A characterization of analytic ruled surfaces

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We address the following question. Consider a sub-manifold of an affine space, defined by its equations $F = 0$: *does there exist a finite characterization of ruled sub-manifolds in terms of derivatives of F via algebraic inequalities and/or equalities ?* This question is related to an open problem in control theory: the finite characterization of flat systems [2] and, more generally, of systems linearizable via dynamic feedback [1]. This note has been motivated by interesting discussions and electronic mails with François Labourie. It presents a characterization of analytic ruled surfaces of $\mathbb{R}^3$: the second and third derivatives of F satisfy one algebraic inequality and one algebraic equality.

Main result

Theorem *Consider a surface Σ of $\mathbb{R}^3$ defined by $x_3 = f(x_1, x_2)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is analytic. The surface Σ is ruled if, and only if, for all $x \in \mathbb{R}^2$, the two following conditions are satisfied:*

- **C1***:* $\det(D^2 f) \leq 0$.
- **C2***: resultant of* $\left\{D^2 f[X, X], D^3 f[X, X, X]\right\} = 0$.

where $X = (X_1, X_2)^T$ are formal variables,

$$D^2 f[X, X] = \sum_{i,j=1}^{2} f_{ij} X_i X_j, \quad D^3 f[X, X, X] = \sum_{i,j,k=1}^{2} f_{ijk} X_i X_j X_k$$

with $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ *and* $f_{ijk} = \dfrac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}$*, for* $i, j, k = 1, 2$.

Resultant calculations for two polynomials can be found in [7, page 84]. The resultant of $D^2 f[X, X]$ and $D^3 f[X, X, X]$ is a real polynomial in the f_{ij}'s and f_{ijk}'s, homogeneous of degree 3 in the f_{ij}'s and homogeneous of degree 2 in the f_{ijk}'s. This resultant is equal to zero, if, and only if, the equations $D^2 f[X, X] = 0$ and $D^3 f[X, X, X] = 0$ admit a non trivial common solution $X = a \in \mathbb{C}^2 / \{0\}$ (see [8] for more details).

Proof

According to [4, proof of theorem 2], **C1** and **C2** are clearly necessary. We will prove now that these conditions are sufficient. Assume that **C1** and **C2** hold.

Case 1

Assume that $\det(D^2 f) \equiv 0$. When $D^2 f \equiv 0$, Σ is an affine space and thus is ruled. Assume additionally that $D^2 f \neq 0$. Since f is analytic, for almost every $x \in \mathbb{R}^2$ (excepted a countable set of isolated points) $D^2 f(x)$ is of rank 1.

Consider $x \in \mathbb{R}^2$ such that $\operatorname{rank} D^2 f(x) = 1$. Then there exists $(a_1(x), a_2(x)) \in \mathbb{R}^2/\{0\}$ such that,

$$D^2 f(x) \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} = 0$$

where, moreover, $x \to a(x)$ is analytic and defined locally around x ($a(x)$ belongs to the kernel of the linear operator $D^2 f(x)$ that depends analytically on x and is of rank 1 around x). We will see that for $k \geq 2$,

$$D^k f(x) \left[\underbrace{\begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix}, \ldots, \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix}}_{k-1 \text{ times}}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right] = 0. \tag{1}$$

The derivation with respect to x of the identity,

$$D^2 f(x) \left[\begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right] = 0$$

leads to

$$D^3 f(x) \left[\begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right] + D^2 f(x) \left[\begin{pmatrix} Da_1(x) \cdot Y \\ Da_2(x) \cdot Y \end{pmatrix}, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right] = 0$$

where $Y = (Y_1, Y_2)$ corresponds to formal variables independent of X. By taking $X_1 = a_1(x)$ and $X_2 = a_2(x)$, we obtain (1) for $k = 3$. An additional derivation leads to (1) for $k = 4$, ... It appears clearly that (1) can be proved by induction on k via successive derivations. For all $k \geq 2$, (1) implies that $D^k f(x)[a(x), \ldots, a(x)] = 0$.

When $\operatorname{rank} D^2 f(x) = 1$, there exists a straight line passing through $(x, f(x))$ and included in Σ: its direction is given by the vector $(a(x), Df(x)[a(x)]) \neq 0$. (the analytic function $\mathbb{R} \ni \lambda \to f(x + \lambda a(x))$ is linear since all its derivatives of order ≥ 2 are equal to 0 for $\lambda = 0$).

By density and compacity arguments, one can prove that, even if x is one of the isolated points where $D^2 f(x) = 0$, there exists a straight line passing through $(x, f(x))$ and contained in Σ. It suffices to consider a series $(x^n)_{n \geq 0}$ converging to x such that $D^2 f(x^n) \neq 0$ (density). Up to an extraction of a convergent sub-series, the corresponding series of directions $(a^n)_{n \geq 0}$ with $\|a^n\| = 1$ converges to $a \neq 0$ (compacity of S^1). Then, $(a, Df(x)[a])$ gives the direction of a straight line included in Σ and passing through x.

Case 2

Assume that $\det(D^2 f) \neq 0$. Since f is analytic, for almost every $x \in \mathbb{R}^2$ (up to a countable set of isolated points) $\det(D^2 f)(x) < 0$.

Consider x such that $\det(D^2 f)(x) < 0$. We have the following decomposition

$$D^2 f[X, X] = M(X)\ N(X) \tag{2}$$

where the homogeneous polynomials of degree 1, $M(X)$ and $N(X)$, correspond to independent linear forms with real coefficients that are analytical functions of x. For $Q(X)$, a polynomial those coefficients are C^1 functions of x, we denote by $Q'(X, Y)$ the polynomial obtained via derivation with respect to x:

$$Q'(X, Y) \stackrel{\text{def}}{=} \frac{\partial}{\partial x}(Q(X)) \cdot Y$$

(as X, $Y = (Y_1, Y_2)^T$ corresponds to formal variables).

By **C2**, $D^3 f[X, X, X]$ admits a common non zero root with $D^2 f[X, X]$. Thus, $D^3 f[X, X, X]$ can be divided by M or N (M and N are of degree 1), says M for example. This gives

$$D^3 f[X, X, X] = A_3(X) M(X)$$

where $A_3(X)$ is an homogeneous polynomial of degree 2. Derivation of (2) with respect to x gives:

$$D^3 f[X, X, Y] = M'(X, Y) N(X) + M(X) N'(X, Y).$$

This implies that $M'(X, X)$ becomes 0 when $M(X)$ becomes 0. Since the degree of M is equal to 1, we have necessarily $M'(X, X) = B(X) M(X)$ where $B(X)$ is of degree 1 (in this special case, the exponent of Hilbert's "Nullstellenstatz" equals 1).

We have obtained

$$D^3 f[X, X, X] = A_3(X)\ M(X), \quad M'(X, X) = B(X) M(X).$$

Derivation with respect to x leads to

$$\begin{aligned} D^4 f[X, X, X, X] &= A_3'(X, X)\ M(X) + A_3(X) M'(X, X) \\ &= (A_3'(X, X) + A_3(X) B(X))\ M(X) = A_4(X) M(X). \end{aligned}$$

By continuing this process, we have, for $k \geq 3$,

$$D^k f[X, \ldots, X] = A_k(X) M(X)$$

where $A_k(X) = A_{k-1}'(X, X) + A_{k-1}(X) B(X)$ is an homogeneous polynomial of degree $k - 1$. Take $a(x) \in \mathbb{R}^2/\{0\}$ such that $M(a(x))$. Then, for all $k \geq 2$, $D^k f(x)[a(x), \ldots, a(x)] = 0$.

It results that, when $\det(D^2 f)(x) < 0$, the straight line passing through $(x, f(x))$ with direction $(a(x), Df(x)[a(x)])$ belongs to Σ.

Similarly to case 1, one can proved that, even if x is one of the isolated points where $\det(D^2 f)$ becomes 0, there exists a straight line passing through $(x, f(x))$ and contained in Σ. ■

We feel that this result can be extended to obtain semi algebraic characterization of ruled sub-manifolds. The tricks introduced during the proof rely on some, probably already existing, mathematical developments. Extensions to C^∞ functions f seem also possible but will certainly require several technical precisions that would complicate the presentation.

1 Control implications

In [4], it is proved that, if the control system

$$\dot{z} = f(z, u)$$

is flat or linearizable via dynamic feedback, then, for each z, the projection onto the affine space of $\dot{z}$ of the sub-manifold $\{(\dot{z}, u)) \mid (\dot{z} = f(z, u)\}$ is a ruled sub-manifold (see also [3, 5]). This means that, when $z = (z_1, z_2, z_3)$, $u = (u_1, u_2)$ and

$$\left\{ \begin{array}{rcl} \dot{z}_1 & = & u_1 \\ \dot{z}_2 & = & u_2 \\ \dot{z}_3 & = & f(z, u_1, u_2), \end{array} \right. \qquad (3)$$

the surface Σ_z of $\mathbb{R}^3$ defined by the equation $x_3 = f(z, x_1, x_2)$ is ruled, for all z.

For an analytic control system of form (3), flatness implies that Σ_z is ruled. According to the theorem here above and the resultant form after Sylvester, we have the following semi-algebraic characterization: for all u and z,

$$(f_{12})^2 - f_{11} f_{22} \geq 0 \quad \text{and} \quad \begin{vmatrix} f_{11} & 2f_{12} & f_{22} & 0 & 0 \\ 0 & f_{11} & 2f_{12} & f_{22} & 0 \\ 0 & 0 & f_{11} & 2f_{12} & f_{22} \\ f_{111} & 3f_{112} & 3f_{122} & f_{222} & 0 \\ 0 & f_{111} & 3f_{112} & 3f_{122} & f_{222} \end{vmatrix} = 0 \qquad (4)$$

where $f_{ij} = \dfrac{\partial^2 f}{\partial u_i \partial u_j}$, $\quad f_{ijk} = \dfrac{\partial^3 f}{\partial u_i \partial u_j \partial u_k}$ for $i, j, k = 1, 2$.

We feel that there must exist a characterization of flatness in finite terms (semi-algebraic set in a suitable jet-space). For (3), conditions (4) are necessary flatness conditions. Clearly, they are not sufficient since they do not imply controllability (take $f = 0$). *If one completes (4) with controllability conditions such as, for example, the strong accessibility ones [6], does one obtain a necessary and sufficient flatness condition ?* This question is still open.

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