Motion planning for the heat equation

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1 Introduction

Motion planning, i.e., the construction of an open-loop control connecting an initial state to a final state, is a fundamental problem of control theory both from a practical and theoretical point of view. For systems governed by *ordinary* differential equations the notion of *flatness* [5, 11] provides a constructive solution to this problem. As noticed in [11], the idea underlying equivalence and flatness –the existence of a one-to-one correspondence between trajectories of systems– can be adapted to partial differential equations [6, 1, 7] with boundary control.

In this paper, which develops ideas introduced in [11, 10], we study in this spirit the heat equation with one space dimension and control on the boundary. We give an explicit parametrization of the trajectories as a power series in the space variable with coefficients involving time derivatives of the "flat" output. This series is convergent when the flat output is restricted to be a Gevrey function (i.e., a smooth function with a "not too divergent" Taylor expansion). This parameterization *explicitely* provides *regular* (i.e., C^2, smooth, Gevrey,...) open-loop controls achieving the approximate motion planning. We then extend some of these results to the general 1-D linear diffusion equation.

Approximate controllability of the linear and semilinear heat equation has already been extensively studied, using for instance duality coupled with BU results in the Hilbertian case (see for instance [3] or [13]), or semigroup theory [4]. Our approach, which focuses more on finding explicit solutions rather than on mathematical generality, is somewhat different and is more related to older works by Holmgren and Gevrey [18, 8] (see also [9] for a more modern presentation).

2 Gevrey functions

The Taylor expansion of a smooth function is not convergent, unless the function is analytic. The notion of Gevrey order is a way of estimating this divergence.

Definition 1. A smooth function $t \in [0,T] \mapsto y(t)$ is *Gevrey of order* α if

$$\exists M, R > 0, \forall m \in \mathbb{N}, \quad \sup_{t\in[0,T]} \left| y^{(m)}(t) \right| \leq M \frac{(m!)^\alpha}{R^m}.$$

By definition, a Gevrey function of order α is also of order β for any $\beta \geq \alpha$. A classical result (the *Cauchy estimates*) asserts that Gevrey functions of order 1 are analytic (entire functions if $\alpha < 1$). Gevrey functions of order $\alpha > 1$ have a divergent Taylor expansion; the larger α, the "more divergent" the Taylor expansion.

Important properties of analytic functions generalize to Gevrey functions of order $\alpha > 1$: the scaling, integration, addition, multiplication and composition of Gevrey functions of order $\alpha > 1$ is of order α [8]. But contrary to analytic functions, functions of order $\alpha > 1$ may be constant on an open set without being constant everywhere. For example the "bump function"

$$\phi_\gamma(t) = \begin{cases} 0 & \text{if } t \notin]0,1[, \\ \exp\left(\dfrac{-1}{\left((1-t)t\right)^\gamma}\right) & \text{if } t \in]0,1[. \end{cases}$$

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is of order $1+1/\gamma$ whatever $\gamma > 0$ [14]. Similarly,

$$\Phi_\gamma(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1 \\ \dfrac{\int_0^t \phi_\gamma(\tau)d\tau}{\int_0^1 \phi_\gamma(\tau)d\tau} & \text{if } t \in]0,1[, \end{cases}$$

that will be used for motion planning, has order $1+1/\gamma$.

In the same way, we can define Gevrey functions of two variables.

Definition 2. A smooth function $(x,t) \in [0,1]\times[0,T] \mapsto y(x,t)$ is *Gevrey of order α in t and β in x* if

$$\exists M,R,S>0, \forall m,n \in \mathbb{N}, \qquad \sup_{(x,t)\in[0,1]\times[0,T]} \left| \frac{\partial^{n+m}\theta}{\partial x^n \partial t^m}(x,t) \right| \leq M \frac{(m!)^\alpha (n!)^\beta}{R^m S^n}.$$

3 The heat equation is "flat"

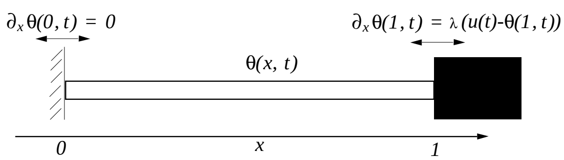

Figure 1: The heated rod

Consider the one-dimensional rod shown in figure 1 [16, example 4.2.2]. Heat is added from a steam chest at $x=1$, while the end at $x=0$ is perfectly insulated. This system is modeled by

$$(1) \qquad \begin{cases} \partial_t\theta(x,t) = \partial_{xx}\theta(x,t), \quad x \in [0,1] \\ \partial_x\theta(0,t) = 0 \\ \partial_x\theta(1,t) = \lambda\big(u(t)-\theta(1,t)\big), \end{cases}$$

where $\theta(x,t)$ is the temperature along the rod and $u(t)$, the temperature of the steam chest is the control input ; λ is a positive constant.

We claim this system is "flat" [5, 11] with

$$y(t) := \theta(0,t)$$

as a "flat" output. In other words, we will show there is (in a certain sense) a $1-1$ correspondence between arbitrary functions $t \mapsto y(t)$ and solutions of (1) (other boundary conditions could be treated with minor adaptations).

Noticing the "inverse" system

$$(2) \qquad \begin{cases} \partial_{xx}\theta(x,t) = \partial_t\theta(x,t) \\ \partial_x\theta(0,t) = 0 \\ \theta(0,t) = y(t), \end{cases}$$

is in Cauchy-Kovalevskaya form, we first seek a formal solution $\theta(x,t) = \sum_{i=0}^{+\infty} a_i(t)\frac{x^i}{i!}$, where the a_i are smooth functions. Using (2), we find

$$\forall i \geq 0, \begin{cases} a_{2i}(t) = y^{(i)}(t) \\ a_{2i+1}(t) = 0, \end{cases}$$

so that

$$(3) \qquad \theta(x,t) = \sum_{i=0}^{+\infty} y^{(i)}(t)\frac{x^{2i}}{(2i)!}.$$

The formal control is then

$$u(t) = \theta(1,t) + \frac{1}{\lambda}\partial_x\theta(1,t) = \sum_{i=0}^{+\infty}\frac{y^{(i)}(t)}{(2i)!} + \frac{1}{\lambda}\sum_{i=1}^{+\infty}\frac{y^{(i)}(t)}{(2i-1)!}. \tag{4}$$

We now give a meaning to this formal solution by restricting $t \mapsto y(t)$ to be Gevrey of a suitable order α.

Theorem 1. *When $y(t)$ is Gevrey of order $\alpha < 2$, the formal solution* (3) *is Gevrey of order α in t and order* 1 *in x (and in particular the formal control* (4) *is Gevrey of order α).*

When $\alpha = 2$, the same result holds provided $R > 4$ (R is the "radius" of $y(t)$ in definition 1).

Proof. We formally differentiate (3),

$$\frac{\partial^{n+m}\theta}{\partial x^n \partial t^m}(x,t) = \sum_{2i\geq n} y^{(i+m)}(t)\frac{x^{2i-n}}{(2i-n)!}, \tag{5}$$

and bound the general term of this power series to find Gevrey estimates:

$$\begin{aligned}
\left|\frac{y^{(i+m)}(t)}{m!^\alpha n!}\frac{x^{2i-n}}{(2i-n)!}\right| &\leq \frac{M}{R^{i+m}}\frac{(i+m)!^\alpha}{(2i-n)!}\frac{1}{m!^\alpha n!}\\
&= \frac{M}{R^{i+m}}\left(\frac{(i+m)!}{i!m!}\right)^\alpha \frac{i!^2 i!^{\alpha-2}}{n!(2i-n)!}\\
&\sim \frac{M}{R^{i+m}}\left(\frac{(i+m)!}{i!m!}\right)^\alpha \frac{\sqrt{\pi i}}{2^{2i}}\frac{(2i)!i!^{\alpha-2}}{n!(2i-n)!}\\
&\leq \frac{M}{\tilde{R}^m}\frac{i!^{\alpha-2}\sqrt{\pi i}}{\tilde{R}^i},
\end{aligned}$$

where we have set $\tilde{R} := \frac{R}{2^\alpha}$ and used $(l+k)! \leq 2^{l+k} l! k!$ and Stirling's formula $k! \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k}$.

Consider now the number

$$\tilde{M} := M \sum_{2i\geq n}\frac{i!^{\alpha-2}\sqrt{\pi i}}{\tilde{R}^i}.$$

Since the general term of this series satisfies

$$\lim_{i\to+\infty}\frac{\frac{(i+1)!^{\alpha-2}\sqrt{\pi(i+1)}}{\tilde{R}^{i+1}}}{\frac{i!^{\alpha-2}\sqrt{\pi i}}{\tilde{R}^i}} = \lim_{i\to+\infty}\frac{i^{\alpha-2}}{\tilde{R}},$$

$\tilde{M}$ is finite when $\alpha < 2$; when $\alpha = 2$, $\tilde{M}$ is finite provided $\tilde{R} > 1$, i.e., $R > 4$. Hence

$$\left|\frac{\partial^{m+n}\theta}{\partial x^n \partial t^m}(x,t)\right| \leq \sum_{2i\geq n}\left|y^{(i+m)}(t)\frac{x^{2i-n}}{(2i-n)!}\right| \leq \tilde{M}\frac{m!^\alpha n!}{\tilde{R}^m 1^n},$$

which means $\theta(x,t)$ is Gevrey of order 1 in x and α in t. This implies in particular that the control $u(t)$ defined by (4) is Gevrey of order α, since the power series defining θ and $\partial_x\theta$ have a radius of convergence > 1.

Notice this result could be improved: it is quite easy to prove that $\theta(x,t)$ is in fact of order $\alpha/2$ in x, i.e., is *entire* in x when $\alpha < 2$. Since we won't make use of this extra regularity in the sequel, we have restricted to a simpler statement. Besides, when $\alpha = 2$, series (5) is uniformly convergent on $[0,1]\times[0,T]$ as soon as $R > 1/4$ for all $m, n \in \mathbb{N}$. □

We have therefore established that the heat equation (1) is flat in the following sense: any Gevrey function $y(t)$ of order $\alpha < 2$ uniquely defines a trajectory $\big(\theta(x,t), u(t)\big)$ of (1) which is Gevrey of order α in t and order 1 in x. Conversely any trajectory of (1) which is Gevrey of order 1 in x and order α in t obviously defines a unique Gevrey function $y(t) := \theta(0,t)$ of order α.

For $\alpha = 2$, this 1–1 correspondence holds between Gevrey functions $y(t)$ of order 2 with $R > 4$ and trajectories which are Gevrey of order 2 in t order 1 in x.

In other words, we have obtained an explicit parametrization of the solutions of the heat equation by curves.

4 Motion planning

The previous developments provide a simple and explicit solution to the problem of (approximate) motion planning. Assuming the initial temperature profile is

$$\forall x \in [0,1], \quad \theta(x,0) = \Theta_0(x), \quad \Theta_0 \in L^2(0,1),$$

we want to find an open-loop control $[0,T] \ni t \mapsto u(t)$ such that at time T the final temperature profile is "arbitrary close" to

$$\forall x \in [0,1], \quad \theta(x,T) = \Theta_T(x), \quad \Theta_T \in L^2(0,1).$$

Of course Θ_0 as well as Θ_T do not in general have a convergent Taylor expansion on even powers of x. Nevertheless, as a direct consequence of the Stone-Weierstrass theorem, the set of polynomials of even degree is dense in $C(0,1)$ (see [17, chap. 7]), hence in $L^2(0,1)$. This means that for all $\varepsilon > 0$ there exists polynomials

$$\Pi_0(x) = \sum_{i=0}^{n} p_i \frac{x^{2i}}{(2i)!}, \qquad p_i \in \mathbb{R}$$
$$\Pi_T(x) = \sum_{i=0}^{n} q_i \frac{x^{2i}}{(2i)!}, \qquad q_i \in \mathbb{R}$$

such that $\|\Theta_0 - \Pi_0\| \leq \varepsilon$ and $\|\Theta_T - \Pi_T\| \leq \varepsilon$ (here and in the sequel $\|.\|$ means the usual norm on $L^2(0,1)$). On the other hand the function

$$Y(t) := \sum_{i=0}^{n} p_i \frac{t^i}{i!} \left(1 - \Phi_\gamma\left(\frac{t}{T}\right)\right) + q_i \frac{(t-T)^i}{i!} \Phi_\gamma\left(\frac{t}{T}\right)$$

is Gevrey of order ≤ 2 when $\gamma \geq 1$ (see section 2) and satisfies

$$\begin{aligned} &Y^{(i)}(0) = p_i, \quad Y^{(i)}(T) = q_i, && i = 0, \dots, n \\ &Y^{(i)}(0) = 0, \quad Y^{(i)}(T) = 0, && i > n. \end{aligned}$$

Guided by (4), we then define the open-loop control

$$U(t) := \sum_{i=0}^{+\infty} \frac{Y^{(i)}(t)}{(2i)!} + \frac{1}{\lambda} \sum_{i=1}^{+\infty} \frac{Y^{(i)}(t)}{(2i-1)!}, \quad t \in [0,1].$$

Consider also an approximate control $\bar{U} \in C^2(0,T)$ such that

$$\sup_{t\in[0,T]} \left|\bar{U}(t) - U(t)\right| \leq \varepsilon \quad \text{and} \quad \sup_{t\in[0,T]} \left|\dot{\bar{U}}(t) - \dot{U}(t)\right| \leq \varepsilon.$$

An exemple of such an approximate control is

$$\bar{U}(t) := \sum_{i=0}^{N} \frac{Y^{(i)}(t)}{(2i)!} + \frac{1}{\lambda} \sum_{i=1}^{N} \frac{Y^{(i)}(t)}{(2i-1)!},$$

obtained by truncating U to some large enough order N (a suitable N does exist, since the series defining U, as well as all its derivatives, is uniformly convergent on $[0,T]$ by theorem 1).

Theorem 2. *The control $[0,T] \ni t \mapsto U(t)$ exactly steers system (1) from the initial state Π_0 at time 0 to the final state Π_T at time T.*

The approximate control $[0,T] \ni t \mapsto \bar{U}(t)$ approximately steers system (1) from the initial state Θ_0 at time 0 to the final state Θ_T at time T. More precisely, there exists $K > 0$ independent of T such that

$$\|\theta(.,T) - \Theta_T\| \leq K\varepsilon.$$

Proof. The proof is a direct consequence of standard results from the theory of strongly continuous semigroups, see for instance [2], [19] or [4].

Setting $\tilde\theta(x,t)=\theta(x,t)-u(t)$, we first transform the boundary control problem (1) into

$$\begin{cases} \partial_t\tilde\theta(x,t)=\partial_{xx}\tilde\theta(x,t)+\dot u(t), & x\in[0,1]\\ \partial_x\tilde\theta(0,t)=0 \\ \partial_x\tilde\theta(1,t)=-\lambda\tilde\theta(1,t). \end{cases} \tag{6}$$

Consider now the differential operator A on $L^2(0,1)$ defined by

$$A(g)=-\frac{d^2g}{dx^2},$$

with domain

$$D(A)=\{g\in L^2(0,1)|\ \frac{dg}{dx}\in L^2(0,1),\ \frac{d^2g}{dx^2}\in L^2(0,1),\ \frac{dg}{dx}(0)=0,\ \frac{dg}{dx}(1)=-\lambda g(1)\}.$$

A is maximal, accretive and symmetric, hence is closed, densely defined and self-adjoint. Hence, as a consequence of the Hille-Yosida theorem [2, theorems VII.4,VII.7 and VII.10], given an initial condition $\tilde\theta_0\in L^2(0,1)$ and a control $u\in C^2(0,T)$, problem (6) has a unique solution

$$t\mapsto\tilde\theta(\cdot,t)\in C\big([0,T];L^2(0,1)\big)\cap C^1\big(]0,T];L^2(0,1)\big)\cap C\big(]0,T];D(A)\big),$$

such that

$$\tilde\theta(\cdot,t)=S_A(t)\tilde\theta_0-\int_0^t S_A(t-\tau)\dot u(\tau)d\tau, \tag{7}$$

where S_A denotes the stongly continuous semigroup with infinitesimal generator $-A$. Moreover for all $t\geq 0$,

$$\|S_A(t)\|\leq e^{-\omega t},$$

where $-\omega$ is the (strictly negative) growth bound of the semigroup.

Clearly, the smooth function

$$\tilde\theta(x,t)=\sum_{i=0}^{+\infty}Y^{(i)}(t)\frac{x^{2i}}{(2i)!}-U(t)$$

is a solution of (6) with control $u=U$ such that $\tilde\theta(.,0)=\Pi_0-U(0)$ and $\tilde\theta(.,T)=\Pi_T-U(T)$. Therefore, it is the unique solution of (6) with initial condition $\tilde\theta_0=\Pi_0-U(0)$. This proves the first claim. Moreover by (7),

$$\Pi_T-U(T)=S_A(T)\big(\Pi_0-U(0)\big)-\int_0^T S_A(T-\tau)\dot U(\tau)d\tau. \tag{8}$$

We next estimate the final error when applying the approximate control $u=\bar U$ to equation (6) starting from the initial condition $\tilde\theta_0=\Theta_0-\bar U(0)$. Setting $\Delta U:=\bar U-U$ and using (7) then (8), we find

$$\begin{aligned}\tilde\theta(.,T)&=S_A(T)\big(\Pi_0-U(0)+\Theta_0-\Pi_0-\Delta U(0)\big)-\int_0^T S_A(T-\tau)\dot U(\tau)d\tau-\int_0^T S_A(T-\tau))\Delta\dot U(\tau)d\tau\\ &=\Pi_T-U(T)+S_A(T)\big(\Theta_0-\Pi_0-\Delta U(0)\big)-\int_0^T S_A(T-\tau)\Delta\dot U(\tau)d\tau\end{aligned}$$

Therefore,

$$\begin{aligned}\|\theta(.,T)-\Theta_T\|&\leq\|\Theta_T-\Pi_T\|+\|\theta(.,T)-\Pi_T\|\\ &\leq\|\Theta_T-\Pi_T\|+\|S_A(T)\|\cdot\|\Theta_0-\Pi_0+\Delta U(0)\|+\sup_{t\in[0,T]}|\Delta\dot u(t)|\int_0^T\|S_A(T-\tau)\|\,d\tau\\ &\leq\|\Theta_T-\Pi_T\|+e^{-\omega T}\big(\|\Theta_0-\Pi_0\|+\|\Delta U(0)\|\big)+\sup_{t\in[0,T]}|\Delta\dot u(t)|\int_0^T e^{-\omega(T-\tau)}d\tau\\ &\leq\left(1+2e^{-\omega T}+\frac{1-e^{-\omega T}}{\omega}\right)\varepsilon\\ &\leq\left(3+\frac{1}{\omega}\right)\varepsilon,\end{aligned}$$

which proves the second claim. □

In other words we have proved with elementary and *constructive* arguments that (1) is *approximately controllable for every time* T, using a regular (smooth, C^2,...) control. Notice we really need to use Gevrey functions of order $1 < \alpha \leq 2$ in the control U: when $\alpha > 2$ the formal control is not a convergent series, while when $\alpha \leq 1$ we cannot build a non-constant function $Y(t)$ with infinitely many zero derivatives (since $Y(t)$ is in this case at least analytic).

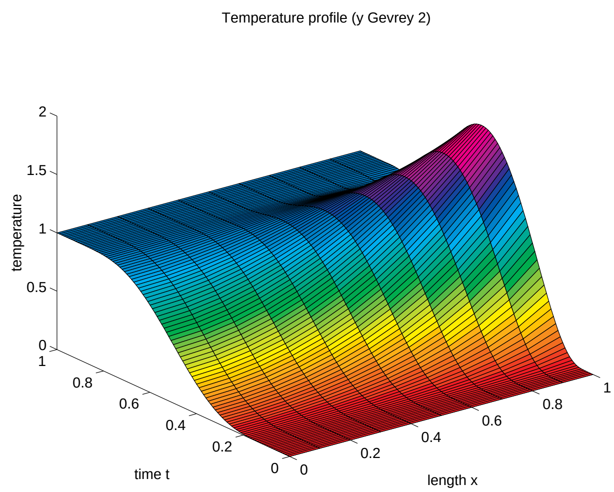

Figure 2: Motion planning using a Gevrey function of order 2.

Figure 2 displays the evolution of the temperature with the control generated by $Y(t) = \Phi_1(t)$, made to steer the system from the uniform profile $\theta = 0$ at $t = 0$ to the uniform profile $\theta = 1$ at $t = 1$ (the approximate control is U truncated to order 10).

5 The linear diffusion equation

We can generalize some of the previous results to the linear diffusion equation

$$\begin{cases} \partial_t\theta = f(x)\partial_{xx}\theta + g(x)\partial_x\theta + h(x)\theta, \quad x \in [0,1] \\ \partial_x\theta(0,t) = 0 \\ \theta(1,t) = u(t), \end{cases}$$

where $f > 0$, g and h are analytic functions.

5.1 The diffusion equation is flat

We first show that $y(t) := \theta(0,t)$ is again a "flat" output. As before, the "inverse" system

$$\begin{cases} \partial_{xx}\theta = \frac{1}{f(x)}\big(\partial_t\theta - g(x)\partial_x\theta - h(x)\theta\big) \\ \partial_x\theta(0,t) = 0 \\ \theta(0,t) = y(t), \end{cases} \tag{9}$$

is in Cauchy-Kovalevskaya form, and we seek a formal solution $\theta(x,t) = \sum_{i=0}^{+\infty} a_i(t)\frac{x^i}{i!}$, where the a_i are smooth functions.

Theorem 3. *The formal solution of (9) is convergent when $y(t)$ is Gevrey of order $1 \leq \alpha \leq 2$.*

Proof. We can assume $f = 1$ and $g = 0$ thanks to the change of variables

$$\begin{aligned} \tilde{x} &:= \int_0^x \frac{1}{f^{\frac{1}{2}}(z)}dz \\ \tilde{\theta}(\tilde{x},t) &:= \frac{\theta(x,t)}{f^{\frac{1}{4}}(x)} \exp\left(\int_0^x \frac{g(z)}{2f(z)}dz\right) \end{aligned} \tag{10}$$

(this is the so-called gauge equivalence of any second order operator to a Schrödinger operator, see [12]). Denoting again by x, θ and h the space variable, the state and the new coefficient, system (9) becomes:

$$\begin{cases} \partial_{xx}\theta = \partial_t\theta - h(x)\theta \\ \partial_x\theta(0,t) = \gamma\theta(0,t) \\ \theta(0,t) = y(t), \end{cases} \tag{11}$$

where γ is a constant coefficient. Writing then $h(x) = \sum_{k\geq 0} h_i \frac{x^i}{i!}$ and using (11), we easily find the a_k are recursively defined by

$$\begin{cases} a_{k+2}(t) = \dot{a}_k(t) - \displaystyle\sum_{i=0}^{k} \frac{k!}{i!(k-i)!} h_{k-i}\, a_i(t) \\ a_0(t) = y(t) \\ a_1(t) = \gamma a_0(t), \end{cases} \tag{12}$$

and we have to show that $|a_k| \leq \frac{\lambda}{\mu^k} k!$ for some $\lambda, \mu > 0$.

The proof is adapted from the classical *method of majorants*: we first replace the sequence a_k by a "majorizing" sequence A_k (in the sense of lemma 1) such that

$$|a_k| \leq \big(A_k(0)\big)^\alpha.$$

This sequence is initialized with $A_0 = A_1 = A$, where

$$\forall t \in [0, r[, \quad A(t) := \frac{m}{1 - \frac{t}{r}},$$

with $m, r > 0$. A obviously satisfies

$$A^{(k)} = \frac{m\ k!}{r^k \left(1 - \frac{t}{r}\right)^{k+1}}$$

and enjoys a nice differential property (lemma 2).

We then estimate the growth of the A_k in terms of the derivatives of A (lemma 3),

$$A_{2k}, A_{2k+1} \leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!} \frac{A^{(k)}}{\rho^k}.$$

We finally conclude $|a_{2k}|, |a_{2k+1}| \leq \frac{\tilde{m}}{(\tilde{r}\tilde{\rho})^k}(2k)!$, which proves the claim. Here and in the sequel we denote $m^\alpha, r^\alpha, \ldots$ by $\tilde{m}, \tilde{r}, \ldots$. ☐

Lemma 1. $\forall \alpha \geq 1$,

$$\begin{cases} A_{k+2} = \dot{A}_k + \displaystyle\sum_{i=0}^{k} \left(\frac{M}{R^{k-i}} \frac{k!}{i!}\right)^{\frac{1}{\alpha}} A_i \\ A_0 = A \\ A_1 = A \end{cases}$$

is a majorant problem of (12) in the sense that

$$\forall k, n \geq 0, \quad \left|a_k^{(n)}\right| \leq \left(A_k^{(n)}(0)\right)^\alpha.$$

Proof. The claim is true at steps 0 and 1 since y is by assumption Gevrey of order α. Assuming it is true till step

$k+1$, we prove it is true at step $k+2$; indeed, since h is analytic,

$$\begin{aligned}\left|a_{k+2}^{(n)}\right| &= \left|\dot{a}_k^{(n)} + \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} a_i^{(n)} h_{k-i}\right| \\ &\leq \left|a_k^{(n+1)}\right| + \sum_{i=0}^{k} \frac{M}{R^{k-i}} \frac{k!}{i!} \left|a_i^{(n)}\right| \\ &\leq \left(A_k^{(n+1)}(0)\right)^{\alpha} + \sum_{i=0}^{k} \frac{M}{R^{k-i}} \frac{k!}{i!} \left(A_i^{(n)}(0)\right)^{\alpha} \\ &\leq \left(A_k^{(n+1)}(0) + \sum_{i=0}^{k} \frac{\tilde{M}}{\tilde{R}^{k-i}} \left(\frac{k!}{i!}\right)^{\frac{1}{\alpha}} A_i^{(n)}(0)\right)^{\alpha} \\ &= \left(A_{k+2}^{(n)}(0)\right)^{\alpha}.\end{aligned}$$

Notice $\sum_q |L_q|^{\alpha} \leq (\sum_q |L_q|)^{\alpha}$ when $\alpha \geq 1$. □

Lemma 2. $\forall n \geq 0, k \geq j \geq 0$,

$$A^{(j+n)} \leq \frac{j!}{k!} r^{k-j} A^{(k+n)}.$$

Proof. As $D(t) := \frac{1}{1-\frac{t}{r}} \leq 1$ on $[0, r[$,

$$\begin{aligned}A^{(j+n)} &= \frac{m\ (j+n)!}{r^{j+n} D^{j+n+1}} \\ &\leq \frac{m\ (j+n)!}{r^{j+n} D^{k+n+1}} \\ &= \frac{(j+n)!}{(k+n)!} r^{k-j} A^{(k+n)} \\ &\leq \frac{j!}{k!} r^{k-j} A^{(k+n)}. \quad \square\end{aligned}$$

Lemma 3. $\forall \alpha \leq 2, k \geq 0, n \geq 0$,

$$A_{2k}^{(n)}, A_{2k+1}^{(n)} \leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!} \frac{A^{(k+n)}}{\rho^k},$$

where $\dfrac{1}{\rho} := \max(\dfrac{r}{\tilde{R}^2}, 1 + r\tilde{M}, r\tilde{M}\tilde{R})$.

Proof. The claim is obvious at step 0 since $A_0 = A_1 = A$. Assume then it is true till step k. By definition,

$$\begin{aligned}A_{2(k+1)}^{(n)} &= \dot{A}_{2k}^{(n)} + \sum_{i=0}^{2k} \underbrace{\frac{\tilde{M}}{\tilde{R}^{2k-i}} \left(\frac{(2k)!}{i!}\right)^{\frac{1}{\alpha}}}_{T_i} A_i^{(n)} \\ &= \dot{A}_{2k}^{(n)} + \sum_{j=0}^{k} T_{2j} A_{2j}^{(n)} + \sum_{j=0}^{k-1} T_{2j+1} A_{2j+1}^{(n)}\end{aligned}$$

Using successively the induction assumption, lemma 2 and $\rho \leq \tilde{R}^2/r$, we find

$$\begin{aligned}T_{2j} A_{2j}^{(n)} &\leq \frac{\tilde{M}}{\tilde{R}^{2k-2j}} \frac{(2k)!^{\frac{1}{\alpha}}}{j!} \frac{A^{(j+n)}}{\rho^j} \\ &\leq \frac{\tilde{M} r}{\rho^j} \left(\frac{r}{\tilde{R}^2}\right)^{k-j} \frac{(2k)!^{\frac{1}{\alpha}}}{(k+1)!} A^{(k+1+n)} \\ &\leq \frac{\tilde{M} r}{\rho^k} \frac{(2k)!^{\frac{1}{\alpha}}}{(k+1)!} A^{(k+1+n)}.\end{aligned}$$

Similarly,

$$T_{2j+1}A_{2j+1}^{(n)} \leq \frac{\tilde{M}\tilde{R}r}{\rho^k}\frac{(2k)!^{\frac{1}{\alpha}}}{(k+1)!}A^{(k+1+n)}.$$

On the other hand the induction assumption implies

$$\dot{A}_{2k}^{(n)} \leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!}\frac{rA^{(k+1+n)}}{\rho^k}.$$

Hence,

$$\begin{aligned} A_{2k+2}^{(n)} &\leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!}\frac{A^{(k+1+n)}}{\rho^k}(1 + r\tilde{M} + r\tilde{M}\tilde{R}) \\ &\leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!}\frac{A^{(k+1+n)}}{\rho^{k+1}} \\ &\leq \frac{(2k+2)!^{\frac{1}{\alpha}}}{(k+1)!}\frac{A^{(k+1+n)}}{\rho^{k+1}}. \end{aligned}$$

Notice $k \mapsto \frac{(2k)!^{\frac{1}{\alpha}}}{k!}$ is increasing when $\alpha \leq 2$.

The proof is the same for the odd terms $A_{2k+1}^{(n)}$. □

It is possible though rather tedious to extend theorem 1 to the present case.

5.2 Rest-to-rest motion

We briefly sketch here how to use the previous result to steer the diffusion equation from a rest profile to another rest profile. A rest profile $\bar{\theta}$ is characterized by

$$\bar{\theta}(x) = \lambda\theta_0(x),$$

where $\lambda \in \mathbb{R}$ and θ_0 is the solution of

$$\begin{aligned} f(x)\theta_0''(x) + g(x)\theta_0'(x) + h(x)\theta_0(x) = 0 \\ \theta_0(0) = 1 \\ \theta_0'(0) = 0. \end{aligned}$$

This is equivalent to all the $a_k(t)$ in (12) being constant or alternatively to all the derivatives of $y(t)$ being 0.

Hence the open-loop control $U(t) := \sum_{k\geq 0} \frac{a_{k(t)}}{k!}$ built from the Gevrey function

$$Y(t) := \lambda + (\mu - \lambda)\cdot\Phi_\gamma(t/T), \qquad \gamma \geq 1,$$

will steer the system from the rest profile $\lambda\theta_0$ at time 0 to the rest profile $\mu\theta_0$ at time T, and an approximate control will steer the system to a neighbourhood of $\mu\theta_0(x)$.

5.3 Further generalization

The previous results could also be extended to linear evolution equations with analytic coefficients of the form

$$\begin{cases} \partial_t^m\theta(x,t) = \sum_{j=0}^{n}\lambda_j(x)\ \partial_x^j\theta(x,t), \quad x \in [0,1] \\ \partial_x^k\theta(1,t) = u(t), \quad k \in \{0,\dots,n-1\} \\ \partial_x^j\theta(0,t) = 0, \qquad j = 1,\dots,n-1, j \neq k. \end{cases}$$

provided $n > m$ (i.e., the order of differentiation in space is strictly greater than the order of differentiation in time), using Gevrey functions of order not greater than n/m. Of course the computations and estimations of the growth of the coefficients are more tedious, but the key point is that all the boundary conditions except the control input are on the same side, so that the inverse problem

$$\begin{cases} \partial_t \theta(x,t) = \mu(x) + \sum_{j=0}^{n} \lambda_j(x)\ \partial_x^j \theta(x,t) \\ \partial_x^k \theta(0,t) = y(t), \quad k \in \{0,\dots,n-1\} \\ \partial_x^j \theta(0,t) = 0, \quad j = 1,\dots,n-1, j \neq k. \end{cases}$$

is in Cauchy-Kovalevskaya form and

$$y(t) := \partial_x^k \theta(0,t)$$

is a flat output (see [11] for an example of an elastic nonlinear control system).

6 Computations with divergent series

When $\alpha > 2$, the series:

$$\theta(x,t) = \sum_{i=0}^{+\infty} a_i(t) \frac{x^i}{i!}$$

is in general *divergent* (with order $\alpha/2$) and has a priori no meaning, though it still defines a *formal* 1–1 correspondence between arbitrary curves $y(t)$ and solution of system (1).

Guided by the Ramis-Sibuya theorem for ordinary differential equations [15], we wonder whether there exists a *smooth solution* $\hat{\theta}$ which is *Gevrey asymptotic of order* β to θ, i.e., whether there exists $C, A > 0$ such that for all $n > 1$,

$$|x|^{-n} \sup_t \left| \theta(x,t) - \sum_{i=0}^{n-1} a_i(t) \frac{x^i}{i!} \right| \leq C(n!)^\beta A^i. \tag{13}$$

In general, such divergent series first converges very fast, and then diverges very fast. Using only the convergent part, i.e., a “smallest term summation”, is numerically very effective and leads to exponentially small error [15]. Indeed for x small of order ϵ, simple estimations via the Stirling formula show that, when the number of terms is chosen to $n \approx 1/(A\epsilon)^{1/\beta}$, we have

$$\begin{aligned} \sup_t \left| \hat{\theta}(x,t) - \sum_{i=0}^{n-1} a_i(t) \frac{x^i}{i!} \right| &\leq C(2\pi)^{1/\beta} \exp(-\beta n) \\ &\sim C(2\pi)^{1/\beta} \exp\left(\frac{-\beta}{(A\epsilon)^{1/\beta}} \right). \end{aligned}$$

This correspond to a ”smallest term summation”: the a_i begin to grow for $i \geq n$.

We thus performed (without any theoretical justification) some numerical experiments on the heat equation: as in section 4 we tried to steer the temperature from the uniform profile $\theta = 0$ at $t = 0$ to the uniform profile $\theta = 1$ at $t = 1$, but using this time $y = \Phi_{2/3}$, i.e., a Gevrey function of order $\alpha = 5/2 > 2$. The open-loop control is obtained by performing for every t a kind of “smallest term summation”,

$$\tilde{U}(t) = \sum_{i=0}^{n_t} \frac{\Phi_{2/3}^{(i)}(t)}{(2i)!} + \frac{1}{\lambda} \sum_{i=1}^{n_t} \frac{\Phi_{2/3}^{(i)}(t)}{(2i-1)!},$$

with n_t defined by

$$\left| \frac{\Phi_\gamma^{(n_t)}(t)}{(2n_t)!} \right| = \min_i \left| \frac{\Phi_\gamma^{(i)}(t)}{(2i)!} \right|.$$

As can be seen on figure 3, the results look numerically correct while the steering control is much “softer”: the maximum is around 1.4, to be compared with 2.0 in the case of Gevrey order 2 (see the figure in section 4). Moreover, we have observed that when y is Gevrey $1 < \alpha \leq 2$, i.e., when the formal solution is convergent, the smaller α is, the more “bumpy” the control. These numerical experiments seem to indicate that divergent series provide more natural and smooth transition profiles than convergent ones.

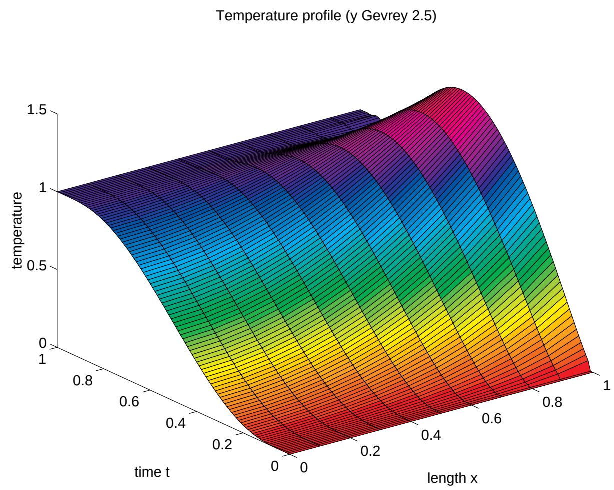

Figure 3: Motion planning using a Gevrey function of order 2.5.

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