Adiabatic elimination for bipartite open quantum systems

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Joint work with R. Azouit, F. Chittaro and A. Sarlette (arXiv.1704.00785)

Slow/fast bipartite master equations

Model reduction and geometric singular perturbations

Geometric singular perturbations for bipartite quantum systems

Several contributions on adiabatic elimination (partial list)

- Lambda systems :
  E. Brion, L.H. Pedersen, K. Mølmer : Adiabatic elimination in a lambda system Journal of Physics A : Mathematical and Theoretical, 2007, 40, 1033.
  M. Mirrahimi, PR :. Singular perturbations and Lindblad-Kossakowski differential equations IEEE Trans. Automatic Control , 2009, 54, 1325-1329
  F. Reiter, A. Sørensen : Effective operator formalism for open quantum systems Phys. Rev. A, 2012, 85, 032111-
- Slow/fast Lindblad dynamics :
  E.M. Kessler : Generalized Schrieffer-Wolff formalism for dissipative systems. Phys. Rev. A, 2012, 86, 012126-
  D. Burgarth et al. : Non-Abelian Phases from a Quantum Zeno Dynamics. Phys. Rev. A 88, 042107 (2013)
  P. Zanardi, L. Campos Venuti : Coherent quantum dynamics in steady-state manifolds of strongly dissipative systems. Phys. Rev. Lett. 113, 240406 (2014)
  K. Macieszczak, M. Guta, I. Lesanovsky,J.P. Garrahan : Towards a Theory of Metastability in Open Quantum Dynamics. Phys. Rev. Lett. 116, 240404 (2016)
  L. Campos Venuti, P. Zanardi : Dynamical Response Theory for Driven-Dissipative Quantum Systems. Phys. Rev. A 93, 032101 (2016)
- Quantum stochastic models :
  J. Gough, R. van Handel : Singular perturbation of quantum stochastic differential equations with coupling through an oscillator model. J. Stat. Phys. 2007, 127 pp :575.
  L. Bouten, A. Silberfarb : Adiabatic elimination in quantum stochastic model, Commun. Math. Phys., 283, 491-505 (2008)
  L. Bouten, R. van Handel, A. Silberfarb : Approximation and limit theorems for quantum stochastic models with unbounded coefficients. Journal of Functional Analysis 254 (2008) 3123-3147.
  O. Cernotik, D. Vasilyev, K. Hammerer : Adiabatic elimination of Gaussian subsystems from quantum dynamics under continuous measurement Phys. Rev. A, , 92, 012124 (2015)

- Sub-system A with Hilbert space $\mathcal{H}_A$ relaxing rapidly towards a unique equilibrium density operator $\overline{\rho}_A$ via the Lindbladian evolution :

$$\frac{d}{dt}\rho_A = \mathcal{L}_A(\rho_A) \text{ with } \lim_{t\mapsto+\infty} \rho_A(t) = \overline{\rho}_A;$$

- Sub-system B with Hilbert space $\mathcal{H}_B$ having a slow Lindbladian evolution

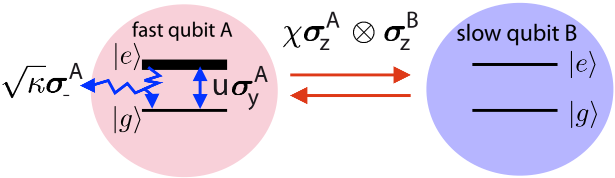
$$\frac{d}{dt}\rho_B = \epsilon\mathcal{L}_B(\rho_B) \text{ with } 0 \leq \epsilon \ll 1$$

- Weak (A, B) coupling via the Hamiltonian $\epsilon\sum_{k=1}^m \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger$

Bipartite Hilbert space (A, B) with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and density operator ρ governed by

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) - i\epsilon\left[\sum_{k=1}^m \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger \, , \, \rho\right] + \epsilon\mathcal{L}_B(\rho)$$

Adiabatic elim. of fast qubit A dispersively coupled to slow qubit B [1]



The slow/fast dynamics

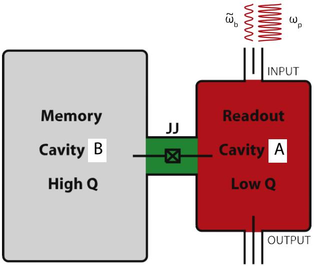
$$\frac{d\rho}{dt} = u\left[\sigma_+^A - \sigma_-^A \,,\, \rho\right] + \kappa\left(\sigma_-^A \rho \sigma_+^A - \frac{\sigma_+^A\sigma_-^A\rho + \rho\sigma_+^A\sigma_-^A}{2}\right) - i\chi\left[\sigma_z^A \otimes \sigma_z^B, \rho\right]$$

Slow dynamics (second order versus $\epsilon = \chi/\kappa$) :

$$\frac{d\rho_s}{dt} = i\frac{\chi\kappa^2}{\kappa^2+8u^2}[\sigma_z, \rho_s] + \frac{(64\kappa\chi^2u^2)(\kappa^2+2u^2)}{(\kappa^2+8u^2)^3}\left(\sigma_z\rho_s\sigma_z - \rho_s\right)$$

Kraus (CPTP) map : $\rho = (I - iQ \otimes \sigma_z)(\overline{\rho}_A \otimes \rho_s)(I + iQ^\dagger \otimes \sigma_z)$ with
$\overline{\rho}_A = \frac{4\kappa u}{\kappa^2+8u^2}\sigma_x - \frac{\kappa^2}{\kappa^2+8u^2}\sigma_z + \frac{1}{2}I$ and $Q = \bullet\sigma_x + \bullet\sigma_y + \bullet\sigma_z + \bullet I$

1. R. Azouit, F. Chittaro, A. Sarlette, P.R., IFAC world congress 2017.



$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) - i[\boldsymbol{H}_{\text{int}}, \rho] + \mathcal{L}_B(\rho) \text{ where}$$

$$\mathcal{L}_A(\rho) = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\mathcal{D}_{\boldsymbol{a}}(\rho)$$

$$\boldsymbol{H}_{\text{int}} = g[\boldsymbol{a}(\boldsymbol{b}^\dagger)^2 + \boldsymbol{a}^\dagger\boldsymbol{b}^2, \rho] + \chi(\boldsymbol{a}^\dagger\boldsymbol{a})(\boldsymbol{b}^\dagger\boldsymbol{b}) + \tfrac{\chi_a}{2}(\boldsymbol{a}^\dagger\boldsymbol{a})^2$$

$$\mathcal{L}_B(\rho) = -i\tfrac{\chi_b}{2}[(\boldsymbol{b}^\dagger\boldsymbol{b})^2, \rho]$$

with $\kappa \gg \max\left(|g|, |\chi|, |\chi_a|, |\chi_b|\right)$.

The slow dynamics (second order approximation, $\alpha = 2u/\kappa$) :

$$\frac{d}{dt}\rho_s = -i\left[\alpha^2\chi\boldsymbol{b}^\dagger\boldsymbol{b} + \tfrac{\chi_b}{2}(\boldsymbol{b}^\dagger\boldsymbol{b})^2, \rho_s\right] - i\alpha g[\boldsymbol{b}^2 + (\boldsymbol{b}^\dagger)^2, \rho_s] + \left(\frac{4g^2}{\kappa}\right)\mathcal{D}_{\boldsymbol{L}_s}(\rho)$$

with $\boldsymbol{L}_s = \boldsymbol{b}^2 + \frac{\alpha}{g}\left(\chi\boldsymbol{b}^\dagger\boldsymbol{b} + \frac{\chi_a(1+2\alpha^2)}{2}\boldsymbol{I}\right)$.

Kraus (CPTP) map : $\rho = (\boldsymbol{I} - i\boldsymbol{M})\Big(|\alpha\rangle\langle\alpha| \otimes \rho_s\Big)(\boldsymbol{I} + i\boldsymbol{M}^\dagger)$ with

$$\boldsymbol{M} = (\boldsymbol{a}^\dagger - \alpha^*) \otimes \left(\frac{2g}{\kappa}\boldsymbol{b}^2 + \frac{2\alpha\chi}{\kappa}\boldsymbol{b}^\dagger\boldsymbol{b} + \frac{2\alpha(1+2\alpha^2)\chi_a}{\kappa}\boldsymbol{I}\right) + \frac{\alpha^2\chi_a}{\kappa}(\boldsymbol{a}^\dagger - \alpha^*)^2 \otimes \boldsymbol{I}.$$

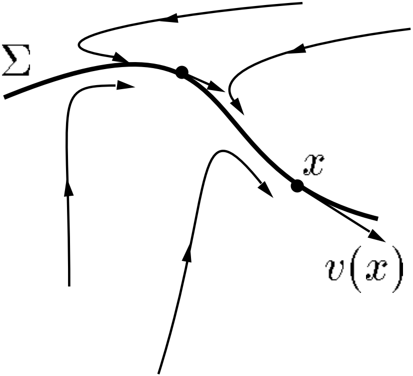
2. M. Mirrahimi, Z. Leghtas, V.V. Albert, S. Touzard, R.J. Schoelkopf, L. Jiang, and M.H. Devoret. Dynamically protected cat-qubits : a new paradigm for universal quantum computation. New J. of Physics, 16 :045014, 2014.

Outline

Slow/fast bipartite master equations

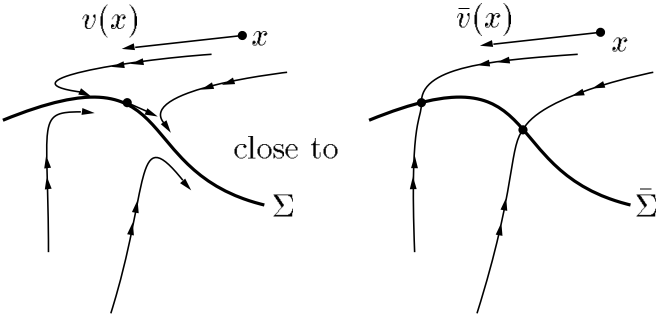
Model reduction and geometric singular perturbations

Geometric singular perturbations for bipartite quantum systems



A possible answer for $\frac{d}{dt}x = v(x)$:

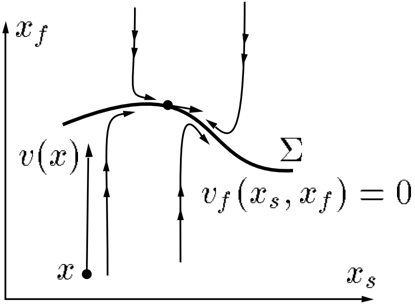
restriction to an attractive invariant manifold Σ.



Geometric definition independent of coordinates due to Fenichel [3] :

- $x \mapsto v(x)$ close to $x \mapsto \overline{v}(x)$.
- $\overline{v}(x) = 0$ define a manifold $\overline{\Sigma}$ of dimension $n_s < n = \dim(x)$ of steady-states for $\overline{v}(x)$.
- $n_f = n - n_s$ eigenvalues of $\left.\frac{\partial \overline{v}}{\partial x}\right|_{\overline{\Sigma}}$ are stable.

3. N. Fenichel : Geometric singular perturbation theory for ordinary differential equations. J. Diff. Equations, 1979, 31, 53-98.

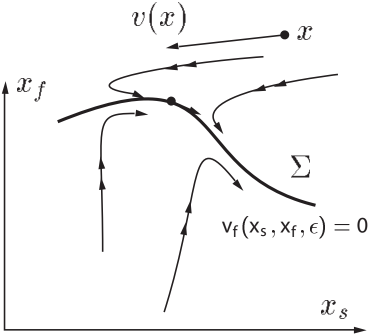


Any slow/fast system, can be put, after a suitable change of coordinates, in to a **quasi-vertical vector field** v :

$$\frac{d}{dt}x_s = v_s(x_s, x_f) = \varepsilon \tilde{v}_s(x_s, x_f, \varepsilon), \qquad \frac{d}{dt}x_f = v_f(x_s, x_f)$$

with $0 < \varepsilon \ll 1$.
The reduced system $\frac{d}{dt}x_s = v_s(x_s, x_f)$ with $0 = v_f(x_s, x_f)$ is correct if $\frac{d}{dt}\xi_f = v_f(x_s, \xi_f)$ stable for any fixed x_s.
In general, modeling variables x are **not** Tikhonov variables.

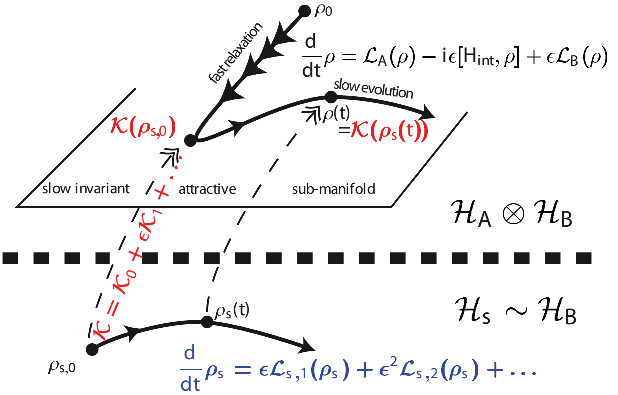
Model reduction with modeling variables



The reduced model of $\frac{d}{dt}x_s = v_s(x_s, x_f, \epsilon)$, $\frac{d}{dt}x_f = v_f(x_s, x_f, \epsilon)$ is [4]

$$\frac{d}{dt}x_s = \left(1 + \frac{\partial v_s}{\partial x_f}\left(\frac{\partial v_f}{\partial x_f}\right)^{-2}\frac{\partial v_f}{\partial x_s}\right)^{-1} v_s(x_s, x_f, \epsilon) + O(\epsilon^2), \quad v_f(x_s, x_f, \epsilon) = 0.$$

4. J. Carr : Application of Center Manifold Theory. Springer, 1981. P. Duchêne, P.R. : Kinetic scheme reduction via geometric singular perturbation techniques. Chem. Eng. Science, 1996, 51, 4661-4672.



Lindbladian slow dynamics in a copy $\mathcal{H}_s$ of $\mathcal{H}_B$

$$\frac{d}{dt}\rho_s = \mathcal{L}_s(\rho_s) = \epsilon\mathcal{L}_{s,1}(\rho_s) + \epsilon^2\mathcal{L}_{s,2}(\rho_s) + \ldots$$

with **Kraus map** to recover the physical density operator ρ from ρ_s :

$$\rho = \mathcal{K}(\rho_s) = \mathcal{K}_0(\rho_s) + \epsilon\mathcal{K}_1(\rho_s) + \ldots$$

5. R. Azouit et al. IEEE CDC 2016.

Plug $\rho = \mathcal{K}(\rho_s) = \overline{\rho}_A \otimes \rho_s + \epsilon \mathcal{K}_1(\rho_s) + \ldots$ and $\frac{d}{dt}\rho_s = \mathcal{L}_s(\rho_s) = \epsilon \mathcal{L}_{s,1}(\rho_s) + \epsilon^2 \mathcal{L}_{s,2}(\rho_s) + \ldots$ into the invariance condition

$$\mathcal{L}_A(\mathcal{K}(\rho_s)) - \epsilon i \left[\boldsymbol{H}_{\text{int}}, \mathcal{K}(\rho_s) \right] + \epsilon \mathcal{L}_B(\mathcal{K}(\rho_s)) = \frac{d}{dt}\rho = \mathcal{K}(\mathcal{L}_s(\rho_s))$$

and identify terms of same order :

order 1 : $\mathcal{L}_A(\mathcal{K}_1(\rho_s)) + \mathcal{L}_{\text{int}}(\mathcal{K}_0(\rho_s)) + \mathcal{L}_B(\mathcal{K}_0(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,1}(\rho_s))$

order 2 : $\mathcal{L}_A(\mathcal{K}_2(\rho_s)) + \mathcal{L}_{\text{int}}(\mathcal{K}_1(\rho_s)) + \mathcal{L}_B(\mathcal{K}_1(\rho_s)) = \mathcal{K}_0(\mathcal{L}_{s,2}(\rho_s)) + \mathcal{K}_1(\mathcal{L}_{s,1}(\rho_s))$

$\ldots$

At each order

1. take the trace versus A to get the correction to $\mathcal{L}_s$
2. compute the correction to $\mathcal{K}$ via $-\mathcal{L}_A^{-1}$, a super operator for zero-trace operators $\boldsymbol{W}$ on $\mathcal{H}_A$

$$-\mathcal{L}_A^{-1}(\boldsymbol{W}) = \int_0^{+\infty} e^{t\mathcal{L}_A}(\boldsymbol{W})dt$$

   that coincides with the restriction to zero-trace operators of a completely positive (CP) map.

First order expansion for a bipartite system : Zeno dynamics [6]

The full dynamics

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) - i\epsilon \left[\sum_{k=1}^{m} \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger \, , \, \rho \right] + \epsilon \mathcal{L}_B(\rho)$$

can be approximated by

$$\frac{d}{dt}\rho_s = -i\epsilon \left[\sum_{k=1}^{m} \mathrm{tr}(\boldsymbol{A}_k \overline{\rho}_A) \boldsymbol{B}_k^\dagger \, , \, \rho_s \right] + \epsilon \mathcal{L}_B(\rho_s) + O(\epsilon^2)$$

$$\rho = \left(\boldsymbol{I} - i\epsilon \boldsymbol{M} \right) \left(\overline{\rho}_{\boldsymbol{A}} \otimes \boldsymbol{\rho_s} \right) \left(\boldsymbol{I} + i\epsilon \boldsymbol{M}^\dagger \right) \; + O(\epsilon^2)$$

where $\boldsymbol{M} = \sum_{k=1}^{m} \boldsymbol{F}_k \otimes \boldsymbol{B}_k^\dagger$ with $\boldsymbol{F}_k$ given by

$$\boldsymbol{F}_k \overline{\rho}_A = -\mathcal{L}_A^{-1} \left(\boldsymbol{A}_k \, \overline{\rho}_A - \mathrm{tr}(\boldsymbol{A}_k \, \overline{\rho}_A) \overline{\rho}_A \right).$$

6. A. Azouit et al. arXiv.1704.00785

Second order dynamics [7]

The full dynamics

$$\frac{d}{dt}\rho = \mathcal{L}_A(\rho) - i\epsilon\left[\sum_{k=1}^{m} \boldsymbol{A}_k \otimes \boldsymbol{B}_k^\dagger\ ,\ \rho\right] + \epsilon\mathcal{L}_B(\rho)$$

can be approximated by

$$\frac{d}{dt}\rho_s = -i\left[\ \epsilon\sum_k \mathrm{tr}(\boldsymbol{A}_k\overline{\rho}_A)\boldsymbol{B}_k + \boldsymbol{\epsilon^2}\sum_{\boldsymbol{k,j}} y_{k,j}\ \boldsymbol{B_k B_j^\dagger}\ ,\ \rho_s\right]$$

$$+\ \epsilon\mathcal{L}_B(\rho_s) + \boldsymbol{\epsilon^2}\sum_{\boldsymbol{k=1}}^{\boldsymbol{m}} \boldsymbol{\mathcal{D}_{L_k}(\rho_s)} + O(\epsilon^3)$$

$$\rho = \left(\boldsymbol{I} - i\epsilon\boldsymbol{M}\right)\left(\overline{\rho}_A \otimes \rho_s\right)\left(\boldsymbol{I} + i\epsilon\boldsymbol{M}^\dagger\right)\ \ + O(\epsilon^2)$$

where $y_{k,j} = \frac{1}{2i}\,\mathrm{tr}\left(\boldsymbol{F}_j\overline{\rho}_A\boldsymbol{A}_k^\dagger - \boldsymbol{A}_j\overline{\rho}_A\boldsymbol{F}_k^\dagger\right)$ and $\boldsymbol{L_k} = \sum_{j=1}^m \lambda_{j,k}\boldsymbol{B}_j$ with matrix λ given by $\lambda\lambda^\dagger = x$ and $x_{k,j} = \mathrm{tr}\left(\boldsymbol{F}_j\overline{\rho}_A\boldsymbol{A}_k^\dagger + \boldsymbol{A}_j\overline{\rho}_A\boldsymbol{F}_k^\dagger\right)$

7. A. Azouit et al. arXiv.1704.00785

Conclusion

Interest of such geometric adiabatic elimination preserving the quantum structure (Lindblad master equation, CPTP maps) :

Some non Markovian dynamics might be modeled via a Lindbladian dynamics on a small Hilbert space and via a CPTP map towards the physical Hilbert space of large dimension.

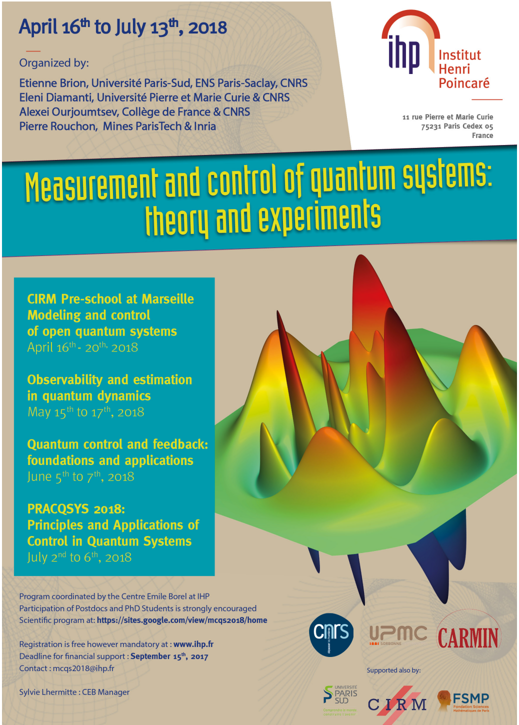
Quantum feedback where the quantum controller is designed faster than the quantum system to be controlled ((S, L, H) theory of Gough/James).

Extension when $\mathcal{H} = \bigoplus_k \mathcal{H}_{A_k} \otimes \mathcal{H}_{B_k}$ and the slow manifold is parameterized via

$$\rho_s = \sum_k \overline{\rho}_{A_k} \otimes \rho_{s,k} \text{ with } \rho_{s,k} \geq 0 \text{ and } \mathrm{tr}(\rho_{s,k}) \in [0, 1]$$

(talk of Katarzyna Macieszczak this Monday).

Conjecture : at any order it is always possible to obtain, up-to higher order terms, Lindbladian dynamics for ρ_s and CPTP maps relating ρ to ρ_s.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