibrium, $x_{1+} = x_3 = 0$. Moreover, since we have a locally Lipschitz system, any solution not starting from an equilibrium cannot reach an equilibrium in finite time. As a result, along any solution $x_{1+}^2 + x_3^2$ remains strictly positive.

Consider now a solution with $x_3(0) \neq 0$. The above remark motivates the introduction of the function

$$\tau(t) = \int_0^t (x_{1+}(s)^2 + x_3(s)^2) ds$$

where $x_{1+}(t)^2 + x_3(t)^2$ is obtained from the solution. The function is strictly increasing and, since

$$|x_3(t)| \geq \exp\left(-\int_0^t x_3(s)^2 ds\right) |x_3(0)|,$$

the integral $\int_0^t x_3(s)^2 ds$ and therefore $\tau(t)$ go to $+\infty$ as t goes to ∞ . Therefore there exists a time t_0 such that $\tau(t)$ is larger than 2π for all $t \geq t_0$.

Using τ , we can express the (x_1, x_2) -components of the solution as

$$x_1(t) + ix_2(t) = \exp(-i\tau(t)) x_1(0) + ix_2(0)$$

where $i^2 = -1$. It follows that, in any interval $[\tau(s), \tau(s) + 2\pi]$, there exists an interval of length $\frac{\pi}{2}$ in which x_1 and therefore x_{1+} is larger than or equal to $\frac{\sqrt{2[x_1(0)^2 + x_2(0)^2]}}{4}$. As a result, for all $t \geq t_0$,

$$\begin{aligned} & \int_0^{\tau(t)} \exp(\tau(s)) x_{1+}(\tau(s))^{9/2} ds \\ & \geq \sum_{k=0}^{K(t)} \exp(2k\pi) \frac{\pi}{2} \left(\frac{\sqrt{2[x_1(0)^2 + x_2(0)^2]}}{4}\right)^{9/2}, \\ & \geq \frac{\pi}{2} \frac{\exp(2(K(t)+1)\pi) - 1}{\exp(2\pi) - 1} \left(\frac{\sqrt{2[x_1(0)^2 + x_2(0)^2]}}{4}\right)^{9/2}, \end{aligned}$$

where $K(t)$ is the largest integer k satisfying $\tau(t) \geq 2k\pi$.

Consider now the identity

$$-x_3^3 + x_{1+}^{9/2} = -(x_{1+}^2 + x_3^2) x_3 + (x_{1+}^2 x_3 + x_{1+}^{9/2})$$

and the bound

$$\frac{x_{1+}^2 x_3 + x_{1+}^{9/2}}{x_{1+}^2 + x_3^2} \geq \frac{x_{1+}^{9/2}}{c^2},$$

yielding

$$\begin{aligned} & \exp(\tau(t)) x_3(t) - x_3(0) \\ & \geq \frac{1}{c^2} \int_0^{\tau(t)} \exp(\tau(s)) x_{1+}(\tau(s))^{9/2} ds \\ & \geq \frac{1}{c^2} \frac{\pi}{2} \frac{\exp(2(K(t)+1)\pi) - 1}{\exp(2\pi) - 1} \left(\frac{\sqrt{2[x_1(0)^2 + x_2(0)^2]}}{4}\right)^{9/2} \end{aligned}$$

Finally, exploiting the conditions

$$\lim_{t \rightarrow +\infty} \tau(t) = +\infty,$$

$$1 \leq \exp([2(K(t) + 1)\pi] - \tau(t)) \leq \exp(2\pi),$$

we conclude

$$\begin{aligned} \liminf_{t \rightarrow +\infty} x_3(t) & \geq \frac{1}{c^2} \frac{\pi}{2} \frac{1}{\exp(2\pi) - 1} \left[\frac{\sqrt{2[x_1(0)^2 + x_2(0)^2]}}{4}\right]^{9/2} \\ & > 0, \end{aligned}$$

which shows that condition (16) does not hold.

Figure 1 displays the state histories of system (15) for the initial condition $(x_1(0), x_2(0), x_3(0)) = (1, 0, 1)$. Note that the x_3 component of the state is bounded away from zero.

VI. CONCLUSION

A weak version of the small-gain theorem has been established. This result relies upon the properties of a set of differential inequalities together with a linear small-gain condition. The paper provides a non-trivial generalization of the results in [2] in which cascaded systems have been studied.

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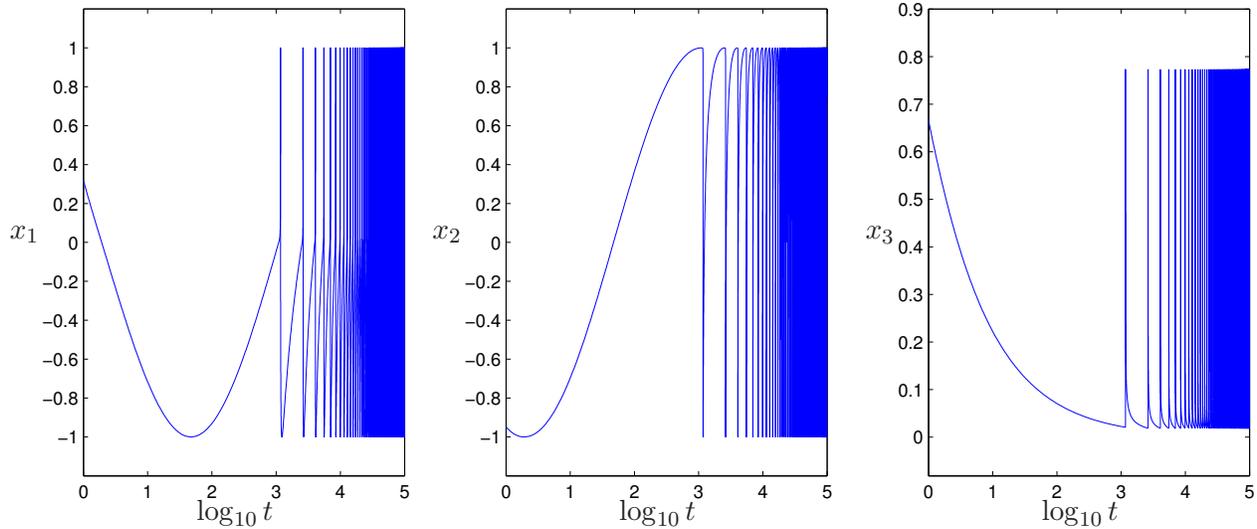


Fig. 1. State histories of the system (15).

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