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Stabilization of a linear hyperbolic system with one boundary controlled transport PDE coupled with n counterconvecting PDEs

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Abstract— **We propose a full-state feedback law to stabilize linear first-order hyperbolic systems featuring n positive and one negative transport speeds on a finite space domain. Only one state, corresponding to the negative velocity, is actuated at the right boundary. The proposed controller guarantees convergence of the whole $(n+1)$–state system to zero in the $\mathcal{L}^2$-sense.**

I. Introduction

We investigate boundary stabilization of a class of linear first-order hyperbolic systems on a finite space domain $x \in [0, 1]$. Transport equations are predominant in modeling of traffic flow [1], water management systems [5] or multiphase flow [6]. The coupling between states traveling in opposite directions, both in-domain and at the boundaries, may induce instability leading to undesirable behaviors. For example, oscillatory two-phase flow regimes occurring on oil and gas production systems directly result, in some cases, from these mechanisms [7].

A first result on exponential stabilization of linear first-order hyperbolic systems has been presented in [4], using a control Lyapunov function to design stabilizing boundary feedback laws. The result deals with 2–state *hetero-directional* systems, i.e. systems of 2 transport equations with opposite transport speeds, but no in-domain coupling. The control laws take the form of static output feedbacks applied at both boundaries. In [2], this result is extended to 2–state hetero-directional systems with space-varying transport speeds and linear source terms. However, the source terms are required to satisfy a restrictive condition on their magnitude, thus limiting the applicability of the result.

In [13], an observer-controller structure is proposed to stabilize 2–state hetero-directional systems with the only restriction that the source terms are bounded in the $\mathcal{L}^\infty$-norm on the space domain. A full-state feedback law is designed using a backstepping approach [9], guaranteeing exponential stability of the zero equilibrium. Based on the same approach, a full-state feedback law is designed for a 3–state first-order hyperbolic system representing gas-liquid flow in oil wells in [8].

In this article, we propose to extend the control designs of [13] and [8] to a broader class of system. More precisely, we consider systems of $(n + 1)$ linearly coupled transport equations, with space-varying transport speeds and source term coefficients. One of the transport speeds remains strictly negative throughout the space domain, while the n others remain strictly positive. The state corresponding to the negative velocity is controlled at the right boundary ($x = 1$). At the left boundary ($x = 0$), reflexivity conditions ensure well-posedness of the system. The system is strongly underactuated, as only one state is controlled, whereas a possibly large number of states (determined by the value of n) are uncontrolled. Yet, we propose a control design that forces all the states to exponentially converge to the zero equilibrium profile. This is the main contribution of the paper.

Following the classical backstepping approach, the system is mapped to a so-called *target* system with desirable stability properties using a Volterra transformation. The target system is designed by removing from the original system the minimum amount of coupling terms required to ensure stability. Then, existence and uniqueness of the transformation kernels mapping the original system to the target system are investigated. The kernels are shown to satisfy a system of first-order hyperbolic PDEs on a triangular domain. After using the method of characteristics to transform them into integral equations, the method of successive approximations is used to prove well-posedness of the kernel equations, and thus the validity of the design.

The article is organized as follows. In Section II, we describe the problem statement and the notations. In Section III, we define the target system and investigate its stability properties. In Section IV, we describe the backstepping transformation and derive the kernel equations. In Section V, we prove existence and uniqueness of the kernels. Finally, the design is summarized in Section VI where the main result is stated. We give conclusions and ideas for future work in Section VII.

II. Problem statement

We consider the following linear hyperbolic system

$$\mathbf{u}_t(t,x) + \boldsymbol{\lambda}(x)\mathbf{u}_x(t,x) + \boldsymbol{\sigma}(x)\mathbf{u}(t,x) + \boldsymbol{\omega}(x)v(t,x) = 0 \quad (1)$$

$$v_t(t,x) - \mu(x)v_x(t,x) + \boldsymbol{\theta}(x)\mathbf{u}(t,x) = 0 \quad (2)$$

where

$$\mathbf{u}(t,x) = \begin{pmatrix} u_1(t,x) & u_2(t,x) & \cdots & u_n(t,x)\end{pmatrix}^T,$$

$$\boldsymbol{\lambda}(x) = \text{diag }\{\lambda_1(x), ..., \lambda_n(x)\}, \quad \boldsymbol{\sigma}(x) = \left(\sigma_{i,j}(x)\right)_{1\le i,j\le n},$$

$$\boldsymbol{\omega}(x) = \begin{pmatrix}\omega_1(x) & \omega_2(x) & \cdots & \omega_n(x)\end{pmatrix}^T,$$

$$\boldsymbol{\theta}(x) = \begin{pmatrix}\theta_1(x) & \theta_2(x) & \cdots & \theta_n(x)\end{pmatrix}$$

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along with the following boundary conditions

$$u(t,0) = Q_0 v(t,0) = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}^T v(t,0) \quad (3)$$

$$v(t,1) = U(t) \quad (4)$$

where $U(t)$ is the control input, and the q_i, $i = 1, \ldots n$ are non-zero. The transport coefficients continuously differentiable on the segment $[0,1]$ and such that[1]

$$\forall x \in [0,1], \quad \forall i = 1, \ldots, n \quad \lambda_i(x) > 0, \quad \mu(x) > 0 \quad (5)$$

Physically, (5) indicates that the states u_i, $i = 1, .., n$, evolve from left to right, while v evolves from right to left. Besides, the source term coefficients are assumed continuous on the segment $[0,1]$. This setup is schematically depicted on Figure 1.

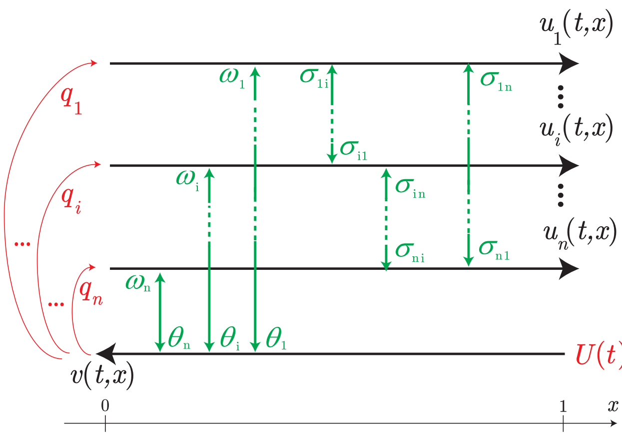

Fig. 1. Schematic view of the $(n+1)$–state hyperbolic system. The space dependence of the coupling coefficients has been omitted for the sake of clarity.

System (1)–(4) is the most general form of $(n+1)$–state linear systems satisfying (5). Such systems arise, e.g., in modelling of open channel flow (see [2], with $n = 1$) or multiphase flow (see [7], with $n = 2$)[2]. Our goal is to find a feedback control law $U(t)$ that exponentially stabilizes the zero equilibrium of the system (1)–(4).

III. Target system

We want to map system (1)-(4) to the following target system

$$\boldsymbol{\alpha}_t(t,x) + \boldsymbol{\lambda}(x)\boldsymbol{\alpha}_x(t,x) + \boldsymbol{\sigma}(x)\boldsymbol{\alpha}(t,x) + \boldsymbol{\omega}(x)\beta(t,x) + \int_0^x \left[\mathbf{c}(x,\xi)\boldsymbol{\alpha}(t,\xi) + \boldsymbol{\kappa}(x,\xi)\beta(t,\xi)\right] d\xi = 0 \quad (6)$$

$$\beta_t(t,x) - \mu(x)\beta_x(t,x) = 0 \quad (7)$$

where $\mathbf{c}(x,\xi) = \begin{bmatrix} c_{1,1}(x) & c_{1,2}(x) & \cdots & c_{1,n}(x) \\ c_{2,1}(x) & c_{2,2}(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1,n}(x) \\ c_{n,1}(x) & \cdots & c_{n,n-1}(x) & c_{n,n}(x) \end{bmatrix}$ and $\boldsymbol{\kappa}(x,\xi) = \begin{pmatrix} \kappa_1(x,\xi) & \cdots & \kappa_{n-1}(x,\xi) & \kappa_n(x,\xi) \end{pmatrix}^T$ are function matrices to be defined on the triangular domain

$$\mathcal{T} = \left\{(x,\xi) \in \mathbb{R}^2 \mid 0 \le \xi \le x \le 1\right\}$$

with the following boundary conditions

$$\alpha(t,0) = Q_0 \beta(t,0) \qquad \beta(t,1) = 0 \quad (8)$$

The target system is a copy of the original system depicted in Figure 1, where the θ_i, $i = 1, \ldots, n$ coupling terms acting on the controlled state were removed, and integral coupling terms between the n uncontrolled states were added. The stability properties of this system are stated in the following lemma.

Lemma 3.1: Under the following assumptions

$$\forall i,j = 1, \ldots, n \quad \lambda_i, \mu \in C^1([0,1], \mathbb{R}_+^*), \quad \sigma_{i,j}, \omega_i, \theta_i \in \mathcal{L}^\infty([0,1])$$
$$\alpha_i^0, \beta^0 \in \mathcal{L}^2([0,1]), \quad c_{i,j}, \kappa_i \in \mathcal{L}^\infty(\mathcal{T})$$

the equilibrium $(\alpha,\beta)^T \equiv (0, \ldots 0, 0)^T$ of system (6)-(7) with boundary conditions (8) and initial conditions $(\alpha^0, \beta^0)^T = (\alpha_1^0, \ldots \alpha_n^0, \beta^0)^T$ is exponentially stable in the $\mathcal{L}^2$ sense.

Proof The details of the proof are omitted for lack of space. Exponential stability is shown by considering the following Lyapunov functional

$$V(t) = \int_0^1 p e^{-\delta x} \sum_{i=1}^n \frac{\alpha_i(t,x)^2}{\lambda_i(x)} dx + \int_0^1 \frac{1+x}{\mu(x)} \beta(t,x)^2 dx \quad (9)$$

where $p > 0$ and $\delta > 0$ are analysis parameters. Picking δ large enough and then $p(\delta)$ small enough yields

$$\dot{V} \le -\eta V \quad (10)$$

for some $\eta > 0$. ∎

In order to map the original system (1)–(4) to the target system (6)–(8), we propose a Volterra transformation of the second kind. In the next section, we derive a set of PDEs that the transformation kernels verify.

IV. Backstepping transformation

We now denote

$$w = \begin{pmatrix} \mathbf{u}(t,x) \\ v(t,x) \end{pmatrix} \qquad \gamma(t,x) = \begin{pmatrix} \boldsymbol{\alpha}(t,x) \\ \beta(t,x) \end{pmatrix}$$

$$\Sigma(x) = \begin{pmatrix} \boldsymbol{\sigma}(x) & \boldsymbol{\omega}(x) \\ \boldsymbol{\theta}(x) & 0 \end{pmatrix} \qquad \Sigma^0(x) = \begin{pmatrix} \boldsymbol{\sigma}(x) & \boldsymbol{\omega}(x) \\ 0_{1\times n} & 0 \end{pmatrix}$$

$$\Lambda(x) = \begin{pmatrix} \boldsymbol{\lambda}(x) & 0_{n\times 1} \\ 0_{1\times n} & -\mu(x) \end{pmatrix} \qquad C(x,\xi) = \begin{pmatrix} \mathbf{c}(x,\xi) & \boldsymbol{\kappa}(x,\xi) \\ 0_{1\times n} & 0 \end{pmatrix}$$

which allows us to rewrite the original system (1)–(2) and the target system (6)–(7) in matrix form, as follows, omitting the time argument for brevity purposes

$$w_t(x) + \Lambda(x) w_x(x) + \Sigma(x) w(x) = 0 \quad (11)$$

$$\gamma_t(x) + \Lambda(x)\gamma_x(x) + \Sigma^0(x)\gamma(x) + \int_0^x C(x,\xi)\gamma(\xi) d\xi = 0 \quad (12)$$

We consider the following backstepping transformation

$$\beta(t,x) = v(t,x) - \int_0^x \mathbf{k}(x,\xi) w(t,\xi) d\xi \quad (13)$$

[1]This ensures well-posedness of system (1),(2) with boundary conditions (3),(4).

[2]Systems corresponding to higher values of n could arise when considering two-fluid models for gas-liquid flow [3], [11], even though no such models are, to the best of our knowledge, developed in the literature.

where the kernel row vector $\mathbf{k}$ has the following form

$$\mathbf{k} = \begin{pmatrix} k^1 & k^2 & \cdots & \cdots & k^{n+1} \end{pmatrix} \tag{14}$$

Besides, we set $\boldsymbol{\alpha} \equiv \mathbf{u}$.Plugging (13) into (11) and (12), one shows that $\mathbf{k}$ and C satisfy a cascade system of PDEs. First, the kernel coefficients k^j, $j = 1, ..., n$ and k^{n+1} satisfy the following $(n+1) \times (n+1)$ system of hyperbolic PDEs

$$\begin{cases} \mu(x)k^j_x - \lambda_j(\xi)k^j_\xi = \lambda_j'(\xi)k^j - \sum_{i=1}^{n} \sigma_{i,j}(\xi)k^i - \omega_j(\xi)k^{n+1} \\ \mu(x)k^{n+1}_x + \mu(\xi)k^{n+1}_\xi = -\mu'(\xi)k^{n+1} - \sum_{i=1}^{n} \theta_i(\xi)k^i \end{cases} \tag{15}$$

with the following boundary conditions

$$\begin{cases} k^j(x,x) = \dfrac{\theta_j(x)}{\lambda_j(x) + \mu(x)}, \quad j = 1, ..., n \\ \mu(0)k^{n+1}(x,0) = \sum_{i=1}^{n} q_j \lambda_j(0) k^j(x,0) \end{cases} \tag{16}$$

Besides, the coefficients of matrix $\boldsymbol{\kappa}$ satisfy the following integral equations, for all $i = 1, .., n$

$$\kappa_i(x,\xi) = \int_\xi^x \kappa_i(x,s)k^{n+1}(s,\xi)ds + \omega_i(x)k^{n+1}(x,\xi) \tag{17}$$

and the coefficients of matrix C are given, for all $i, j = 1, .., n$, by

$$c_{i,j}(x,\xi) = \int_\xi^x \kappa_i(x,s)k^j(s,\xi)ds + \omega_i(x)k^j(x,\xi) \tag{18}$$

In the next section, we investigate the well-posedness of System (15) with boundary conditions (16).

V. Well-posedness of the kernel equations

In this section, we investigate the existence, uniqueness and continuity of the solution to system (15) with boundary conditions (16). After giving some preliminary results, we convert the system of hyperbolic PDEs into integral equations, using the method of characteristics. Then, we use the method of successive approximations to construct a solution to the integral equations in the form of a converging series.

A. Preliminary results

To convert hyperbolic PDEs into integral equations, one must define characteristic curves in the (t, x)–plane along which the equations are integrated. To do so, we use the two following lemmas[3].

Lemma 5.1: Let $(y_0, z_0) \in \mathbb{R}$ be such that $0 \le y_0 \le z_0 \le 1$ and $h \in C^1([0,1])$ be such that $\forall x \in [0,1] \quad h(x) < 0$. Then, if y and z are the maximal solutions of the following Cauchy problems

$$y'(s) = h(y(s)), \quad y(0) = y_0, \quad z'(s) = h(z(s)), \quad z(0) = z_0 \tag{19}$$

then, there exists $T > 0$ such that $y(T) = 0$ and $z(T) \ge 0$.

[3]Another way to define these is to give an implicit solution to the characteristics equations as is done in [13]

Lemma 5.2: Let $(y_0, z_0) \in \mathbb{R}$ be such that $0 \le y_0 \le z_0 \le 1$ and $h, g \in C^1([0,1])$ be such that $\forall x \in [0,1] \quad g(x) > 0$ and $h(x) < 0$. Then, if y and z are the maximal solutions of the following Cauchy problems

$$y'(s) = g(y(s)), \quad y(0) = y_0, \quad z'(s) = h(z(s)), \quad z(0) = z_0$$

then, there exists $T > 0$ such that $y(T) = z(T)$.
The interpretation and usefulness of these Lemmas will appear clearly in Section V-C. Their proofs are omitted for lack of space. In the next section, we state the main theorem regarding the existence of the kernel coefficients.

B. Existence of the kernel

For clarity purposes, we re-write the kernel equations (15),(16) using simpler notations. We show well-posedness of the following generic hyperbolic $(n+1)\times(n+1)$ system. For $i = 1, ..., n$, the system equations read

$$\mu(x)F^i_x - \lambda_i(\xi)F^i_\xi = a_i(x,\xi)G + \sum_{j=1}^{n} b_{i,j}(x,\xi)F^j \tag{20}$$

$$\mu(x)G_x + \mu(\xi)G_\xi = d(x,\xi)G + \sum_{i=1}^{n} e_i(x,\xi)F^i \tag{21}$$

evolving on the domain $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$, with boundary conditions

$$\forall i = 1, ..., n \quad \forall x \in [0,1] \quad F^i(x,x) = f_i(x) \tag{22}$$

$$\forall i = 1, ..., n \quad \forall x \in [0,1] \quad G(x,0) = \sum_{i=1}^{n} g_i(x)F^i(x,0) \tag{23}$$

The following Theorem discusses existence and uniqueness of the solutions to equations (20)–(23).

Theorem 5.3: Under the following assumptions

$$\forall i,j = 1,..,n \quad a_i, b_{i,j}, d, e_{i,j} \in \mathcal{L}^\infty(\mathcal{T}), \quad f_i, g_i \in \mathcal{L}^\infty([0,1])$$
$$\forall i = 1,..,n \quad \lambda_i, \mu \in C^0([0,1], \mathbb{R}_+^*)$$

system (20)–(23) admits a unique continuous solution on $\mathcal{T}$. The proof of Theorem 5.3 is contained in the next two sections. First, we transform system (20)–(23) into integral equations using the method of characteristics.

C. Transformation to integral equations

For equation (21), we define the characteristic curves (χ, ζ) along which the equations can be integrated as the solutions of the following Cauchy problems

$$\begin{cases} \dfrac{d}{ds}\chi(x,\xi;s) = \mu(\chi(x,\xi;s)), & s \in [0, s^F(x,\xi)], \\ \chi(x,\xi;s^F(x,\xi)) = x, & \chi(x,\xi;0) = \chi^0(x,\xi) \end{cases} \tag{24}$$

$$\begin{cases} \dfrac{d}{ds}\zeta(x,\xi;s) = \mu(\xi(x,\xi;s)), & s \in [0, s^F(x,\xi)], \\ \zeta(x,\xi;s^F(x,\xi)) = \xi, & \zeta(x,\xi;0) = 0 \end{cases} \tag{25}$$

For each $(x,\xi) \in \mathcal{T}$, the existence of $s^F(x,\xi)$ such that there exists such solutions is proved by applying Lemma 5.1 with

$$h(x) = -\mu(x), \qquad s^F(x,\xi) = T \tag{26}$$

$$\chi(x,\xi;s) = z(s^F(x,\xi) - s) \quad \zeta(x,\xi;s) = y(s^F(x,\xi) - s) \tag{27}$$

In other words, Lemmas 5.1 ensures that, when solving the characteristic equations (24),(25) backwards from a given point (x, ξ) in $\mathcal{T}$, one "hits" the boundary $\xi = 0$ of the triangular domain. Similarly, for each equation of system (20), we define the characteristics curves (x_i, ξ_i) as the solutions of the following Cauchy problems

$$\begin{cases} \dfrac{d}{ds} x_i(x, \xi; s) = \mu(x_i(x, \xi; s)), & s \in [0, s_i^F(x, \xi)], \\ x_i(x, \xi; s_i^F(x, \xi)) = x, & x_i(x, \xi; 0) = x_i^0(x, \xi) \end{cases} \quad (28)$$

$$\begin{cases} \dfrac{d}{ds} \xi_i(x, \xi; s) = -\lambda_i(\xi_i(x, \xi; s)), & s \in [0, s_i^F(x, \xi)], \\ \xi_i(x, \xi; s_i^F(x, \xi)) = \xi, & \zeta(x, \xi; 0) = x_i(x, \xi; 0) \end{cases} \quad (29)$$

Again, for each $(x, \xi) \in \mathcal{T}$ and each $i = 1, ..., n$, the existence of $s_i^F(x, \xi)$ such that there exists such solutions is proved by applying Lemma 5.2 with

$$h(x) = -\mu(x), \quad g(x) = \lambda_i(x), \quad s_i^F(x, \xi) = T, \quad (30)$$

$$x_i(x, \xi; s) = z(s_i^F(x, \xi) - s), \xi_i(x, \xi; s) = y(s_i^F(x, \xi) - s) \quad (31)$$

Again, Lemma 5.2 ensures that, when solving the characteristics equations backwards from a given point (x, ξ) in $\mathcal{T}$, one "hits" the boundary $x = \xi$ of the triangular domain. Integrating equations (20) along there respective characteristic lines defined by (28),(29), between 0 and $s_i^F(x, \xi)$ and using the boundary conditions (22) yields, for all $i = 1, ..., n$

$$\begin{aligned} F^i(x, \xi) = f_i(x_i^0(x, \xi)) + \int_0^{s_i^F(x,\xi)} & \Big[a_i(x_i(x, \xi; s), \xi_i(x, \xi; s)) \\ \times G(x_i(x, \xi; s), \xi_i(x, \xi; s)) + \sum_{j=1}^{n} & b_{i,j}(x_i(x, \xi; s), \xi_i(x, \xi; s)) \\ & \times F^j(x_i(x, \xi; s), \xi_i(x, \xi; s))\Big] ds \end{aligned} \quad (32)$$

Similarly, integrating equations (21) along the characteristic lines defined by (24)-(25) between 0 and $s^F(x, \xi)$, using boundary conditions (22) and the expression of F^i given by (32) yields, for all $(x, \xi) \in \mathcal{T}$ and $i = 1, ..., n$

$$\begin{aligned} G(x, \xi) = \sum_{i=1}^{n} & g_i(\chi^0(x, \xi)) \Big[f_i(x_i^0(\chi^0(x, \xi), 0)) \\ & + \int_0^{s_i^F(\chi^0(x,\xi),0)} \Big[a_i(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s)) \\ & \quad \times G(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s)) \\ & \quad + \sum_{j=1}^{n} b_{i,j}(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s)) \\ & \quad \times F^j(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s))\Big] ds\Big] \\ & + \int_0^{s^F(x,\xi)} [d(\chi(x, \xi; s), \zeta(x, \xi; s)) G(\chi(x, \xi; s), \zeta(x, \xi; s)) \\ & + \sum_{i=1}^{n} e_i(\chi(x, \xi; s), \zeta(x, \xi; s)) F^i(\chi(x, \xi; s), z_i(x, \xi; s)) \end{aligned} \quad (33)$$

In the next section, we solve equations (32), for $i = 1, ..., n$ and (33) using the method of successive approximations.

D. Solution of the integral equations via a successive approximation series

The successive approximation method can be used to solve the integral equations. Define first, for $i = 1, ..., n$

$$\varphi_i(x, \xi) = f_i(x_i^0(x, \xi))$$

$$\psi(x, \xi) = \sum_{i=1}^{n} g_i(\chi^0(x, \xi)) f_i(x_i^0(\chi^0(x, \xi), 0))$$

Besides, denoting

$$\mathbf{H} = \begin{bmatrix} F^1 & \cdots & F^n & G \end{bmatrix}^T$$

$$\boldsymbol{\phi}(x, \xi) = \begin{bmatrix} \varphi_1(x, \xi) & \cdots & \varphi_n(x, \xi) & \psi \end{bmatrix}^T$$

we define the following functionals acting on $\mathbf{H}$

$$\begin{aligned} \Phi_i[\mathbf{H}](x, \xi) = & \\ \int_0^{s_i^F(x,\xi)} & \Big[a_i(x_i(x, \xi; s), \xi_i(x, \xi; s)) G(x_i(x, \xi; s), \xi_i(x, \xi; s)) \\ & + \sum_{j=1}^{n} b_{i,j}(x_i(x, \xi; s), \xi_i(x, \xi; s)) F^j(x_i(x, \xi; s), \xi_i(x, \xi; s))\Big] ds \end{aligned} \quad (34)$$

$$\begin{aligned} \Psi[\mathbf{H}](x, \xi) = & \sum_{i=1}^{n} g_i(\chi^0(x, \xi)) \\ & \times \int_0^{s_i^F(\chi^0(x,\xi),0)} \Big[a_i(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s)) \\ & \quad \times G(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s)) \\ & \quad + \sum_{j=1}^{n} b_{i,j}(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s)) \\ & \quad \times F^j(x_i(\chi^0(x, \xi), 0; s), \xi_i(\chi^0(x, \xi), 0; s))\Big] ds \\ & + \int_0^{s^F(x,\xi)} [d(\chi(x, \xi; s), \zeta(x, \xi; s)) G(\chi(x, \xi; s), \zeta(x, \xi; s)) \\ & + \sum_{i=1}^{n} e_i(\chi(x, \xi; s), \zeta(x, \xi; s)) F^i(\chi(x, \xi; s), z_i(x, \xi; s))\Big] ds \end{aligned} \quad (35)$$

Define then the following sequence

$$\begin{aligned} \mathbf{H}^0(x, \xi) &= 0, \\ \mathbf{H}^m(x, \xi) &= \boldsymbol{\phi}(x, \xi) + \mathbf{\Phi}[\mathbf{H}^{m-1}](x, \xi) \\ &= \begin{bmatrix} \phi_1(x, \xi) + \Phi_1[\mathbf{H}^{m-1}](x, \xi) \\ \vdots \\ \phi_n(x, \xi) + \Phi_n[\mathbf{H}^{m-1}](x, \xi) \\ \psi(x, \xi) + \Psi[\mathbf{H}^{m-1}](x, \xi) \end{bmatrix} = \begin{bmatrix} H_1^m(x, \xi) \\ \vdots \\ H_n^m(x, \xi) \\ H_{n+1}^m(x, \xi) \end{bmatrix} \end{aligned} \quad (36)$$

Finally, define for $n \geq 1$ the increment $\Delta\mathbf{H}^m = \mathbf{H}^m - \mathbf{H}^{m-1}$, with $\Delta\mathbf{H}^0 = \boldsymbol{\phi}$ by definition. Since the functional $\mathbf{\Phi}$ is linear, the following equation $\Delta\mathbf{H}^m(x, \xi) = \mathbf{\Phi}[\mathbf{H}^{m-1}](x, \xi)$ holds.

If the limit exists, then $\mathbf{H} = \lim_{m \to +\infty} \mathbf{H}^m(x, \xi)$ is a solution of the integral equations, and thus solves the original hyperbolic system. Using the definition of $\Delta\mathbf{H}^m$, it follows that if the sum $\sum_{m=0}^{+\infty}\Delta\mathbf{H}^m(x,\xi)$ is finite, then

$$\mathbf{H}(x,\xi)=\sum_{m=0}^{+\infty}\Delta\mathbf{H}^m(x,\xi) \tag{37}$$

We now prove convergence of the series. First, define

$$\bar{f}=\max\left\{1,\max_{(x,\xi)\in\mathcal{T},\,i,j=1,\ldots,n}|f_{i,j}(x,\xi)|\right\},\quad \text{for } f=a,b,d,e,g \tag{38}$$

$$\bar{\phi}=\max_{(x,\xi)\in\mathcal{T},\,i=1,\ldots,n}\{\phi(x,\xi),\psi(x,\xi)\} \tag{39}$$

$$M=n\bar{g}\left(\bar{a}+n\bar{b}\right)+\bar{d}+n\bar{e} \tag{40}$$

Lemma 5.4: For $i=1,\ldots,n$, $p\geq 1$, $(x,\xi)\in\mathcal{T}$, and $s_i^F(x,\xi)$, $\zeta_i^F(x,\xi)$, $x_i(x,\xi,\cdot)$, $\chi_i(c,\xi,\cdot)$ defined as in (24),(25),(28),(29), the following inequalities holds

$$\int_0^{s_i^F(x,\xi)}x_i^m(x,\xi;s)ds\leq M_\lambda\frac{x^{m+1}}{m+1} \tag{41}$$

$$\int_0^{\zeta_i^F(x,\xi)}\chi^m(x,\xi;s)ds\leq M_\lambda\frac{x^{m+1}}{m+1} \tag{42}$$

Proof We first prove (41). Consider the following change of integration variable $\varsigma=x_i(x,\xi;s)$. Then,

$$d\varsigma=\frac{d}{ds}x_i(x,\xi;s)ds=\mu(x_i(x,\xi;s))ds \tag{43}$$

Thus, the left-hand-side of (41) rewrites

$$\int_0^{s_i^F(x,\xi)}x_i^m(x,\xi;s)ds=\int_{x_i^0(x,\xi)}^x\frac{\varsigma^m}{\lambda(\varsigma)}d\varsigma\leq M_\lambda\int_0^x\varsigma^m d\varsigma = M_\lambda\frac{x^{m+1}}{m+1}$$

Inequality (42) is proved the same way using change of integration variable $\varsigma=\chi(x,\xi;s)$. ∎

Lemma 5.5: For $m\geq 1$, assume that, for all $(x,\xi)\in\mathcal{T}$, and $i=1,\ldots,n$

$$\left|\Delta F^i(x,\xi)\right|\leq\bar{\phi}\frac{M^m x^m}{m!}\quad\text{and}\quad|\Delta G(x,\xi)|\leq\bar{\phi}\frac{M^m x^m}{m!} \tag{44}$$

then, it follows that for all $(x,\xi)\in\mathcal{T}$, and $i=1,\ldots,n$

$$|\Phi_i[\Delta\mathbf{H}](x,\xi)|\leq\bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!},\quad|\Psi[\Delta\mathbf{H}](x,\xi)|\leq\bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!} \tag{45}$$

Proof Assume that (44) holds. Then, for all $i=1,..,n$ and $(x,\xi)\in\mathcal{T}$ one has, using the expression of Φ_i given by (34) and the inequality (44)

$$|\Phi_i[\Delta\mathbf{H}](x,\xi)|\leq\bar{a}\int_0^{s_i^F(x,\xi)}\bar{\phi}M^m\frac{x_i(x,\xi;s)^m}{m!}ds+\sum_{j=1}^n\bar{b}\int_0^{s_i^F(x,\xi)}\bar{\phi}M^m\frac{x_i(x,\xi;s)^m}{m!}ds$$

Using Lemma 5.4, and the fact that $x_i(x,\xi;s)\leq x$, this yields

$$|\Phi_i[\Delta\mathbf{H}](x,\xi)|\leq\bar{\phi}\left[\bar{a}+n\bar{b}\right]M^m\frac{x^{m+1}}{(m+1)!}\leq\bar{\phi}M^{m+1}\frac{x^{m+1}}{(m+1)!}$$

using the definition of M given by (40). Similarly, using the expression of Ψ given by (35), the inequality (44) and Lemma 5.4, one has

$$|\Psi[\Delta\mathbf{H}](x,\xi)|\leq\bar{\phi}M^{m+1}\frac{x^{m+1}}{(m+1)!}$$

which concludes the proof.

Finally, we prove that (37) converges.

Proposition 5.6: Consider the sequence $\mathbf{H}^m$, $m\geq 0$ defined by (36). For $i=1,..,2n$, one has

$$\forall(x,\xi)\in\mathcal{T}\quad\left|\sum_{m=0}^{+\infty}\Delta H_i^m(x,\xi)\right|\leq\bar{\phi}e^{Mx}$$

Proof The result follows if we show that for all $m\geq 0$, one has

$$\forall i=1,\ldots,2n\quad\left|\Delta H_i^m(x,\xi)\right|\leq\bar{\phi}\frac{M^m x^m}{m!} \tag{46}$$

We prove this result by induction. For $m=0$, it follows directly from the fact that $\Delta\mathbf{H}^0=\phi$ and the definition of $\bar{\phi}$ given by (39). Assume that (46) holds for $m\geq 1$. Then, for $i=1,...n$, one has

$$\begin{aligned}|\Delta H_i^{m+1}(x,\xi)|&=|\Phi_i[\Delta\mathbf{H}^m](x,\xi)|\quad\text{by definition of }\Delta H^{m+1}\\&\leq\bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!}\qquad\text{using Lemma 5.5.}\end{aligned}$$

Similarly, using the definition of ΔH_{n+1}^{m+1} and Lemma 5.5, one has

$$|\Delta H_{n+1}^{m+1}(x,\xi)|\leq\bar{\phi}\frac{M^{m+1}x^{m+1}}{(m+1)!}$$

which concludes the proof. ∎

The proof of uniqueness and continuity of the solutions is identical to the one in [13]. For this reason and brevity purposes, we will not detail it here. We now assess the invertibility of transformation (13) and the existence of the coefficients κ_i, $c_{i,j}$, $i,j=1,\ldots,n$.

E. Inverse transformation and target system coefficients

Since $\boldsymbol{\alpha}\equiv\mathbf{u}$, transformation (13) rewrites

$$v(t,x)-\int_0^x k^{n+1}(x,\xi)v(t,\xi)d\xi=\Gamma(t,x) \tag{47}$$

with $\Gamma(t,x)=\beta(t,x)+\sum_{i=1}^n\int_0^x k^i(x,\xi)\alpha_i(t,\xi)d\xi$. Since k^{n+1} is continuous, there exists a unique continuous inverse kernel l^{n+1} defined on $\mathcal{T}$ and such that (see, e.g., [12])

$$v(t,x)=\Gamma(t,x)+\int_0^x l^{n+1}(x,\xi)\Gamma(t,\xi)d\xi \tag{48}$$

which yields the following inverse transformation

$$v(t,x) = \beta(t,x) + \int_0^x l^{n+1}(x,\xi)\beta(t,\xi)d\xi + \sum_{i=1}^n \left(k^i(x,\xi) + \int_\xi^x k^i(x,\xi) l^{n+1}(\xi,s)ds\right)\alpha(t,\xi)d\xi \quad (49)$$

Besides, the continuity (and thus, the boundedness) of K also implies the existence and continuity of the solutions to the Volterra equations of the second kind (17) (see, e.g., [10, Theorem 3.1, p.30]). Therefore, the functions κ_i and $c_{i,j}$ (defined by (18)), for $i,j = 1,...,n$ are continuous on $\mathcal{T}$. In the next section, we summarize the control design in Theorem 6.1.

VI. Control law and main result

We now state the main result of the paper.

Theorem 6.1: Consider system (1),(2) with boundary conditions (3),(4), initial conditions $\mathbf{u}^0$, v^0 and the following control law

$$U(t) = \int_0^1 \left[\sum_{i=1}^n k^i(x,\xi)u_i(t,\xi) + k^{n+1}(x,\xi)v(t,\xi)\right]d\xi \quad (50)$$

where, for $i = 1,...,n+1$, the k^i satisfy System (15) with boundary conditions (16). Then, under the assumption that for all $i,j = 1,...,n$

$$\lambda_i, \mu \in C^1([0,1]),\ \sigma_{i,j}, \omega_i, \theta_i \in \mathcal{L}^\infty([0,1]),\ \mathbf{u}^0, v^0 \in \mathcal{L}^2([0,1])$$

the equilibrium $\mathbf{w} \equiv 0$ is exponentially stable in the $\mathcal{L}^2$ sense

Proof The existence of the kernel coefficients verifying (15) with boundary conditions (16) is proved by applying Theorem 5.3 with, for all $i,j = 1,...,n$

$$F^i(x,\xi) = k^i(x,\xi), \qquad G(x,\xi) = k^{n+1}(x,\xi)$$

$$a_i(x,\xi) = -\omega_i(\xi), \qquad b_{i,j}(x,\xi) = \begin{cases} \lambda_i'(\xi) & \text{if } i=j \\ -\sigma_{j,i}(\xi) & \text{otherwise} \end{cases}$$

$$d(x,\xi) = -\mu'(\xi), \qquad e_i(x,\xi) = -\theta_i(\xi),$$

$$f_i(x) = -\frac{\theta_i(x)}{\lambda_i(x)+\mu(x)}, \qquad g_i(x) = \frac{q_i\lambda_i(0)}{\mu(0)}$$

The existence of the direct and inverse transformations (respectively given by (13) and (49)) guarantees that the exponential stability of the target system (6),(7) with boundary conditions (8), investigated in Lemma 3.1, is equivalent to that of the original system (1),(2) with boundary conditions (3),(4), which concludes the proof. ∎

VII. Conclusion and future work

We have presented a control design for a class of linear first-order hyperbolic systems, which guarantees exponential stability of the zero equilibrium. The control gains may be computed, indifferently, by solving the hyperbolic system of equations (15) with boundary conditions (16), or by truncating the infinite sum (37).

The resulting control law is a full-state static feedback which requires measurements or estimates of all the states over the entire spatial domain. Thus, the design of a boundary observer generalizing the one presented in [13] for the special case $n = 1$ is an important focusing point for current and future investigations. The main difficulty of such a design is the necessity to cancel coupling terms between homo-directional states (i.e. states traveling in the same direction), which may result in ill-posed equations for the corresponding backstepping kernels.

Another direction for future investigations is to further generalize the control and observer designs to a broader class of systems, by considering arbitrary numbers of states in either direction, i.e. $(n+m)$-state systems with n positive transport speeds and m negative ones. Again, the necessary cancellation of coupling terms between homo-directional states severely complicates the control design.

References

[1] Saurabh Amin, Falk Hante, and Alexandre Bayen. *On stability of switched linear hyperbolic conservation laws with reflecting boundaries*, pages 602–605. Springer-Verlag, hybrid systems: comp edition, 2008.

[2] G. Bastin and J.-M. Coron. Further results on boundary feedback stabilization of 2x2 hyperbolic systems over a bounded interval. *Proceedings of IFAC Nolcos 2010, Bologna, Italy*, 2010.

[3] K. Bendiksen, D. Maines, R. Moe, and S. Nuland. The dynamic two-fluid model OLGA: Theory and application. *SPE production engineering*, 6(2):171–180, 1991.

[4] J.-M. Coron. *Control and Nonlinearity*. American Mathematical Society, 2007.

[5] J.-M. Coron, B. d'Andréa Novel, and G. Bastin. A lyapunov approach to control irrigation canals modeled by saint-venant equations. *Proceedings of the 1999 European Control Conference, Karlsruhe, Germany*, 1999.

[6] F. Di Meglio. *Dynamics and control of slugging in oil production*. PhD thesis, MINES ParisTech, 2011.

[7] F. Di Meglio, G.-O. Kaasa, N. Petit, and V. Alstad. Slugging in multiphase flow as a mixed initial-boundary value problem for a quasilinear hyperbolic system. *2011 American Control Conference (to appear)*, 2011.

[8] F. Di Meglio, R. Vazquez, M. Krstic, and N. Petit. Backstepping stabilization of an underactuated 3 × 3 linear hyperbolic system of fluid flow transport equations. *2012 American Control Conference, Montreal, Canada (to appear)*, 2012.

[9] M. Krstic and A. Smyshlyaev. Backstepping boundary control for first-order hyperbolic pdes and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9):750 – 758, 2008.

[10] Peter Linz. *Analytical and Numerical Methods for Volterra Equations*. Society for Industrial and Applied Mathematics, Philadephia, PA, 1985.

[11] E. Storkaas. *Control solutions to avoid slug flow in pipeline-riser systems*. PhD thesis, Norwegian University of Science and Technology, 2005.

[12] R. Vazquez. *Boundary Control Laws and Observer Design for Convective, Turbulent and Magnetohydrodynamic Flows*. PhD thesis, University of California, San Diego, 2006.

[13] R. Vazquez, M. Krstic, and J.-M. Coron. Backstepping boundary stabilization and state estimation of a 2x2 linear hyperbolic system. *in Proceedings of the 50th Conference on Decision and Control, Orlando (to appear)*, 2011.