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Explicit Control Laws for the Periodic Motion Planning of Controllable Driftless Systems on SU(n)

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Abstract— **In a previous work, the authors have introduced and treated the periodic motion planning problem for right-invariant driftless systems that evolve on SU(n), which in turn corresponds to an approximate motion planning control problem. Although the only assumption was the controllability of the system, the control laws are not explicitly known in general. By using analogous Lyapunov-like control techniques that were developed in that work, this paper complements its results and proves that if the system satisfies a more restrictive form of controllability, then the control laws are given by explicit and analytical expressions. Simulation results are presented for $n = 4$ and the generation of the C-NOT (Controlled-NOT) quantum logic gate.**

I. INTRODUCTION

In a previous work ([3]), the authors have introduced and treated the periodic motion planning problem for right-invariant driftless systems the form

$$\dot{X}(t) = \sum_{k=1}^{m} u_k(t) H_k X(t), \quad X(0) = I, \tag{1}$$

where $X \in \mathrm{SU}(n)$ is the state, $H_k \in \mathfrak{su}(n)$, $\mathfrak{su}(n)$ is the Lie algebra associated to the special unitary group SU(n), $u_k \in \mathbb{R}$ are the controls, and I is the identity matrix of SU(n). The periodic motion planning problem for system (1) is formulated as: given a *goal state* $X_\infty \in \mathrm{SU}(n)$ and $T > 0$, find a continuously differentiable *reference trajectory* X_r: $\mathbb{R}_+ \to \mathrm{SU}(n)$ with period T and $X_r(0) = X_\infty$, and determine piecewise-continuous control laws u_k: $\mathbb{R}_+ \to \mathbb{R}$, $1 \leq k \leq m$, so that

$$\lim_{t\to\infty} X^\dagger(t) X_r(t) = I. \tag{2}$$

($X^\dagger(t)$ is the conjugate transpose of $X(t)$). In particular, $\lim_{\ell\to\infty} X(\ell T) = X_\infty$ ($\ell \in \mathbb{N}$).

It was proved in [3] that when system (1) is regular (in the sense of [3, Definition 1]), the periodic motion planning problem always has a solution. Based on Coron's Return Method ([1]), one has that the regularity of (1) is in fact equivalent to its controllability on SU(n) (i.e. Lie$(H) = \mathfrak{su}(n)$, where Lie(H) is the Lie algebra generated by the set $H = \{H_1, \ldots, H_m\} \subset \mathfrak{su}(n)$). Loosely speaking, by specifying an adequate periodic trajectory $(X_r(t), u_1^r(t), \ldots, u_m^r(t), t \in \mathbb{R}_+)$ of (1) with period T and initial condition $X_r(0) = X_\infty$, and developing Lyapunov-like convergence results, the authors formulated in [3] an algorithm that determined, in a finite number of steps, piecewise-continuous open-loop controls $u_k(t)$ that assure the validity of (2) in case (1) is regular (see [3, Algorithm 1 and Theorem 2]). The controls u_k: $\mathbb{R}_+ \to \mathbb{R}$ are obtained by numerical integration of (1) and they depend on the period T and on the desired goal state X_∞. Although the existence of such X_r and u_k^r is established by Coron's Return Method, they are in general not completely known. This paper overcomes such drawback by providing explicit and analytical expressions for X_r and u_k^r when the control system (1) is p-controllable (in the sense of Definition 1 in Section II), which is a more restrictive form of controllability on SU(n). The desired controls u_k are then given by explicit and analytical expressions. This is the main result of this paper (see Theorem 1 and Remark 1). The key ingredient is that the "reference controls" u_k^r perform a periodic switching among the H_k in (1) so as to generate $\mathfrak{su}(n)$ according to the p-controllable property.

The paper is organized as follows. By assuming that (1) is p-controllable and paralleling the arguments and results required in the proof of [3, Theorem 1] for the regular case, Section II establishes a solution to the periodic motion planning control problem with explicit and analytical expressions for the controls laws u_k. The mathematical proof of the solution presented is given in Section III. In Section IV, the control strategy is applied to the same quantum system ($n = 4$) that was considered in [3]. Once again, the aim is the generation of the C-NOT (Controlled-NOT) quantum logic gate. Two lemmas that are used in Section III as well as the proof of Proposition 1 are presented in Appendix.

II. MAIN RESULT

One denotes by M^n the real Banach space of n-square matrices with complex entries endowed with the Euclidean norm $\|\cdot\|$, and by tr(X) the trace of $X \in M^n$. If $X \in M^n$ and $\Omega \subset M^n$ is nonempty, $d(X, \Omega) \triangleq \inf_{Y\in\Omega} \|Y - X\|$. As usual, one defines $\mathrm{ad}_A^0 B = B$ and $\mathrm{ad}_A^{j+1} B = [A, \mathrm{ad}_A^j B]$, for $A, B \in M^n$ and $j \in \mathbb{N}$ ($\mathbb{N}$ includes zero). Moreover, $\Re(z)$ is the real part of $z \in \mathbb{C}$, $\Im(z)$ is its imaginary part, $\iota \in \mathbb{C}$ is the imaginary unit, and $[p] = \{1, \ldots, p\}$, where $p \in \mathbb{N}$ with $p \geq 1$. Let $H = \{H_1, \ldots, H_m\} \subset \mathfrak{su}(n)$, where H_k, $k \in [m]$, are as in (1). For simplicity, it will be assumed throughout

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this paper that $T > 0$ and $X_\infty \in \mathrm{SU}(n)$ (goal state) are fixed. One recalls a well-known result: every $W \in \mathrm{SU}(n)$ can be decomposed as $W = M\mathrm{diag}(\lambda_1, \dots, \lambda_n)M^\dagger$, where $M \in \mathrm{SU}(n)$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $|\lambda_i| = 1$ and $\prod_{i=1}^n \lambda_i = 1$.

In order to define the reference trajectory $X_r(t)$, one introduces:

Definition 1: The control system (1) is *p-controllable* when there exist $\overline{H}_1, \dots \overline{H}_p \in \mathrm{span}(H)$ such that

$$\mathfrak{su}(n) = \mathrm{span}\{\mathrm{ad}^j_{\overline{H}_\ell} H_k, \text{for all } j \in \mathbb{N}, k \in [m], \ell \in [p]\}. \tag{3}$$

Note that if (1) is p-controllable, then $\mathrm{Lie}(H) = \mathfrak{su}(n)$, that is, (1) is controllable on $\mathrm{SU}(n)$. In synthesis,

$$p\text{-controllable} \Rightarrow \mathrm{Lie}(H) = \mathfrak{su}(n) \Leftrightarrow \text{regular}.$$

The equivalence above follows from the results of Coron's Return Method in [1]. The next proposition provides algebraic conditions for the validity of $\mathrm{Lie}(H) = \mathfrak{su}(n) \Rightarrow$ p-controllable. Its proof is given in Appendix.

Proposition 1: Define $\mathbf{H} = \mathrm{span}(H)$. Suppose that there exists a base G of $\mathrm{Lie}(H)$ such that $G \subset \mathrm{span}\{[X_1, [X_2, X_3]], ad^j_Y X_4, \text{ for all } X_1, \dots, X_4 \in H, Y \in \mathbf{H}, j \in \mathbb{N}\}$. Then, there exist $\overline{H}_1, \dots, \overline{H}_p \in \mathbf{H}$ such that $\mathrm{Lie}(H) = \mathrm{span}\{\mathrm{ad}^j_{\overline{H}_\ell} H_k, \text{ for all } j \in \mathbb{N}, k \in [m], \ell \in [p]\}$. In particular, if in addition one has that $\mathrm{Lie}(H) = \mathfrak{su}(n)$, then (1) is p-controllable.

It is clear that $(X_r(t), u_1^r(t), \dots, u_m^r(t), t \in \mathbb{R})$ is a (continuous) trajectory of (1) with initial condition $X_r(0) = X_\infty \in \mathrm{SU}(n)$ if and only if it is a trajectory of the *reference system*

$$\dot{X}_r(t) = \sum_{k=1}^m u_k^r(t) H_k X_r(t), \quad X_r(0) = X_\infty. \tag{4}$$

Now, assume that (1) is p-controllable. Then, there exist $\overline{H}_1, \dots \overline{H}_p \in \mathrm{span}(H)$ such that (3) holds with $\overline{H}_\ell = \sum_{k=1}^m c_{\ell k} H_k$, where $c_{\ell k} \in \mathbb{R}$. Let

$$T_p = T/p. \tag{5}$$

The functions $u_k^r\colon \mathbb{R} \to \mathbb{R}$ in (4), $k \in [m]$, are chosen as

$$u_k^r(t) = \sin(2\pi t/T_p) c_{jk}, \;\; \ell T + (j-1)T_p \le t < \ell T + jT_p, \tag{6}$$

for all $j \in [p], \ell \in \mathbb{Z}$. Hence (4) becomes

$$\dot{X}_r(t) = F_r(t, X_r(t)) \triangleq \sin(2\pi t/T_p) \overline{H}_r(t) X_r(t), \tag{7}$$

where $\overline{H}_r\colon \mathbb{R} \to \mathfrak{su}(n)$ is given by

$$\overline{H}_r(t) = \begin{cases} \overline{H}_1, & \ell T \le t < \ell T + T_p, \\ \overline{H}_2, & \ell T + T_p \le t < \ell T + 2T_p, \\ \vdots & \\ \overline{H}_p, & \ell T + (p-1)T_p \le t < (\ell+1)T, \end{cases} \tag{8}$$

for $\ell \in \mathbb{Z}$. It is clear that the (continuously differentiable) solution $X_r\colon \mathbb{R} \to \mathrm{SU}(n)$ of (7)–(8) with $X_r(0) = X_\infty$ is

$$X_r(t) = \exp((T_p/2\pi)[1 - \cos(2\pi t/T_p)]\overline{H}_j) X_\infty, \quad \text{for } \ell T + (j-1)T_p \le t < \ell T + jT_p, \tag{9}$$

for all $j \in [p], \ell \in \mathbb{Z}$. Thus $X_r(t)$ has period T and

$$X_r(\ell T_p) = X_\infty, \quad \text{for all } \ell \in \mathbb{Z}. \tag{10}$$

It is worth emphasizing that X_r and u_k^r are given in an explicit and analytical manner. Note that $\overline{H}_r$ in (8) is bounded and has period T. Furthermore, although $\overline{H}_r$ is piecewise-constant (on every compact interval in $\mathbb{R}$) with the discontinuities precisely at $t = \ell T_p$, for all $\ell \in \mathbb{Z}$, the function $\sin(2\pi t/T_p)$ assures that $F_r\colon \mathbb{R} \times M^n \to M^n$ in (7) is continuous. In the same way, u_k^r in (6) is continuous with period T. Note that: (i) u_k^r in (6) corresponds to an amplitude modulation of the rectangular pulses determined by the c_{jk}, and (ii) the definition of $\overline{H}_r$ in (8) corresponds to a periodic switching with period T among the elements $\overline{H}_1, \dots, \overline{H}_p \in \mathrm{span}(H)$ that satisfy (3). The switching instants are at $t = \ell T_p$, for $\ell \in \mathbb{Z}$. This type of behavior plays a major role in the proof of Theorem 2 in Section III.

Definition 2: Assume that (1) is p-controllable. Then the reference trajectory $X_r(t)$ is chosen as in (9) and the "reference controls" $u_k^r(t)$ are chosen as in (6).

Suppose that (1) is p-controllable. Following [3], it is straightforward to verify from (1) and (4) that the time-dependent change of coordinates

$$W = W(t, X) = X^\dagger X_r(t), \quad \text{for all } (t, X) \in \mathbb{R} \times M^n,$$

along with the time-varying control shift

$$v_k(t) = u_k^r(t) - u_k(t), \quad \text{for all } t \in \mathbb{R}, \; k \in [m],$$

determine the left-invariant system

$$\dot{W}(t) = W(t) X_r^\dagger(t) \sum_{k=1}^m v_k(t) H_k X_r(t), \tag{11}$$

$(t, W) \in \mathbb{R} \times \mathrm{SU}(n)$. If one finds piecewise-continuous functions $v_k\colon \mathbb{R}_+ \to \mathbb{R}$ in (11) such that

$$\lim_{t\to\infty} W(t) = \lim_{t\to\infty} X^\dagger(t) X_r(t) = I, \tag{12}$$

where $W\colon \mathbb{R}_+ \to \mathrm{SU}(n)$ is the solution of system (11) with initial condition $W(0) = X^\dagger(0) X_r(0) = X_\infty$, then the periodic motion planning control problem will be solved with controls

$$u_k(t) = u_k^r(t) - v_k(t), \quad \text{for all } t \in \mathbb{R}_+, \; k \in [m]. \tag{13}$$

As in [3], one considers the continuous linear function $V\colon M^n \to \mathbb{R}$ defined by

$$V(W) = \Re(tr(W)), \quad \text{for all } W \in M^n, \tag{14}$$

and one chooses

$$v_k(t) = v_k(t, W(t)) \triangleq f_k^2 V(W(t) X_r^\dagger(t) H_k X_r(t)) \in \mathbb{R}, \tag{15}$$

where $f_k \in \mathbb{R}$ are nonzero constants. Hence,

$$\dot{W}(t) = W(t) X_r^\dagger(t) \sum_{k=1}^m f_k a_k(t, W(t)) H_k X_r(t), \tag{16}$$

$(t, W) \in \mathbb{R} \times \mathrm{SU}(n)$, where $f_k \in \mathbb{R}$ are nonzero and

$$a_k(t, W) \triangleq v_k(t, W)/f_k = f_k V(W X_r^\dagger(t) H_k X_r(t)) \in \mathbb{R}, \tag{17}$$

for $(t, W) \in \mathbb{R} \times M^n$. By construction,

$$\dot{V}(t, W) = \sum_{k=1}^{m} a_k(t, W)^2 \geq 0, \quad (t, W) \in \mathbb{R} \times M^n. \tag{18}$$

Moreover, given $W \in \mathrm{SU}(n)$, one has $|V(W)| \leq n$, $V(W) = n \Leftrightarrow W = I$. Therefore, V qualifies as a Lyapunov-like function candidate. Loosely speaking, in order to solve the control problem, the idea is to show that $\lim_{t\to\infty} V(W(t)) = I$, and then that $\lim_{t\to\infty} W(t) = I$.

Proposition 2: Let $G = \{x \in \mathbb{R} : x = \sum_{i=1}^{n} \Re(\lambda_i)$, for some $\lambda_i \in \mathbb{C}$ such that $|\lambda_i| = 1$, $\prod_{i=1}^{n} \lambda_i = 1$, $\Im(\lambda_1) = \cdots = \Im(\lambda_n)\}$. Then: (i) G is a finite set, (ii) $n \in G$, and (iii) $n = \max(G)$. In particular, $\delta = \max(G \setminus \{n\})$ is well-defined (if $G = \{n\}$, one defines $\delta = -\infty$).

Proof: See the first part of the proof of [3, Theorem 1]. ■

The main result of this paper is presented below. Its proof is given in the next section.

Theorem 1: Assume that (1) is p-controllable and let δ be as in Proposition 2. Choose any nonzero $f_1, \ldots, f_m \in \mathbb{R}$. Then,

$$V(X_\infty) > \delta \Rightarrow \lim_{t\to\infty} W(t) = \lim_{t\to\infty} X^\dagger(t) X_r(t) = I,$$

where $W(t)$ is the solution of (16)–(17) with initial condition $W(0) = X_\infty$. Furthermore, $\lim_{t\to\infty} a_k(t, W(t)) = 0$, for $k \in [m]$. In other words, the periodic motion planning problem of (1) is locally solved on an open neighborhood of the identity matrix I by the continuous controls laws

$$u_k(t) = u_k^r(t) - f_k^2 V(X^\dagger(t) H_k X_r(t)),$$

$t \in \mathbb{R}_+$, which are obtained by numerical integration of (1). Moreover,

$$\lim_{t\to\infty} (u_k(t) - u_k^r(t)) = 0, \quad k \in [m]. \tag{19}$$

Proof: Follows from (13), (17) and Theorem 3 with $q = (0, X_\infty) \in \mathbb{R} \times \mathrm{SU}(n)$. ■

Remark 1: The result above solves the periodic motion planning control problem in case $V(X_\infty) > \delta$. Nonetheless, since [3, Algorithm 1] for the regular case can still be applied in the p-controllable case here considered, one has that the control problem is globally solved, i.e. for any desired goal state $X_\infty \in \mathrm{SU}(n)$. More precisely, if $V(X_\infty) \leq \delta$, then the *off-line* execution of [3, Algorithm 1] determines, in a finite number of steps, a piecewise-constant map γ: $\mathbb{R}_+ \to \mathrm{SU}(n)$ such that the piecewise-continuous controls $u_k(t) = u_k^r(t) - f_k^2 \mathrm{V}(\gamma^\dagger(t) X^\dagger(t) H_k X_r(t))$ assure that (2) and (19) are met. The map γ assumes only a finite number of values and it depends on T and X_∞.

III. PROOF OF THE MAIN RESULT

For simplicity, one fixes $q = (t_0, W_{t_0}) \in \mathbb{R} \times \mathrm{SU}(n)$ throughout the rest of the paper, and W_q: $\mathbb{R} \to \mathrm{SU}(n)$ will denote the solution of (16)–(17) with initial condition $W_q(t_0) = W_{t_0}$.

The nontrivial results in Theorem 2 and Lemma 2 presented in the sequel play a role analogous to the Lyapunov-like convergence results [3, Theorem 3] and [3, Lemma 3], respectively, for the regular case. Although (3) is not needed in the proof of Theorem 2, (8) and (10) are fundamental. Of crucial importance is also the more general version of Barbalat's Lemma given below. It is required because $\overline{H}_r(t)$ in (8) is piecewise-constant.

Lemma 1 (Barbalat's Lemma): Let α: $[0, \infty) \to \mathbb{R}$ be a function that satisfies the following hypothesis: (i) α is continuously differentiable; (ii) $\lim_{t\to\infty} \alpha(t)$ exists; and (iii) there exists a bounded function β: $[0, \infty) \to \mathbb{R}$ such that, for any $b > 0$, there exists a finite set $N_b \subset [0, b]$ in a manner that the restriction $\beta|[0, b]$ is piecewise-continuous and $\ddot{\alpha}(t) = \beta(t)$, for each $t \in [0, b] \setminus N_b$. Then, $\lim_{t\to\infty} \dot{\alpha}(t) = 0$. (For simplicity, one writes $\ddot{\alpha} = \beta$ whenever applying this lemma).

Proof: Assumptions (i) and (iii) along with the Mean Value Theorem (see e.g. [2, p. 43]) imply that $\dot{\alpha}$ is uniformly continuous. The result then follows from the usual version of Barbalat's Lemma (see e.g. [4, Lemma 4.2]). ■

Theorem 2: Suppose that (1) is p-controllable. Consider the set $E_C = \{W \in \mathrm{SU}(n) : V(W X_\infty^\dagger (\mathrm{ad}_{\overline{H}_\ell}^j H_k) X_\infty) = 0$, for $j \in \mathbb{N}, k \in [m], \ell \in [p]\}$, where $\overline{H}_\ell$ is as in (8). Then, E_C is nonempty, $\lim_{t\to\infty} V(W_q(t) X_r^\dagger(t) H_k X_r(t)) = 0$, for $k \in [m]$, and $\lim_{t\to\infty} d(W_q(t), E_C) = 0$.

Proof: The proof of this result parallels [3, proof of Theorem 3]. Let $\Omega(W_q)$ be the limit set of the solution W_q. Then $\Omega(W_q) \subset \mathrm{SU}(n)$ is nonempty because $\mathrm{SU}(n)$ is compact. Since $\lim_{t\to\infty} d(W_q(t), \Omega(W_q)) = 0$ (see e.g. [5, Lemma 34, p. 153]), it suffices to show that $\Omega(W_q) \subset E_C$. First of all, since V is a continuous linear function (see (14)), there exists $c > 0$ such that $|V(X)| \leq c\|X\|$, for $X \in M^n$. Furthermore, it follows from (7)–(8), (9), (16)–(17) and the compactness of $\mathrm{SU}(n)$ in M^n that each and every one of the mappings $\overline{H}_r, X_r, X_r^\dagger, W_q, \dot{X}_r, \dot{X}_r^\dagger, \dot{W}_q, \mathrm{ad}_{\overline{H}_r}^j H_k$: $\mathbb{R} \to M^n$, for $j \in \mathbb{N}$, $k \in [m]$, is bounded. Let α: $\mathbb{R} \to \mathbb{R}$, b_k^j: $\mathbb{R} \times M^n \to M^n$, β_k^j: $\mathbb{R} \to \mathbb{R}$ be defined as $\alpha(t) = V(W_q(t))$, for $t \in \mathbb{R}$, $b_k^j(t, W) = V(W X_r^\dagger(t) (\mathrm{ad}_{\overline{H}_r(t)}^j H_k) X_r(t))$, for $(t, W) \in \mathbb{R} \times M^n$, $\beta_k^j(t) = b_k^j(t, W_q(t))$, for $t \in \mathbb{R}$, respectively, for $j \in \mathbb{N}$, $k \in [m]$. It will be shown by induction that, for $j \in \mathbb{N}$, $k \in [m]$,

$$\lim_{t\to\infty} \beta_k^j(t) = \lim_{t\to\infty} b_k^j(t, W_q(t)) = 0. \tag{20}$$

From (14), (16)–(17), (7)–(8) and the definition of b_k^0, one has $\dot{V}(t, W) = \sum_{k=1}^{m} [f_k b_k^0(t, W)]^2 \geq 0$ and $\ddot{V}(t, W) = 2 \sum_{k=1}^{m} f_k^2 b_k^0(t, W) \dot{b}_k^0(t, W)$, where $\dot{b}_k^0(t, W) = V(W X_r^\dagger(t) [\sum_{\ell=1}^{m} f_\ell^2 b_\ell^0(t, W) H_\ell] H_k X_r(t)) - \sin(2\pi t/T_p) b_k^1(t, W)$, for $(t, W) \in \mathbb{R} \times M^n$, and $f_k \in \mathbb{R}$ is nonzero. Note that b_k^0, $\dot{b}_k^0$, $\dot{V}$ and $\ddot{V}$ are continuous, for $k \in [m]$. Since $\dot{V}$ is a nonnegative function, one concludes that α is a nondecreasing function bounded from above such that $\dot{\alpha}$ is continuously differentiable and $\ddot{\alpha}$ is bounded. Hence $\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} V(W_q(t)) = \overline{\alpha}$, where $\overline{\alpha} \in \mathbb{R}$. This relation along with Lemma 1 establishes that $\lim_{t\to\infty}\dot{\alpha}(t) = \lim_{t\to\infty}\dot{V}(t, W_q(t)) = \sum_{k=1}^{m}[f_k b_k^0(t, W_q(t))]^2 = 0$. Therefore, since $f_k \neq 0$, $\lim_{t\to\infty}\beta_k^0(t) = \lim_{t\to\infty} V(W_q(t)X_r^\dagger(t)H_k X_r(t)) = 0$, for $k \in [m]$. Thus (16)–(17) give $\lim_{t\to\infty}\dot{W}_q(t) = 0$. Now, consider the induction hypothesis $\lim_{t\to\infty}\beta_k^j(t) = \lim_{t\to\infty} V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^j_{\overline{H}_r(t)}H_k)X_r(t)) = 0$, for every $k \in [m]$, where $j \in \mathbb{N}$ is fixed. Let $k \in [m]$. Define $c_k^j\colon \mathbb{R}\times M^n \to M^n$ as $c_k^j(t, W) \triangleq \sin^2(2\pi t/T_p)b_k^j(t, W) = V(WX_r^\dagger(t)\sin^2(2\pi t/T_p)(\mathrm{ad}^j_{\overline{H}_r(t)}H_k)X_r(t))$, for every $(t, W) \in \mathbb{R}\times M^n$, with $T_p = T/p$ (see (5)). The product of $\sin^2(2\pi t/T_p)$ and b_k^j assures that $c_k^j(t, W_q(t))$ meets the assumptions of Lemma 1. The details are as follows. From (8), one sees that $\overline{H}_r$ is piecewise-continuous with the discontinuities precisely at $t = \ell T_p$, for each $\ell \in \mathbb{Z}$. Hence c_k^j is continuously differentiable at each $(t, W) \in \mathbb{R}\times M^n$ such that $t \neq \ell T_p$, with $\ell \in \mathbb{Z}$. Let $t = \ell T_p$, with $\ell \in \mathbb{Z}$. Using $(d/dt)|_{t=\ell T_p}\sin^2(2\pi t/T_p) = 0$, it is easy to verify that $\partial c_k^j/\partial t_+(t, W) = \partial c_k^j/\partial t_-(t, W)$, for $W \in M^n$. Thus $\partial c_k^j/\partial t(t, W)$ exists for all $(t, W) \in \mathbb{R}\times M^n$. Immediate computations using (7)–(8) give that $\partial c_k^j/\partial t$ is continuous (even though $\mathrm{ad}^j_{\overline{H}_r(t)}H_k$ and $\mathrm{ad}^{j+1}_{\overline{H}_r(t)}H_k$ are piecewise-constant). Since $\partial c_k^j/\partial W$ is continuous, it follows that c_k^j is continuously differentiable with $\dot{c}_k^j(t, W) = V(WX_r^\dagger(t)d_k(t, W)X_r(t))$, where

$$
\begin{aligned}
d_k(t, W) = &\sin^2(2\pi t/T_p)[\sum_{\ell=1}^{m} f_\ell^2 b_\ell^0(t, W)H_\ell]\mathrm{ad}^j_{\overline{H}_r(t)}H_k \\
&+ (4\pi/T_p)\cos(2\pi t/T_p)\sin(2\pi t/T_p)\mathrm{ad}^j_{\overline{H}_r(t)}H_k \\
&- \sin^3(2\pi t/T_p)\mathrm{ad}^{j+1}_{\overline{H}_r(t)}H_k,
\end{aligned}
$$

for every $(t, W) \in \mathbb{R}\times M^n$. Hence (8) assures that $\ddot{c}_k^j\colon \mathbb{R}\times M^n \to M^n$ is well-defined when considering the partial derivative with respect to t from the right. Define $\gamma_k\colon \mathbb{R}\to\mathbb{R}$ as $\gamma_k(t) \triangleq c_k^j(t, W_q(t)) = \sin^2(2\pi t/T_p)\beta_k^j(t)$, for each $t \in \mathbb{R}$. Straightforward computations show that γ_k is continuously differentiable and that $\delta_k\colon \mathbb{R}\to\mathbb{R}$ given as $\delta_k(t) = \ddot{c}_k^j(t, W_q(t))$, for $t \in \mathbb{R}$, is bounded and piecewise-continuous. Furthermore, $\ddot{\gamma}_k = \delta_k$ (cf. the convention in Lemma 1). The induction hypothesis and Lemma 1 imply that $\lim_{t\to\infty}\gamma_k(t) = \lim_{t\to\infty}\dot{\gamma}_k(t) = \lim_{t\to\infty}\dot{c}_k^j(t, W_q(t)) = 0$. Thus $\lim_{t\to\infty}\sin^3(2\pi t/T_p)V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^{j+1}_{\overline{H}_r(t)}H_k)X_r(t)) = 0$ (see the expression of $\dot{c}_k^j(t, W)$). Hence Lemma 5 provides that $\lim_{t\to\infty}\beta_k^{j+1}(t) = \lim_{t\to\infty}V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^{j+1}_{\overline{H}_r(t)}H_k)X_r(t)) = 0$, showing that (20) holds. Lemma 4 then gives $\Omega(W_{t_0}) \subset E_C$. ■

Lemma 2: Suppose that (1) is p-controllable. Let $F = \{W \in \mathrm{SU}(n) : V(W) = \sum_{i=1}^{n}\Re(\lambda_i)$, where $\lambda_i \in \mathbb{C}$ are such that $|\lambda_i| = 1, \prod_{i=1}^{n}\lambda_i = 1, \Im(\lambda_1) = \cdots = \Im(\lambda_n)\}$. Then, $I \in F$, $E_C \subset F$, $\lim_{t\to\infty} d(W_q(t), F) = 0$, $\lim_{t\to\infty}V(W_q(t)X_r^\dagger(t)H_k X_r(t)) = 0$, for $k \in [m]$, where E_C is as in Theorem 2.

Proof: Assume that (1) is p-controllable. By Theorem 2, it suffices to show that $E_C \subset F$. Let $W \in E_C \subset \mathrm{SU}(n)$ and decompose it as $W = M\mathrm{diag}(\lambda_1, \ldots, \lambda_n)M^\dagger$, where $M \in \mathrm{SU}(n)$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $|\lambda_i| = 1$ and $\prod_{i=1}^{n}\lambda_i = 1$. Hence one obtains from (14) that $V(W) = \sum_{i=1}^{n}\Re(\lambda_i)$ and $V(WX_\infty^\dagger(\mathrm{ad}^j_{\overline{H}_\ell}H_k)X_\infty) = V(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)(X_\infty M)^\dagger\ (\mathrm{ad}^j_{\overline{H}_\ell}H_k)(X_\infty M)) = 0$, for $j \in \mathbb{N}$, $k \in [m]$, $\ell \in [p]$, where $\overline{H}_\ell$ is as in (8). Since $X_\infty M$ is unitary, it is clear that $N\colon \mathfrak{su}(n) \to \mathfrak{su}(n)$ defined by $N(Y) = (X_\infty M)^\dagger Y(X_\infty M)$, for every $Y \in \mathfrak{su}(n)$, is a linear surjective isomorphism. The rest of the proof is identical to the one of [3, Lemma 3], and the details are as follows. By assumption, (1) is p-controllable and (3) is met. Hence, $\mathrm{V}(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)X) = 0$, for every $X \in \mathfrak{su}(n)$, and thus $\mathrm{V}(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)D_\ell) = 0$, for each $\ell \in [n]$, where $D_1 = \mathrm{diag}(\imath, -\imath, 0, \ldots, 0)$, $D_2 = \mathrm{diag}(0, \imath, -\imath, 0, \ldots, 0), \ldots, D_{n-1} = \mathrm{diag}(0, \ldots, \imath, -\imath)$ and $D_n = \mathrm{diag}(\imath, 0, \ldots, 0, -\imath)$ are the canonical diagonal matrices of $\mathfrak{su}(n)$. From the diagonal structure of $D_1, \ldots, D_n$, we conclude that $\lambda_1, \ldots, \lambda_n$ must satisfy $\Im(\lambda_1) = \cdots = \Im(\lambda_n)$. This implies that $W \in F$ and therefore $E_C \subset F$. ■

Theorem 3: Suppose that (1) is p-controllable. Given, $q = (t_0, W_{t_0}) \in \mathbb{R}\times\mathrm{SU}(n)$, let $W_q\colon \mathbb{R}\to\mathrm{SU}(n)$ be the solution of (16)–(17) with initial condition $W_q(t_0) = W_{t_0}$. Then, $\lim_{t\to\infty}V(W_q(t)X_r^\dagger(t)H_k X_r(t)) = 0$, for $k \in [m]$. Furthermore, let δ be as in Proposition 2. Then,

$$V(W_q(t_0)) > \delta \Rightarrow \lim_{t\to\infty} W_q(t) = I.$$

Proof: The proof is identical to the second part of the proof of [3, Theorem 1], but with [3, Lemma 3] replaced by Lemma 2 above. ■

IV. QUANTUM MECHANICAL EXAMPLE

In this section, one returns to the quantum mechanical control system analyzed in [3, Section III] with $n = 4$, and one picks it up at [3, eq. (15)] (after applying the rotating wave approximation (RWA) in the interaction picture):

$$\dot{X} = (u_1H_{14}^R + u_2H_{14}^I + u_3H_{13}^R + u_4H_{13}^I + u_5H_{12}^R + u_6H_{12}^I)X \tag{21}$$

with initial condition $X(0) = I$, where $H_{ij}^R = (h_{k\ell}^{R,ij})$, $H_{ij}^I = (h_{k\ell}^{I,ij}) \in \mathfrak{su}(4)$ are the matrices with entries

$$
\begin{aligned}
&h_{ij}^{R,ij} = 1,\ h_{ji}^{R,ij} = -1,\ h_{k\ell}^{R,ij} = 0, \quad \text{for } k, \ell \neq i, j,\\
&h_{ij}^{I,ij} = h_{ji}^{I,ij} = \imath, \qquad h_{k\ell}^{I,ij} = 0, \quad \text{for } k, \ell \neq i, j,
\end{aligned}
$$

respectively, for all $1 \leq i < j \leq 4$.

One has that $[H_1, H_2] = 2\,\mathrm{diag}(\imath, 0, 0, -\imath)$, $[H_1, H_3] = H_{34}^R$, $[H_1, H_4] = -H_{34}^I$, $[H_1, H_5] = H_{24}^R$, $[H_1, H_6] = -H_{24}^I$, $[H_3, H_4] = 2\,\mathrm{diag}(\imath, 0, -\imath, 0)$, $[H_3, H_5] = H_{23}^R$, $[H_3, H_6] = -H_{23}^I$, $[H_5, H_6] = 2\,\mathrm{diag}(\imath, -\imath, 0, 0)$. Therefore, the referred control system is p-controllable with $p = 4$ and $\overline{H}_1 = H_1$, $\overline{H}_2 = H_2$, $\overline{H}_3 = H_3$, $\overline{H}_4 = H_5$. Hence Theorem 1 can be applied in order to generate once again the C-NOT quantum logic gate considered in [3]. There it was obtained that $G = \{-4, 0, 4\}$ ($\delta = 0$) and $V(X_\infty) = \Re(\mathrm{tr}(X_\infty)) = 2 > 0$, where $X_\infty \in \mathrm{SU}(4)$ corresponds to the C-NOT gate. Choose $T = 40$ and $f_k = 1$. By Theorem 1, the continuous inputs $u_k(t) = u_k^r(t) - f_k^2 V(X^\dagger(t)H_k X_r(t))$, for $t \in \mathbb{R}_+$, $1 \leq k \leq 6$, assure that

(2) and (19) hold, where $X_r(t)$ is given by (9) and $u_k^r(t)$ by (6) with $T_p = T/p = 10$, i.e. $u_k^r(t) = \sin(2\pi t/10)c_{jk}$, for $40\ell + (j-1)10 \leq t < 40\ell + j10$, $1 \leq j \leq 4$, $\ell \in \mathbb{Z}$. Here, $c_{jk} \in \mathbb{R}$ are the corresponding entries of the 4×6 matrix $C = (c_{jk}) = (e_1\ e_2\ e_3\ 0\ e_4\ 0)$, where e_1, e_2, e_3, e_4 are the canonical (column) vectors of $\mathbb{R}^4$. Note that $u_4^r = u_6^r = 0$ and that u_k^r can be seen as an amplitude modulation of the rectangular pulses determined by c_{jk}. The obtained simulation results are now presented. Fig. 1 shows the convergence of $\|X(t) - X_r(t)\|$ to zero (Euclidean norm on M^4). Although unexpected, notice that: (i) the norm is (apparently) nonincreasing, and (ii) only one period ($T = 40$) is required to achieve a reasonable convergence. Recall that $X_r(10\ell) = X_\infty$, for $\ell \in \mathbb{Z}$ (see (10)). Fig. 2 exhibits the controls u_k (dashed) and the "reference controls" u_k^r (solid) on the time interval $[0, 40]$ (over one period $T = 40$). One sees that u_k^r indeed corresponds to an amplitude modulation of rectangular pulses and that $\lim_{t\to\infty}(u_k^r(t) - u_k(t)) = 0$.

Although not shown here due to lack of space, if one chooses $T = 40$ in the control strategy used in [3] (regular case) for this same quantum control problem, one has that: (i) the signal energy of the controls $u_k(t)$ on $[0, 40]$ for the p-controllable case are smaller than the ones for the regular case, and (ii) the tracking error in the p-controllable case exhibits a "faster" convergence to zero in comparison to the regular case.

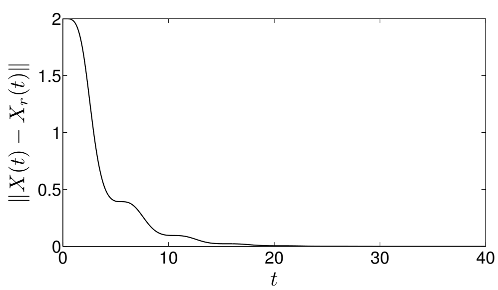

Fig. 1. Convergence of the norm of the tracking error to zero.

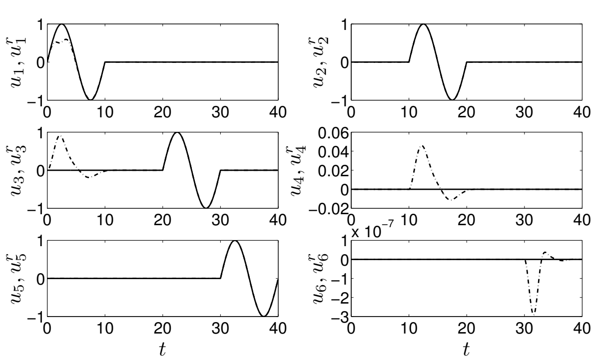

Fig. 2. Controls u_k (dashed) and "reference controls" u_k^r (solid) on $[0, 40]$.

V. CONCLUDING REMARKS

The results in this paper have complemented the ones in [3] by establishing further controllability conditions under which explicit expressions for the control laws are provided. Simulations made for the considered quantum system have suggested that as the "feedback gains" f_k^2 get "closer" to the maximum magnitudes of u_k^r, the "higher" the convergence rate of $\|X(t) - X_r(t)\|$ to zero. This was the reason why one specified $f_k = 1$ in the quantum example. This remains to be investigated.

VI. ACKNOWLEDGMENTS

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APPENDIX

The proof of Proposition 1 is given below.

Proof: Let $L = \text{span}\{[X_1, [X_2, X_3]], ad_Y^j X_4$, for all $X_1, \ldots, X_4 \in H, Y \in \mathbf{H}, j \in \mathbb{N}\}$. By assumption, $G \subset L \subset \text{Lie}(H)$ is finite and $\text{Lie}(H) = \text{span}(G)$. Therefore, it suffices to show that, given $X, Y, Z \in H = \{H_1, \ldots, H_m\}$, there exist $\overline{H}_1, \ldots, \overline{H}_5 \in \mathbf{H}$ such that $[X, [Y, Z]] \in \text{span}\{\text{ad}^2_{\overline{H}_\ell} X, \text{ad}^2_{\overline{H}_\ell} Y, \text{ad}^2_{\overline{H}_\ell} Z$, for all $1 \leq \ell \leq 5\}$. Indeed, if this holds, then there exist $\overline{H}_1, \ldots, \overline{H}_p \in \mathbf{H}$ such that $G \subset \text{span}\{\text{ad}^j_{\overline{H}_\ell} H_k$, for all $j \in \mathbb{N}$, $k \in [m]$, $\ell \in [p]\} \subset L \subset \text{Lie}(H)$. Hence $\text{Lie}(H) \subset \text{span}\{\text{ad}^j_{\overline{H}_\ell} H_k$, for all $j \in \mathbb{N}$, $k \in [m]$, $\ell \in [p]\} \subset \text{Lie}(H)$, from which one concludes that $\text{Lie}(H) = \text{span}\{\text{ad}^j_{\overline{H}_\ell} H_k$, for all $j \in \mathbb{N}$, $k \in [m]$, $\ell \in [p]\}$. Now, let $X, Y, Z \in H$. Since $ad^2_{X+Z} Y = ad^2_X Y + ad^2_Z Y - [X, [Y, Z]] + [Z, [X, Y]]$, $ad^2_{X-Y} Z = ad^2_X Z + ad^2_Y Z - [X, [Y, Z]] + [Y, [Z, X]]$, the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ yields $-3[X, [Y, Z]] = ad^2_{X+Z} Y + ad^2_{X-Y} Z - (ad^2_X Y + ad^2_Z Y + ad^2_X Z + ad^2_Y Z)$. Thus choosing $\overline{H}_1 = X$, $\overline{H}_2 = Y$, $\overline{H}_3 = Z$, $\overline{H}_4 = X + Z$, $\overline{H}_5 = X - Y$, ends the proof. ∎

Lemma 3: [3, Lemma 1] Let $W: \mathbb{R} \to M^n$ be a continuously differentiable mapping with $\lim_{t\to\infty} \dot{W}(t) = 0$. Suppose that $\{t_m\}$ is a real sequence such that $\lim_{m\to\infty} t_m = \infty$ and $\lim_{m\to\infty} W(t_m) = \overline{W}$. Then, for every $\epsilon \in \mathbb{R}$, one has that $\lim_{m\to\infty} W(t_m + \epsilon) = \overline{W}$.

Lemma 4: Suppose that (1) is p-controllable and denote by $\Omega(W_q)$ the limit set of the solution W_q. Let $\overline{W} \in \Omega(W_q)$, $j \in \mathbb{N}$, $k \in [m]$. Assume that $\lim_{t\to\infty} \dot{W}_q(t) = 0$, $\lim_{t\to\infty} V(W_q(t) X_r^\dagger(t)(\text{ad}^j_{\overline{H}_r(t)} H_k) X_r(t)) = 0$, with $\overline{H}_r(t)$ as in (8). Then, $V(\overline{W} X_\infty^\dagger (\text{ad}^j_{\overline{H}_\ell} H_k) X_\infty) = 0$, for $\ell \in [p]$.

Proof: Let $\overline{W} \in \Omega(W_q)$. By definition, there exists a real sequence $\{t_m\}$ such that $\lim_{m\to\infty} t_m = \infty$, $\lim_{m\to\infty} W_q(t_m) = \overline{W}$. Now, for each $m \in \mathbb{N}$, there exists $\ell_m \in \mathbb{Z}$ such that $s_m = t_m - \ell_m T \in [0, T)$. Since $[0, T]$ is a compact set, there exists a subsequence $\{s_{m_i}\}$ of $\{s_m\}$ in which $\lim_{i\to\infty} s_{m_i} = \theta$, where $\theta \in [0, T]$. Let $\{t_{m_i}\}$ be the corresponding subsequence $\{t_m\}$. For each $0 \leq \ell \leq p$, define the real sequences $\{s^\ell_{m_i}\}$ and $\{t^\ell_{m_i}\}$ by $s^\ell_{m_i} = s_{m_i} - \theta + \ell T_p$, $t^\ell_{m_i} \triangleq t_{m_i} - \theta + \ell T_p = s_{m_i} + \ell_{m_i} T - \theta + \ell T_p = s^\ell_{m_i} + \ell_{m_i} T$, respectively, with $T_p = T/p$ (see (5)). Hence $s^{\ell+1}_{m_i} = s^\ell_{m_i} + T_p$, for $i \in \mathbb{N}$, $0 \leq \ell \leq p - 1$. By hypothesis, $\lim_{t\to\infty} \dot{W}_q(t) = 0$. Therefore, Lemma 3 implies

$$\lim_{i\to\infty} (s^\ell_{m_i} + \epsilon) = \ell T_p + \epsilon, \quad \lim_{i\to\infty} W_q(t^\ell_{m_i} + \epsilon) = \overline{W}, \quad (22)$$

for $\epsilon \in \mathbb{R}$, $0 \leq \ell \leq p$. Define β: $\mathbb{R} \to \mathbb{R}$ as $\beta(t) = V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^j_{\overline{H}_r(t)} H_k)X_r(t))$, for $t \in \mathbb{R}$. By assumption, $\lim_{t\to\infty} \beta(t) = 0$. Let $0 \leq \ell \leq p-1$. From (8), one has that $\overline{H}_r$ is constant on the interval $[\ell T_p, (\ell+1)T_p)$ with $\overline{H}_r(\ell T_p) = \overline{H}_{\ell+1}$. Since X_r is continuous and $\overline{H}_r$, X_r have period T (see (8) and (9)), one gets

$$\begin{array}{l} \lim_{i\to\infty} X_r(t^\ell_{m_i} + \epsilon) = X_r(\ell T_p + \epsilon), \\ \lim_{i\to\infty} \overline{H}_r(t^\ell_{m_i} + \epsilon) = \overline{H}_r(\ell T_p + \epsilon) = \overline{H}_{\ell+1}, \end{array} \tag{23}$$

for all $\epsilon \in (0, T_p)$. Define γ: $(0, T_p) \to \mathbb{R}$ by $\gamma(\epsilon) \triangleq \lim_{i\to\infty} \beta(t^\ell_{m_i} + \epsilon) = \lim_{i\to\infty} \beta(t_{m_i} - \theta + \ell T_p + \epsilon)$, for $\epsilon \in (0, T)$. It follows from (22), $\lim_{t\to\infty} \beta(t) = 0$, (23) and the continuity of V (see (14)) that $\gamma(\epsilon) = V(\overline{W} X_r^\dagger(\ell T_p + \epsilon)(\mathrm{ad}^j_{\overline{H}_{\ell+1}} H_k)X_r(\ell T_p + \epsilon)) = 0$, for $\epsilon \in (0, T)$. Therefore, one concludes from the continuity of X_r and (10) that $\lim_{\epsilon\to 0} \gamma(\epsilon) = V(\overline{W} X^\dagger_\infty(\mathrm{ad}^j_{\overline{H}_{\ell+1}} H_k)X_\infty) = 0$. ■

Lemma 5: Suppose that (1) is p-controllable. Let $j \in \mathbb{N}$, $k \in [m]$, and let α: $\mathbb{R} \to \mathbb{R}$ be a continuous function with period T such that the of zeros of the restriction $\alpha|[0,T]$ are finite in number. Assume that $\lim_{t\to\infty} \alpha(t)V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^j_{\overline{H}_r(t)} H_k)X_r(t)) = 0$, $\lim_{t\to\infty} \dot{W}_q(t) = 0$, where $\overline{H}_r(t)$ is as in (8). Then, $\lim_{t\to\infty} V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^j_{\overline{H}_r(t)} H_k)X_r(t)) = 0$. (It is as if one could divide the first limit above by $\alpha(t)$).

Proof: This is proved by contradiction. Define β: $\mathbb{R} \to \mathbb{R}$ by $\beta(t) = V(W_q(t)X_r^\dagger(t)(\mathrm{ad}^j_{\overline{H}_r(t)} H_k)X_r(t))$, for $t \in \mathbb{R}$, and suppose that β does not converge to zero as $t \to \infty$. One has that W_q: $\mathbb{R} \to \mathrm{SU}(n)$, X_r: $\mathbb{R} \to \mathrm{SU}(n)$, $\mathrm{SU}(n)$ is compact, $\overline{H}_r$: $\mathbb{R} \to M^n$ is bounded and V is continuous (see (14)). Thus there exists a real sequence $\{t_m\}$ such that

$$\begin{array}{l} \lim_{m\to\infty} t_m = \infty, \quad \lim_{m\to\infty} W_q(t_m) = \overline{W}, \\ \lim_{m\to\infty} X_r(t_m) = \overline{X}_r, \quad \lim_{m\to\infty} \overline{H}_r(t_m) = \overline{A}_r, \\ \lim_{m\to\infty} \beta(t_m) = V(\overline{WX}^\dagger_r(\mathrm{ad}^j_{\overline{A}_r} H_k)\overline{X}_r) = \overline{\beta} \neq 0, \end{array} \tag{24}$$

where $\overline{W}, \overline{X}_r \in \mathrm{SU}(n)$, $\overline{A}_r \in M^n$, $\overline{\beta} \in \mathbb{R}$. Given $m \in \mathbb{N}$, there exists $\ell_m \in \mathbb{Z}$ such that $s_m = t_m - \ell_m T \in [0, T)$. Since $[0, T]$ is compact, there exists a subsequence $\{s_{m_i}\}$ of $\{s_m\}$ such that $\lim_{i\to\infty} s_{m_i} = \theta$, where $\theta \in [0, T]$. Let $\{t_{m_i}\}$ be the corresponding subsequence of $\{t_m\}$. By hypothesis, α is continuous, has period T and $\lim_{t\to\infty} \alpha(t)\beta(t) = 0$. Hence $\lim_{i\to\infty} \alpha(t_{m_i})\beta(t_{m_i}) = \lim_{i\to\infty} \alpha(s_{m_i})\beta(t_{m_i}) = \alpha(\theta)\overline{\beta} = 0$, and since $\overline{\beta} \neq 0$, one has $\alpha(\theta) = 0$, that is, θ belongs to the set Z of zeros of $\alpha|[0, T]$. However, Z is a finite set by assumption. Therefore, θ is an isolated point of $Z \subset \mathbb{R}$. Furthermore, (24), the definitions above and Lemma 3 imply that, for $\epsilon \in \mathbb{R}$,

$$\lim_{i\to\infty} (s_{m_i} + \epsilon) = \theta + \epsilon, \quad \lim_{i\to\infty} W_q(t_{m_i} + \epsilon) = \overline{W}. \tag{25}$$

One shall now consider two distinct cases depending on the value of $\theta \in [0, T]$. Recall that $T_p = T/p$ (see (5)).

Case 1: $\theta \neq \ell T_p$, for all $0 \leq \ell \leq p$. Since θ is an isolated point of $Z \subset \mathbb{R}$ and using (8), one knows that there exists $\epsilon_0 > 0$ such that $\alpha(\epsilon) \neq 0$, for $\epsilon \in (\theta, \theta + \epsilon_0)$, and that $\overline{H}_r$ is constant on $[\theta, \theta + \epsilon_0)$ and continuous at $t = \theta$. One has that X_r is continuous and $\overline{H}_r$, X_r have period T (see (8) and (9)). Hence (24) and (25) give that

$$\begin{array}{l} \lim_{i\to\infty} X_r(t_{m_i} + \epsilon) = X_r(\theta + \epsilon), \quad X_r(\theta) = \overline{X}_r, \\ \lim_{i\to\infty} \overline{H}_r(t_{m_i} + \epsilon) = \overline{H}_r(\theta + \epsilon) = \overline{H}_r(\theta) = \overline{A}_r, \end{array} \tag{26}$$

for each $\epsilon \in [0, \epsilon_0)$. Define γ: $(0, \epsilon_0) \to \mathbb{R}$ as $\gamma(\epsilon) = \lim_{i\to\infty} \beta(t_{m_i} + \epsilon)$, for $\epsilon \in (0, \epsilon_0)$. Let $\epsilon \in (0, \epsilon_0)$. From (25) and (26), one gets $\gamma(\epsilon) = V(\overline{W} X_r^\dagger(\theta + \epsilon)(\mathrm{ad}^j_{\overline{A}_r} H_k)X_r(\theta + \epsilon))$, and $\lim_{t\to\infty} \alpha(t)\beta(t) = 0$ implies $\lim_{i\to\infty} \alpha(t_{m_i} + \epsilon)\beta(t_{m_i} + \epsilon) = \lim_{i\to\infty} \alpha(s_{m_i} + \epsilon)\beta(t_{m_i} + \epsilon) = \alpha(\theta + \epsilon)\gamma(\epsilon) = 0$. But $\alpha(\theta + \epsilon) \neq 0$. Thus $\gamma = 0$, and (26) gives $\lim_{\epsilon\to 0} \gamma(\epsilon) = V(\overline{WX}^\dagger_r(\mathrm{ad}^j_{\overline{A}_r} H_k)\overline{X}_r) = \overline{\beta} = 0$, which contradicts $\overline{\beta} \neq 0$.

Case 2: $\theta = \ell T_p$, for some $0 \leq \ell \leq p$. Consider the sets $L_+ = \{i \in \mathbb{N} : s_{m_i} \geq \theta\}$, $L_- = \{i \in \mathbb{N} : s_{m_i} < \theta\}$. It is clear that these sets are disjoint and $L_+ \cup L_- = \mathbb{N}$. Furthermore, L_+ is either finite or infinite. Suppose that L_+ is infinite. Hence there exists a subsequence $\{s_{m_{i_j}}\}$ of $\{s_{m_i}\}$ such that $s_{m_{i_j}} \geq \theta$, for $j \in \mathbb{N}$. Let $\{t_{m_{i_j}}\}$ be the corresponding subsequence of $\{t_{m_i}\}$. One can then proceed in an identical manner as in Case 1, replacing $\{s_{m_i}\}$ by $\{s_{m_{i_j}}\}$, $\{t_{m_i}\}$ by $\{t_{m_{i_j}}\}$ and the continuity of $\overline{H}_r$ at $t = \theta$ by the continuity from the right of $\overline{H}_r$ at $t = \theta = \ell T_p$ (see (8)), and thus conclude that $\overline{\beta} = 0$. Now, suppose that L_+ is finite. Hence L_- is infinite and there exists a subsequence $\{s_{m_{i_k}}\}$ of $\{s_{m_i}\}$ such that $s_{m_{i_k}} < \theta$, for $k \in \mathbb{N}$. Note that $\theta \neq 0$, that is, $\ell \neq 0$, because $s_m \in [0, T)$, for $m \in \mathbb{N}$. Since $\theta = \ell T_p$ is an isolated point of $Z \subset \mathbb{R}$, it follows from (8) that there exists $\epsilon_0 > 0$ such that $\alpha(\epsilon) \neq 0$ and $\overline{H}_r(\epsilon) = \overline{H}_r(\ell T_p)$, for $\epsilon \in (\theta - \epsilon_0, \theta)$. By (24) and (25),

$$\begin{array}{l} \lim_{k\to\infty} X_r(t_{m_{i_k}} - \epsilon) = X_r(\theta - \epsilon), \quad X_r(\theta) = \overline{X}_r, \\ \lim_{k\to\infty} \overline{H}_r(t_{m_{i_k}} - \epsilon) = \overline{H}_r(\theta - \epsilon) = \overline{H}_r(\ell T_p) = \overline{A}_r, \end{array} \tag{27}$$

for each $\epsilon \in [0, \epsilon_0)$. Define δ: $(0, \epsilon_0) \to \mathbb{R}$ as $\delta(\epsilon) = \lim_{k\to\infty} \beta(t_{m_{i_k}} - \epsilon)$, for $\epsilon \in (0, \epsilon_0)$. Using (25), (27) and proceeding similarly as in Case 1, one obtains $\overline{\beta} = 0$. ■

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