

# Flatness and stick-slip stabilization

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The origin of this report lies in the very interesting discussions with Challamel of the CGES at Ecole des Mines de Paris and with Hugues Mounier of Ecole des Mines de Nantes.

## 1 Introduction

We consider here the flexible system (torsion dynamics) displayed on figure 1. After some scaling the system is described by

$$\begin{aligned}\partial_t^2 \theta &= \partial_x^2 \theta, & x \in [0, 1] \\ \partial_x \theta(0, t) &= -u(t) \\ \partial_x \theta(1, t) &= F(\partial_t \theta(1, t))\end{aligned}\tag{1}$$

where

- $[0, 1] \ni x \mapsto \theta(x, t)$  is the torsion profile at time  $t$ ,
- $u$  is the control (the torque at the top),
- $v \mapsto F(v)$  is the friction law at the bottom.

To each angular velocity  $\omega_r$  corresponds a uniform rotation at constant speed,

$$\theta(x, t) = a + \omega_r t + F(\omega_r)x, \quad u = -F(\omega_r),$$

solution of (1).

Challamel has noticed that, when  $dF/dv(\omega_r) > 0$ , this stationary motion is unstable. Such an unstable rotation coincides, as displayed on figure 1, with

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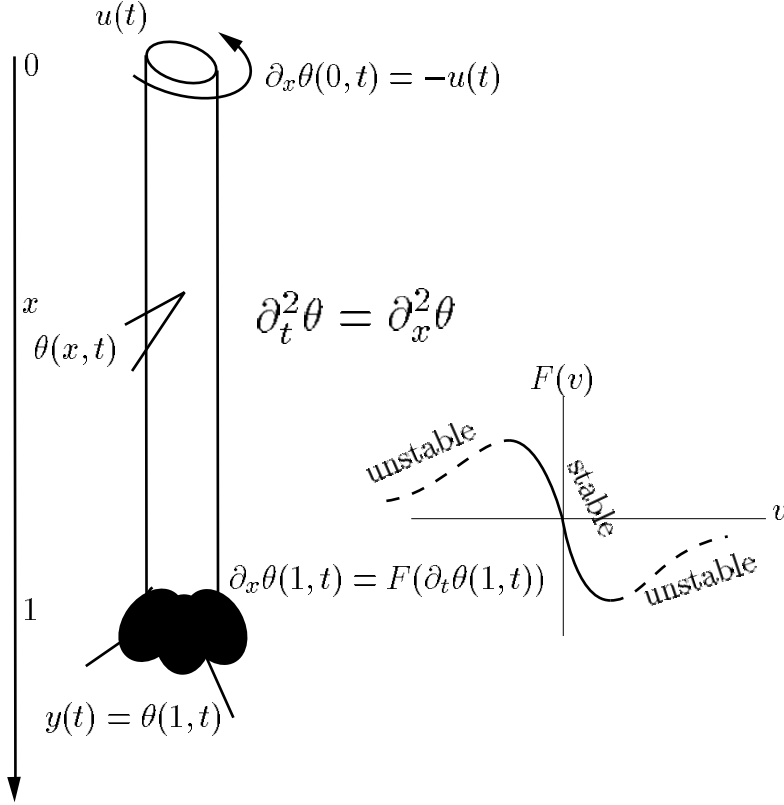


Figure 1: The drilling system with torsion stick-slip instabilities

the increasing part of the bottom friction law. This is related to the well-known stick-slip instabilities.

We propose here a method for controlling and stabilizing (1) by using only the top velocity measure:

$$\omega(t) := \partial_t \theta(0, t).$$

The following simple control law (a nonlinear PI control with distributed delays) stabilizes the rotation  $\omega_r$ :

$$u(t) = -\omega(t) + v(t) - F(v(t)) \quad (2)$$

with

$$v(t) = \omega_r(t) - \lambda I(t) - \frac{\lambda}{2} \left( \int_{t-2}^t v(\tau) d\tau \right), \quad \dot{I}(t) = \omega(t) - \omega_r(t). \quad (3)$$

In (3),  $\omega_r(t)$  is the velocity set-point that can be time-varying. The design parameter  $\lambda$  must be positive and ensures a closed-loop dynamics for the bottom angle  $y(t) = \theta(1, t)$ , the flat output of the system, as follows:

$$\frac{d}{dt}(y(t+1) - y_r(t+1)) = -\lambda(y(t+1) - y_r(t+1))$$

where the reference trajectory  $t \mapsto y_r(t)$  is such that  $\dot{y}_r(t+1) = \omega_r(t)$ .

The method used to construct this nonlinear controller is based on the fact that the trajectories of (1) can be explicitly parameterized via the bottom angle  $y(t) = \theta(1, t)$ . As already noticed for a very similar system (see, e.g., [4]), we have the following explicit parameterization:

$$2\theta(x, t) = y(t+x-1) + y(t-x+1) - \int_{t-x+1}^{t+1-x} F(\dot{y}(\tau)) d\tau. \quad (4)$$

This means in particular that stabilizing the entire profile  $x \mapsto \theta(x, t)$  is equivalent to stabilize the bottom position  $y(t)$ , that can be seen as the flat output by analogy with [2, 3].

Notice that (3) involves distributed delay on the auxiliary control  $v$ . This is a nonlinear natural extension of the stabilization method proposed by Brethé and Loiseau for delay systems [1].

In the sequel, we first prove (4), i.e., the flatness of the system. Then we design the stabilizing controller (2,3) using only the top measure  $\omega$ .

Simulations for  $F(v) = v/2$  around  $v = 0$  show the robustness of the approach versus modeling and measure errors. The Matlab simulation code can be obtained upon request at the Email [rouchon@cas.ensmp.fr](mailto:rouchon@cas.ensmp.fr).

## 2 Flatness

The general solution of  $\partial_t^2 \theta = \partial_x^2 \theta$  is given by the d'Alembert formulae:

$$\theta(x, t) = \phi(t+x) + \psi(t-x)$$

where  $\sigma \mapsto \phi(\sigma)$  and  $\sigma \mapsto \psi(\sigma)$  are arbitrary functions corresponding to waves of velocity  $-1$  and  $1$ , respectively. Plugging this into the boundary conditions, we have

$$-u(t) = \phi'(t) - \psi'(t), \quad F(\dot{y}(t)) = \phi'(t+1) - \psi'(t-1)$$

with  $y(t) := \theta(1, t) = \phi(t+1) + \psi(t-1)$  and where  $'$  denotes derivation with respect to  $\sigma$ . Assume now that  $t \mapsto y(t)$  is known. The three above equations

becomes now a square system that can be solved as follows:

$$\begin{aligned} u(t) &= \frac{\dot{y}(t+1) - F(\dot{y}(t+1))}{2} - \frac{\dot{y}(t-1) + F(\dot{y}(t-1))}{2} \\ \phi'(\sigma) &= \frac{\dot{y}(\sigma-1) + F(\dot{y}(\sigma-1))}{2} \\ \psi'(\sigma) &= \frac{\dot{y}(\sigma+1) - F(\dot{y}(\sigma+1))}{2}. \end{aligned}$$

Thus

$$\begin{aligned} \phi(\sigma) &= a + \frac{y(\sigma-1) + \int_0^{\sigma-1} F(\dot{y}(\tau)) d\tau}{2} \\ \psi(\sigma) &= b + \frac{y(\sigma+1) - \int_0^{\sigma+1} F(\dot{y}(\tau)) d\tau}{2} \end{aligned}$$

with  $a$  and  $b$  constants such that  $a + b = 0$  since  $y(t) = \phi(t+1) + \psi(t+1)$ .

Coming back to the deformation profile  $\theta$ , we have the following parameterization:

$$2\theta(x, t) = y(t+x-1) + y(t-x+1) - \int_{t-1+x}^{t+1-x} F(\dot{y}(\tau)) d\tau.$$

This parameterization is explicit with respect to any arbitrary function  $t \mapsto y(t)$  corresponding to the bottom angle trajectory. In the sequel, we will call the bottom angle  $y$  the flat output: to any control  $t \mapsto u(t)$  and initial condition  $(\theta(x, 0), \partial_x \theta(x, 0))_{0 \leq x \leq 1}$  correspond  $t \mapsto y(t)$  by integration of (1). To any flat-output  $t \mapsto y(t)$  corresponds via the above explicit formulae a control  $t \mapsto u(t)$  and a torsion profile  $t \mapsto \theta(x, t)$  satisfying (1). Thus, there is a one-to-one correspondence between the solutions of (1) and  $t \mapsto y(t)$ .

### 3 Motion planning and tracking

Let us consider a reference trajectory  $t \mapsto y_r(t)$  for the flat output  $y$ . Then the reference control  $t \mapsto u_r(t)$  is just given by

$$u_r(t) = \frac{\dot{y}_r(t+1) - F(\dot{y}_r(t+1))}{2} - \frac{\dot{y}_r(t-1) + F(\dot{y}_r(t-1))}{2}.$$

In practice the precise position of the angle is not important. Only velocities are to be considered: we are interested in velocity tracking. We denote by  $t \mapsto \omega_r(t)$  the reference velocity relative to  $\dot{y}_r$ .

For  $\omega_r$  where the derivative of  $F$  is positive, the system is unstable (see figure 1). Thus, the previous open-loop control strategy cannot be used. Stabilizing feedback is necessary. We will construct now a global stabilizing controller around any reference  $t \mapsto \omega_r(t)$ . The feedback depends only on the top velocity measure  $\omega(t) = \partial_t(\theta(0, t))$  commonly available. Stabilizing the system means stabilizing the flat output.

Since

$$\omega(t) = \frac{\dot{y}(t+1) - F(\dot{y}(t+1))}{2} + \frac{\dot{y}(t-1) + F(\dot{y}(t-1))}{2}$$

we have

$$\omega(t) + u(t) = \dot{y}(t+1) - F(\dot{y}(t+1)).$$

Thus setting

$$u(t) = -\omega(t) + v(t) - F(v(t))$$

where  $v(t)$  is the new control yields

$$\dot{y}(t+1) = v(t)$$

when the slope of  $F$  at  $v(t)$  is not equal to 1, i.e. when  $v \mapsto v - F(v)$  is locally a bijection. Notice that when  $dF/dv$  is close to 1, the system is strongly unstable: for  $F(v) = (1 + \epsilon)v$  with  $\epsilon$  small and positive, the spectrum of the Cauchy problem (1),

$$\{0\} \cup \left\{ -\frac{1}{2} \log \left( \frac{\epsilon}{2 + \epsilon} \right) + n\pi\sqrt{-1}, \quad n \in \mathbb{Z} \right\},$$

admits eigenvalues with very large positive real parts.

Consider  $y_r(t+1)$  such that  $\dot{y}_r(t+1) = \omega_r(t)$ . Set

$$v(t) = \omega_r(t) - \lambda(y(t+1) - y_r(t+1)) \tag{5}$$

with  $\lambda > 0$ . Then the closed-loop error dynamics are stable:

$$\dot{y} - \dot{y}_r = -\lambda(y - y_r).$$

The problem is now to predict the future value  $y(t+1)$  with respect to the measure  $\omega$  and control  $u$  at times  $\leq t$ . From  $\dot{y}(t) = v(t-1)$  we obtain

$$y(t+1) = y(t-1) + \int_{t-2}^t v(\tau) d\tau.$$

Since  $\theta(0, t) = (y(t+1) - y(t-1))/2$ , we can eliminate  $y(t-1)$  to have

$$y(t+1) = \theta(0, t) + \frac{1}{2} \int_{t-2}^t v(\tau) d\tau.$$

Thus (5) reads

$$v(t) = \omega_r(t) - \lambda(\theta(0, t) - y_r(t+1)) - \frac{\lambda}{2} \int_{t-2}^t v(\tau) d\tau.$$

Since  $\dot{y}_r(t+1) = \omega_r(t)$  and  $\partial_t \theta(0, t) = \omega(t)$ , we can compute  $\theta(0, t) - y_r(t+1)$  as an error integral:

$$I := \theta(0, t) - y_r(t+1), \quad \dot{I} = \omega(t) - \omega_r(t).$$

Finally we have obtained the nonlinear PI controller with distributed delays corresponding to (2,3).

Notice that standard Lyapounov technic (passivity arguments based of the mechanical energy) cannot be used directly here: the time derivative of mechanical based Lyapounov functions is the sum of two terms; the control appears only in one term; no obvious factorization can be performed.

## 4 Simulations

The simulations of figures 2, 3 and 4 correspond to the stabilization of (1) around  $\omega_r = 0$  and with  $F(v) = v/2$ . These simulations illustrate the robustness of the control. The typical open-loop instability time-constant is around 4 time unit. For the simulation we consider an approximation with 20 mass/spring small systems.

For the control computation, we use a sampling time of 0.05 time unit. Thus the integral in (3) is computed with  $2/0.05 = 40$  values of  $v$ . We also introduce an error in the friction law in (2) by using  $F(v) = 0.55 v$  instead of  $F(v) = v/2$ . We also add measure errors: the control is computed with a wrong top velocity,  $0.9 \omega + 0.2$ , instead of  $\omega$ .

## References

- [1] David Brethé. *Contribution à l'étude de la stabilisation des systèmes linéaires à retards*. PhD thesis, Ecole Centrale de Nantes, France, 1997.
- [2] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.
- [3] Ph Martin, R. Murray, and P. Rouchon. Flat systems. In *Proc. of the 4th European Control Conf.*, pages 211–264, Brussels, 1997. Plenary lectures and Mini-courses.
- [4] H. Mounier. *Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques*. PhD thesis, Université Paris Sud, Orsay, 1995.

## Simulation figures

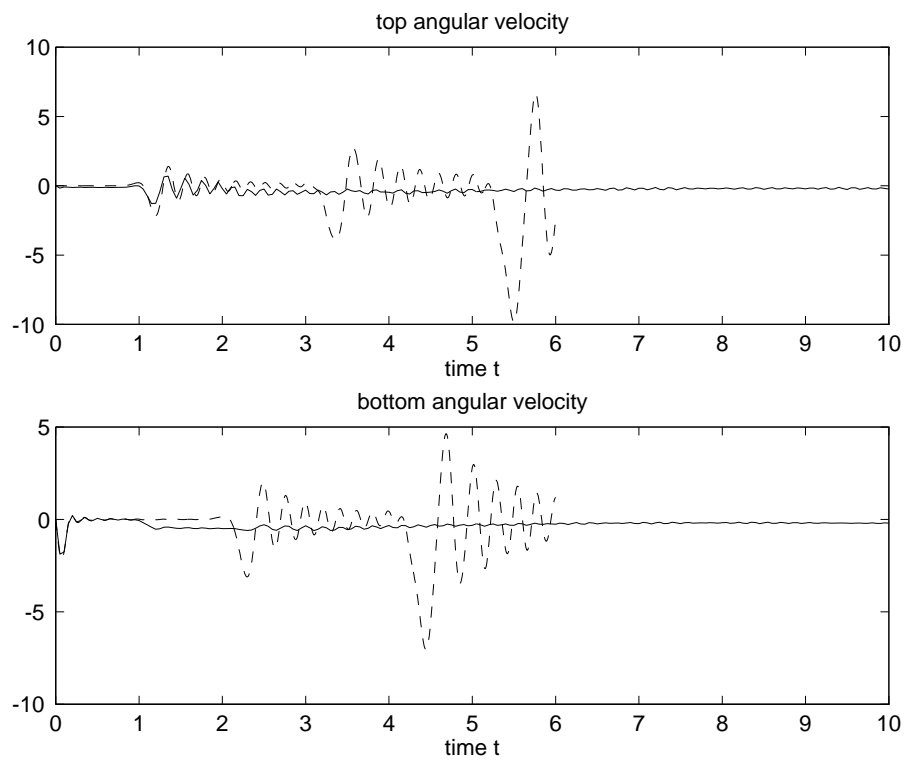


Figure 2: open-loop  $u = 0$  (dashed line) and closed-loop velocities.

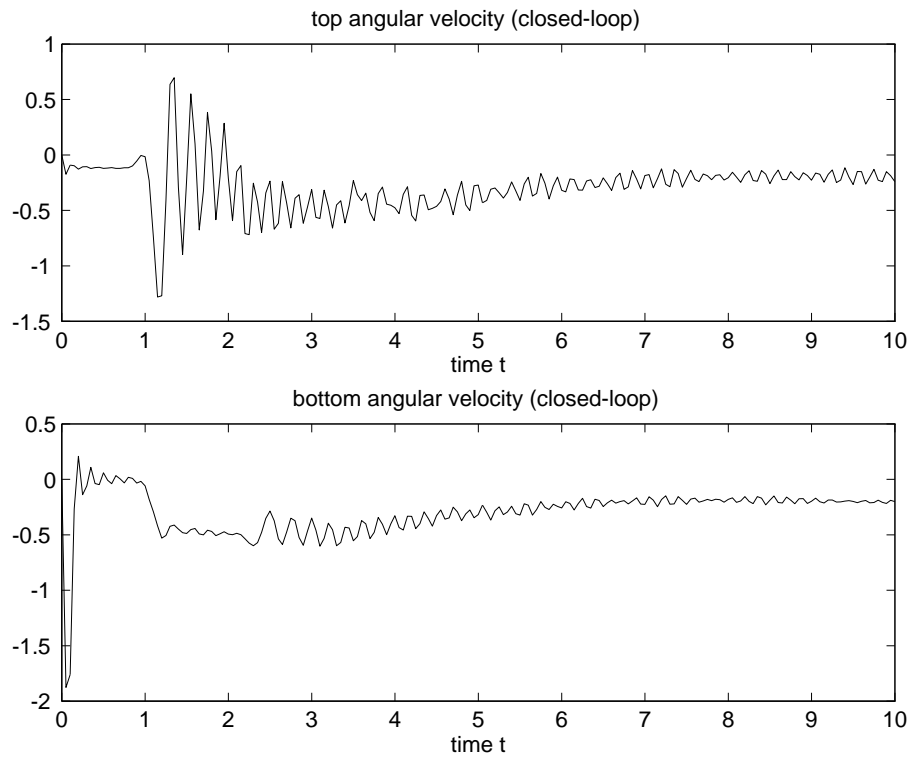


Figure 3: closed-loop velocities.



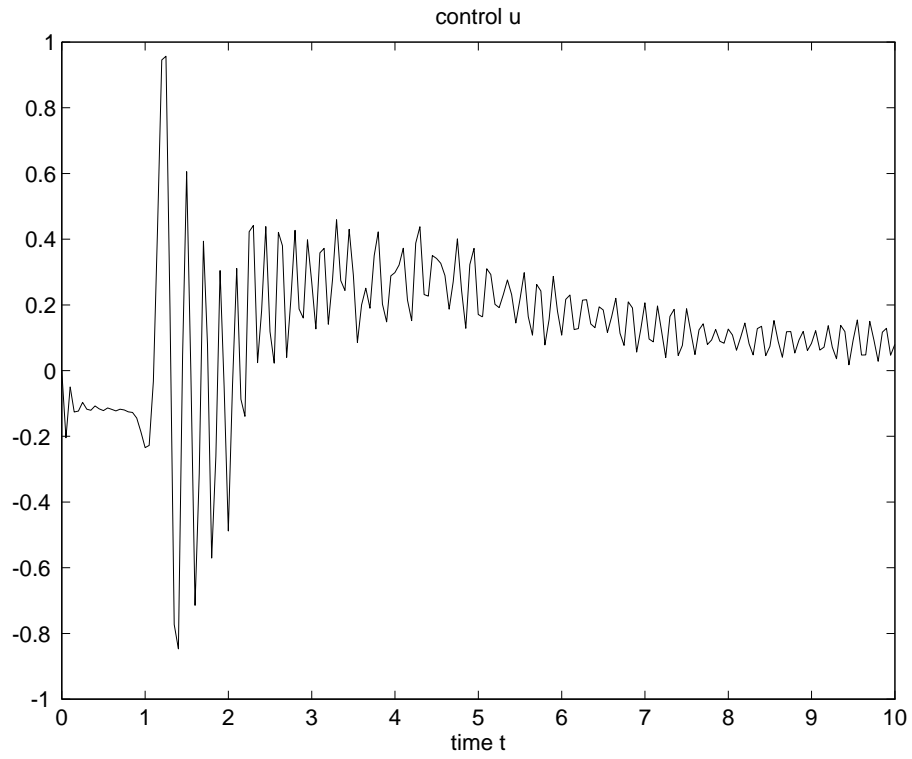


Figure 4: the stabilizing control  $u$ .