Quantum tomography based on quantum trajectories

Quantum Control Theory:
Mathematical Aspects and Physical Applications
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Quantum tomography versus quantum filtering

Likelihood function calculations via adjoint states

- Discrete time case
- Continuous time case

MaxLike estimations with experimental data

- Process tomography for QND measurement of photons
- State tomography of a quantum Maxwell demon

Fisher information and low-rank MaxLike estimates

Appendix: asymptotics for multi-dimension Laplace integrals and boundary corrections

Quantum state tomography based on POVM, $\sum_j \pi_j = \boldsymbol{I}$

- Tomography of ρ via N independent measurements $\boldsymbol{Y}$ associated to POVM: probability $\text{Tr}\,(\rho\pi_j)$ of each measurement outcome j given by π_j; for $\boldsymbol{N_j}$ the number of j outcomes, $\boldsymbol{Y \equiv (N_j)}$ with $\sum_j \boldsymbol{N_j} = N$, the number of measurements.
- Several estimation methods:

MaxEnt: ρ_{ME} maximizes $-\text{Tr}\,(\rho\log(\rho))$ under the constraints $|\text{Tr}\,(\rho\pi_j) - \boldsymbol{N_j}/N| \leq \epsilon$ (Bužek et al, Ann. Phys. 1996).

Compress Sensing: ρ_{CS} minimizes $\text{Tr}\,(\rho)$ under the constraints $|\text{Tr}\,(\rho\pi_j) - \boldsymbol{N_j}/N| \leq \epsilon$ (Gross et al PRL2010)

MaxLike: ρ_{ML} maximizes the likelihood function, $\rho \mapsto \mathbb{P}(\boldsymbol{Y} \mid \rho) = \prod_j \left(\text{Tr}\,(\rho\pi_j)\right)^{\boldsymbol{N_j}}$ (see, e.g., Lvovsky/Raymer RMP 2009)

Bayesian Mean: $\rho_{BM} \propto \int \rho\mathbb{P}(\boldsymbol{Y} \mid \rho)\mathbb{P}_0(\rho)d\rho$ where $\mathbb{P}_0$ is some prior distribution $\mathbb{P}_0(\rho)d\rho$ (see, e.g., Blume-Kohout NJP2010).

Low rank, high dimensional systems: see, e.g, PhD thesis "Efficient and Robust Methods for Quantum Tomography" of Charles Heber Baldwin, University of New Mexico, December 2016.

Filtering: estimation of the quantum state ρ_t at time $t > 0$ from the measurement trajectory $[0, t[\ni \tau \mapsto \boldsymbol{y}_\tau$ and the initial state ρ_0; see Belavkin semilar contributions (links with Monte-Carlo quantum-trajectories).

State tomography: estimation of the initial state $\rho_0 = \boldsymbol{\rho}$ from a collection of N measurement trajectories: $\boldsymbol{Y} = \left(\boldsymbol{y}_t^{(n)}\right)$ with $n \in \{1, \ldots, N\}$ and $t \in [0, T]$.

Process tomography: estimation of a parameter $\mathbf{p}$ from a known initial state $\boldsymbol{\rho}$ and a collection of N measurement trajectories $\boldsymbol{Y}$.

This talk: MaxLike estimation with decoherence and measurement imperfections (PhD thesis of Pierre Six, November 2016):

1. How to compute the likelihood function $\mathbb{P}\big(\boldsymbol{Y}/\boldsymbol{\rho}, \mathbf{p}\big)$ and its gradient from the stochastic master equation governing filtering (P. Six et al. PRA 2016).
2. For state estimation: variance computation based on asymptotic expansions of Laplace integrals for low rank MaxLike estimates (P. Six /PR, chapter in Lecture Notes in Control and Information Sciences no 473, April 2017).

Log-likelihood function $f(p) = \log\left(\mathbb{P}(\boldsymbol{Y} \mid p)\right)$ admits a unique maximum at p_{ML} ($\nabla f(p_{ML}) = 0$) with a negative definite Hessian ($\nabla^2 f(p_{ML}) < 0$).

f coming from N independent realisations: $f(p) \equiv N\bar{f}(p)$ with asymptotics for $N \mapsto +\infty$ of the Laplace integrals connecting

- Bayesian Mean p_{BM} and MaxLike estimation p_{ML}:

$$p_{BM} = \frac{\int p\, e^{N\bar{f}(p)} \mathbb{P}_0(p) dp}{\int e^{N\bar{f}(p)} \mathbb{P}_0(p) dp} = p_{ML} + O(1/N).$$

  with any smooth prior distribution $\mathbb{P}_0(p)dp$

- Bayesian variance and Fisher information $\overline{F}_{ML} = -\nabla^2 \bar{f}(p_{ML})$:

$$\frac{\int \|p - p_{ML}\|^2\, e^{N\bar{f}(p)} \mathbb{P}_0(p) dp}{\int e^{N\bar{f}(p)} \mathbb{P}_0(p) dp} = \text{Tr}\left(\left(\overline{F}_{ML}\right)^{-1}\right)/(2N) + O(1/N^2).$$

Confidence intervals based on $-\nabla^2 f(p_{ML})$.

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Four features[1]:

1. Bayes law: $\mathbb{P}(\mu'/\mu) = \mathbb{P}(\mu/\mu')\mathbb{P}(\mu') \,/\, \left(\sum_{\nu'} \mathbb{P}(\mu/\nu')\mathbb{P}(\nu')\right)$,
2. Schrödinger equations defining unitary transformations.
3. Randomness, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
4. Entanglement and tensor product for composite systems.

$\Rrightarrow$ **Discrete-time models**

Take a set of operators $\boldsymbol{M}_\mu$ satisfying $\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$ and a left stochastic matrices $(\eta_{\boldsymbol{y}_t,\mu})$. Consider the following Markov process of state ρ (density op.) and measured output $\boldsymbol{y}$:

$$\boldsymbol{\rho_{t+1}} = \frac{\boldsymbol{K}_{\boldsymbol{y}_t}(\boldsymbol{\rho_t})}{\mathrm{Tr}\left(\boldsymbol{K}_{\boldsymbol{y}_t}(\boldsymbol{\rho_t})\right)}, \text{ with proba. } \mathbb{P}_{\boldsymbol{y}_t}(\rho_t) = \mathrm{Tr}\left(\boldsymbol{K}_{\boldsymbol{y}_t}(\rho_t)\right)$$

with $\boldsymbol{K}_{\boldsymbol{y}}(\rho) = \sum_{\mu=1}^m \eta_{\boldsymbol{y},\mu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$. It is associated to the Kraus map (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{t+1}|\rho_t\right) = \boldsymbol{K}(\rho_t) = \sum_{\boldsymbol{y}} \boldsymbol{K}_{\boldsymbol{y}}(\rho_t) = \sum_\mu \boldsymbol{M}_\mu \rho_t \boldsymbol{M}_\mu^\dagger.$$

[1] See the book of S. Haroche and J.M. Raimond.

Computation of the likelihood function via the adjoint state (1)

- Denote by $\mathbb{P}_n(\boldsymbol{\rho}, \mathbf{p})$ the probability of getting measurement trajectory n, $(\boldsymbol{y}_t^{(n)})_{t=0,\ldots,T}$, knowing the initial state $\rho_0^{(n)} = \boldsymbol{\rho}$ and parameter $\mathbf{p}$.

- Since $\rho_{t+1}^{(n)} = \frac{\boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_t^{(n)}}\left(\rho_t^{(n)}\right)}{\operatorname{Tr}\left(\boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_t^{(n)}}\left(\rho_t^{(n)}\right)\right)}$ with $\operatorname{Tr}\left(\boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_t^{(n)}}\left(\rho_t^{(n)}\right)\right)$ the probability of having detected $\boldsymbol{y}_t^{(n)}$ knowing $\rho_t^{(n)}$ and $\mathbf{p}$, a direct use of Bayes law yields

$$\mathbb{P}_n(\boldsymbol{\rho}, \mathbf{p}) = \prod_{t=0}^{T} \operatorname{Tr}\left(\boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_t^{(n)}}\left(\rho_t^{(n)}\right)\right) = \operatorname{Tr}\left(\boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_T^{(n)}} \circ \ldots \circ \boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_0^{(n)}}(\boldsymbol{\rho})\right).$$

Computation of the likelihood function via the adjoint state (2)

- With adjoint map $\boldsymbol{K}_{\boldsymbol{y}}^{\mathbf{p}*}$ ($\forall A, B,\ \mathrm{Tr}\left(\boldsymbol{K}_{\boldsymbol{y}}^{\mathbf{p}}(A)B\right) \equiv \mathrm{Tr}\left(A\boldsymbol{K}_{\boldsymbol{y}}^{\mathbf{p}*}(B)\right)$):

$$\mathbb{P}_n(\boldsymbol{\rho}, \mathbf{p}) = \mathrm{Tr}\left(\boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_T^{(n)}} \circ \ldots \circ \boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_0^{(n)}}(\boldsymbol{\rho}) \quad \boldsymbol{I}\right) = \mathrm{Tr}\left(\boldsymbol{\rho} \quad \boldsymbol{K}^{\mathbf{p}*}_{\boldsymbol{y}_0^{(n)}} \circ \ldots \circ \boldsymbol{K}^{\mathbf{p}*}_{\boldsymbol{y}_T^{(n)}}(\boldsymbol{I})\right).$$

- Normalized adjoint quantum filter[2] $E_t^{(n)} = \dfrac{\boldsymbol{K}^{\mathbf{p}*}_{\boldsymbol{y}_t^{(n)}}\left(E_{t+1}^{(n)}\right)}{\mathrm{Tr}\left(\boldsymbol{K}^{\mathbf{p}*}_{\boldsymbol{y}_t^{(n)}}\left(E_{t+1}^{(n)}\right)\right)}$ with

  $E_{T+1}^{(n)} = \boldsymbol{I}/\ \mathrm{Tr}(\boldsymbol{I})$, we get

$$\mathbb{P}_n(\boldsymbol{\rho}, \mathbf{p}) = \prod_{t=T}^{0} \mathrm{Tr}\left(\boldsymbol{K}^{\mathbf{p}*}_{\boldsymbol{y}_t^{(n)}}\left(E_{t+1}^{(n)}\right)\right) \mathrm{Tr}\left(\boldsymbol{\rho} E_0^{(n)}\right) \triangleq g_n(\boldsymbol{Y}, \mathbf{p})\ \mathrm{Tr}\left(\boldsymbol{\rho} E_0^{(n)}\right).$$

- A simple expression of the gradients:

$$\nabla_{\boldsymbol{\rho}} \log \mathbb{P}_n = \frac{E_0^{(n)}}{\mathrm{Tr}\left(\boldsymbol{\rho} E_0^{(n)}\right)}, \quad \nabla_{\mathbf{p}} \log \mathbb{P}_n \cdot \delta\mathbf{p} = \sum_{t=0}^{T} \frac{\mathrm{Tr}\left(E_{t+1}^{(n)}\ \nabla_{\mathbf{p}} \boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_t^{(n)}}\left(\rho_t^{(n)}\right) \cdot \delta\mathbf{p}\right)}{\mathrm{Tr}\left(E_{t+1}^{(n)}\ \boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y}_t^{(n)}}\left(\rho_t^{(n)}\right)\right)},$$

[2] M. Tsang. Time-symmetric quantum theory of smoothing. PRL 2009.

MaxLike tomography based on N trajectories data $\boldsymbol{Y} = \left(\boldsymbol{y_t^{(n)}}\right)$

From $\mathbb{P}_n(\boldsymbol{\rho}, \mathbf{p}) = g_n(\boldsymbol{Y}, \mathbf{p}) \operatorname{Tr}\left(\boldsymbol{\rho} E_0^{(n)}\right)$ we have

$$\mathbb{P}(\boldsymbol{\rho}, \mathbf{p}) \triangleq \prod_{n=1}^{N} \mathbb{P}_n(\boldsymbol{\rho}, \mathbf{p}) = \left(\prod_{n=1}^{N} g_n(\boldsymbol{Y}, \mathbf{p})\right) \left(\prod_{n=1}^{N} \operatorname{Tr}\left(\boldsymbol{\rho} E_0^{(n)}\right)\right).$$

- MaxLike **state tomography:** $\mathbf{p}$ is known and $\boldsymbol{\rho}_{ML}$ maximizes

$$\boldsymbol{\rho} \mapsto \sum_{n=1}^{N} \log\left(\operatorname{Tr}\left(\boldsymbol{\rho} E_0^{(n)}\right)\right)$$

  a concave function on the convex set of density operators ρ:
  a well structured convex optimization problem.

- MaxLike **process tomography:** $\boldsymbol{\rho}$ is known and $\mathbf{p}_{ML}$ maximizes $\mathbf{p} \mapsto f(\mathbf{p}) = \log \mathbb{P}(\boldsymbol{\rho}, \mathbf{p})$ those gradient is given by

$$\nabla_{\mathbf{p}} f(\mathbf{p}) \cdot \delta\mathbf{p} = \sum_{n=1}^{N} \sum_{t=0}^{T} \frac{\operatorname{Tr}\left(E_{t+1}^{(n)} \ \nabla_{\mathbf{p}} \boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y_t^{(n)}}}\left(\rho_t^{(n)}\right) \cdot \delta\mathbf{p}\right)}{\operatorname{Tr}\left(E_{t+1}^{(n)} \ \boldsymbol{K}^{\mathbf{p}}_{\boldsymbol{y_t^{(n)}}}\left(\rho_t^{(n)}\right)\right)},$$

  The Hessian $\nabla^2_{\mathbf{p}} f$ can be computed similarly (Fisher information).

Discrete-time models: Markov chains $\rho_{t+1} = \frac{K_{y_t}(\rho_t)}{\text{Tr}\left(K_{y_t}(\rho_t)\right)}$, with
$K_{y_t}(\rho_t) = \sum_{\mu=1}^{m} \eta_{y_t,\mu} M_\mu \rho_t M_\mu^\dagger$, and proba. $\mathbb{P}_{y_t}(\rho_t) = \text{Tr}\left(K_{y_t}(\rho_t)\right)$.
Ensemble averages correspond to Kraus linear maps

$$\mathbb{E}\left(\rho_{t+1}|\rho_t\right) = K(\rho_t) = \sum_{y} K_y(\rho_t) = \sum_{\mu} M_\mu \rho_t M_\mu^\dagger \quad \text{with} \quad \sum_{\mu} M_\mu^\dagger M_\mu = I$$

Continuous-time models: stochastic differential systems (see, e.g., Barchielli/Gregoratti, 2009)

$$d\rho_t = \left(-\frac{i}{\hbar}[H,\rho_t] + \sum_{\nu} L_\nu \rho_t L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu)\right) dt + \sum_{\nu} \sqrt{\eta_\nu}\left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \text{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}$$

driven by Wiener processes $dW_{\nu,t}$, with measurements $dy_{\nu,t}$,
$dy_{\nu,t} = \sqrt{\eta_\nu}\,\text{Tr}\left((L_\nu + L_\nu^\dagger)\,\rho_t\right) dt + dW_{\nu,t}$, detection efficiencies $\eta_\nu \in [0,1]$ and Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_{\nu} L_\nu \rho L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu)$$

Continuous/discrete-time diffusive SME

The Belavkin quantum filter

$$d\rho_t = \left(-\frac{i}{\hbar}[\boldsymbol{H},\rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt + \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right)\rho_t\right) \boldsymbol{dW_{\nu,t}}$$

with $\boldsymbol{dW_{\nu,t}} = \boldsymbol{dy_{\nu,t}} - \sqrt{\eta_\nu}\operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) dt$ given by the measurement signal $\boldsymbol{dy_{\nu,t}}$, is always a stable filtering process.[3]
Using Itō rules, it can be written as a "discrete-time" Markov model[4]

$$\rho_{t+dt} = \boldsymbol{K}_{\boldsymbol{dy_t}}(\rho_t) / \operatorname{Tr}\left(\boldsymbol{K}_{\boldsymbol{dy_t}}(\rho_t)\right)$$

with partial Kraus maps $\boldsymbol{K}_{\boldsymbol{dy_t}}(\rho_t) = \boldsymbol{M}_{\boldsymbol{dy_t}} \rho_t \boldsymbol{M}_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu)\boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger dt$

$$\boldsymbol{M}_{\boldsymbol{dy_t}} = \boldsymbol{I} + \left(-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\left(\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu\right)\right) dt + \sum_\nu \sqrt{\eta_\nu}\,\boldsymbol{dy_{\nu,t}}\boldsymbol{L}$$

where the probability of outcome $\boldsymbol{dy_t} = (dy_{\nu,t})$ reads:

$$\mathbb{P}\left(\boldsymbol{dy_t} \in \prod_\nu [\xi_\nu, \xi_\nu + d\xi_\nu] \;/\; \rho_t\right) = \operatorname{Tr}\left(\boldsymbol{K}_{\boldsymbol{\xi}}(\rho_t)\right) \prod_\nu e^{-\xi_\nu^2/2dt} \frac{d\xi_\nu}{\sqrt{2\pi dt}}$$

[3] H. Amini et al., Russian J. of Math. Physics, 2014, 21, 297-315.

[4] PR, J. Ralph PRA2015; see also PhD thesis of Ph. Campagne-Ibracq (2015) and of P. Six (2016).

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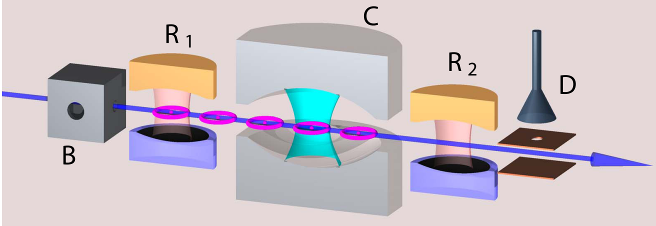


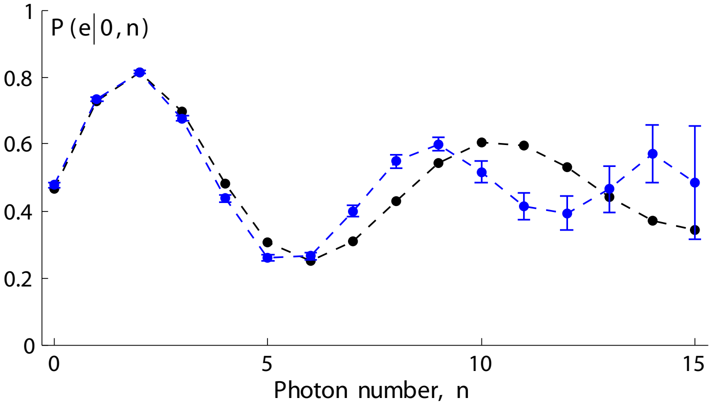
- The probability $\mathbb{P}(y \mid \phi_R, n)$ to get $y \in \{g, e\}$ knowing the Ramsey angle ϕ_R and the number of photon(s) $n \in \{0, 1, 2, \ldots\}$:

  $$\mathbb{P}(y \mid \phi_R, n) = 1 + \epsilon_y \big(A + B_c(n) \cos\phi_R + B_s(n) \sin\phi_R \big) \text{ with } \epsilon_{e/g} = \pm 1.$$

  depends on the parameters $\mathbf{p} = (B_c(n), B_s(n))_{n \in \{0,1,\ldots,\}}$.

- The Kraus maps $\boldsymbol{K}_y^{\mathbf{p}}$ based on known cavity decay and thermal photons.

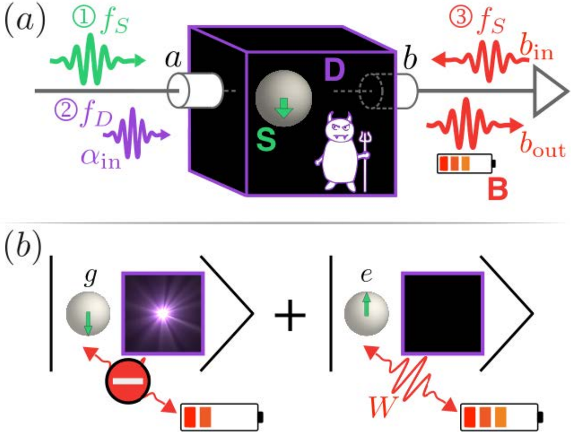
A priori calibration[5] (black dots) versus MaxLike (blue dots)



MaxLike estimation of 32 parameters **p** based on $N = 8000$ trajectories of $T = 6000$ outcome measurements.

[5] T. Rybarczyk, B. Peaudecerf, M. Penasa, S. Gerlich, B. Julsgaard, K. Mølmer, S. Gleyzes, M. Brune, J. M. Raimond, S. Haroche, and I. Dotsenko. Forward-backward analysis of the photon-number evolution in a cavity. PRA 2015.

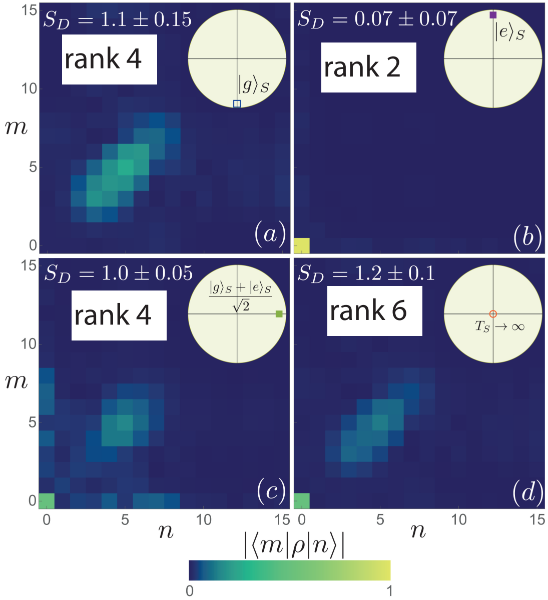
A quantum Maxwell demon experiment arXiv:1702.01917v1



(a) After preparation in a thermal or quantum state the system S (superconducting qubit) state is recorded into the demon's quantum memory D (microwave cavity) via a pulse that populates the cavity mode only if the qubit is in the ground state. This information is used to extract work which charges a battery with one extra photon: system S emits this photon only when the demon's cavity is empty. The memory reset is performed by cavity relaxation.

(b) When the system starts in a quantum superposition of the demon and system are entangled after the record step.

Tomography of the demon after the work extraction step



Computations are based on a truncation to 20 photons

How to define the confidence intervals for low rank ρ_{ML}?

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Asymptotic when ρ_{ML} is on the boundary

- To bypass boundary problem, consider Bayesian estimate instead of MaxLike ones

$$\rho_{BM} = \frac{\int \rho \, \mathbb{P}(\boldsymbol{Y} \mid \rho)\mathbb{P}_0(\rho)d\rho}{\int \mathbb{P}(\boldsymbol{Y} \mid \rho)\mathbb{P}_0(\rho)d\rho}$$

  with some prior distribution $\mathbb{P}_0(\rho)d\rho$.

- When the likelihood $\exp(f(\rho)) \equiv \mathbb{P}(\boldsymbol{Y} \mid \rho)$ is concentrated ($f = \boldsymbol{N}\bar{f}$ with $N \gg 1$) around its maximum ρ_{ML} that lies on the boundary (ρ_{ML} not full rank), how to compute the first terms of an asymptotic expansion versus $\boldsymbol{N}$ of

$$\int \mathrm{Tr}\,(\rho A)^r \exp(\boldsymbol{N}\bar{f}(\rho))\mathbb{P}_0(\rho)d\rho$$

  for any operator A and exponent r and for some prior distribution $\mathbb{P}_0(\rho)d\rho$ (e.g., Gausssian unitary ensemble).

- Since all functions are analytic such an asymptotic expansion versus $\boldsymbol{N}$ always exists: Integration by parts, Watson's lemma, Laplace's method, stationary phase, steepest descents, Hironaka's resolution of singularities [6], "singular learning" [7]

[6] An important reference: V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. Singularities of Differentiable Maps, Vol. II. Birkhäuser, Boston, 1985

[7] S. Watanabe: Algebraic Geometry and Statistical Learning Theory, Cambridge University Press, 2009.

Assume that ρ_{ML} is an argument of the maximum of

$$f : \mathcal{D} \ni \rho \mapsto \sum_{\mu \in \mathcal{M}} \log \left(\text{Tr} \left(\rho E^{(\mu)} \right) \right) \in [-\infty, 0]$$

over $\mathcal{D}$ (the set of density operators, $E^{(\mu)} \in \mathcal{D}$.). Then necessarily, ρ_{ML} satisfies the following conditions:

- $\text{Tr}\left(\rho_{ML} E^{(\mu)}\right) > 0$ for each $\mu \in \mathcal{M}$;
- $\left[\rho_{ML} \, , \, \nabla f|_{\rho_{ML}}\right] = 0$, where $\nabla f|_{\rho_{ML}} = \sum_{\mu \in \mathcal{M}} \frac{E^{(\mu)}}{\text{Tr}\left(\rho_{ML} E^{(\mu)}\right)}$ is the gradient of f at ρ_{ML} for the Frobenius scalar product;
- there exists $\lambda_{ML} > 0$ such that $\lambda_{ML} P_{ML} = P_{ML} \, \nabla f|_{\rho_{ML}}$ and $\nabla f|_{\rho_{ML}} \leq \lambda_{ML} I$, where P_{ML} is the orthogonal projector on the range of ρ_{ML} and I is the identity operator.

These conditions are also sufficient and characterize the unique maximum when, additionally, the vector space spanned by the $E^{(\mu)}$'s coincides with the set of Hermitian matrices.

For any Hermitian operator A, its Bayesian mean and variance read:

$$I_A(N) = \frac{\int_{\mathcal{D}} \mathrm{Tr}\,(\rho A)\, e^{Nf(\rho)}\, \mathbb{P}_0(\rho)\, d\rho}{\int_{\mathcal{D}} e^{Nf(\rho)}\, \mathbb{P}_0(\rho)\, d\rho}, \quad V_A(N) = \frac{\int_{\mathcal{D}} \left(\mathrm{Tr}\,(\rho A) - I_A(N)\right)^2 e^{Nf(\rho)}\, \mathbb{P}_0(\rho)\, d\rho}{\int_{\mathcal{D}} e^{Nf(\rho)}\, \mathbb{P}_0(\rho)\, d\rho}.$$

Denote by ρ_{ML} the unique maximum of f on $\mathcal{D}$ and by P_{ML} the orthogonal projector on its range. In addition to the necessary and sufficient geometric conditions above, assume that $\ker\left(\lambda_{ML} I - \nabla f|_{\rho_{ML}}\right) = \ker(I - P_{ML})$.

$$I_A(N) = \mathrm{Tr}\,(A\rho_{ML}) + O(1/N), \quad \boldsymbol{V_A(N) = \mathrm{Tr}\left(A_{\|}\ (F_{ML})^{-1}(A_{\|})\right)/N} + O(1/N^2)$$

where $B_{\|}$ is an orthogonal projection

$$B_{\|} = B - \frac{\mathrm{Tr}\,(BP_{ML})}{\mathrm{Tr}\,(P_{ML})} P_{ML} - (I - P_{ML})B(I - P_{ML});$$

and where $\boldsymbol{F_{ML}}$ is a linear super-operator, corresponds to the Hessian at ρ_{ML} of some restriction of f and **generalizes the Fisher information matrix**:

$$\boldsymbol{F_{ML}}(X) = \sum_{\mu} \frac{\mathrm{Tr}\left(XE_{\|}^{(\mu)}\right)}{\mathrm{Tr}^2\,(\rho_{ML} E^{(\mu)})} E_{\|}^{(\mu)} + \left(\lambda_{ML} I - \nabla f|_{\rho_{ML}}\right) X \rho_{ML}^{+} + \rho_{ML}^{+} X \left(\lambda_{ML} I - \nabla f|_{\rho_{ML}}\right)$$

with ρ_{ML}^{+} the Moore-Penrose pseudo-inverse of ρ_{ML}.

- Low-rank approximations and efficient numerical schemes for computations of ρ_{ML}, the adjoint states $E^{(n)}$, ...
- Asymptotics when the log-likelihood function is not strongly concave, when $\ker\left(\lambda_{ML} I - \nabla f|_{\rho_{ML}}\right) \neq \ker(I - P_{ML})$ ...
- Process tomography: log-likelihood function not concave ...
- Parameter estimation along quantum trajectories (in real-time) ...
- Thematic quarter at Institut Henri Poincaré in Paris next Spring 2018 gathering experimental physicists and theoreticians.

April 16th to July 13th, 2018

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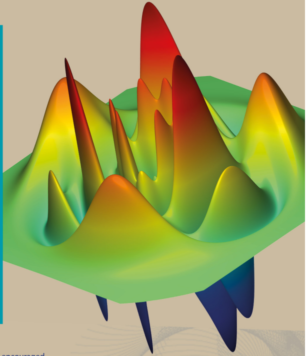

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Participation of Postdocs and PhD Students is strongly encouraged
Scientific program at: **https://sites.google.com/view/mcqs2018/home**

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Theorem (interior max) $\mathcal{I}_g(N) = \int_{z\in(-1,1)^n} g(z) \exp\left(Nf(z)\right) \, \mathrm{d}z$
with f and g analytic functions of z on a compact neighbourhood of $\overline{\mathcal{D}}$, the closure of $\mathcal{D}$. Assume that f admits a unique maximum on $\overline{\mathcal{D}}$ at $z = 0$ with $\left.\frac{\partial^2 f}{\partial z^2}\right|_0$ negative definite.
If $g(0) \neq 0$, we have the following dominant term in the asymptotic expansion of $\mathcal{I}_g(N)$ for large N:

$$\mathcal{I}_g(N) = \left(\frac{g(0)\ (2\pi)^{n/2}\ e^{Nf(0)} N^{-n/2}}{\sqrt{\left|\det\left(\left.\frac{\partial^2 f}{\partial z^2}\right|_0\right)\right|}} \right) + O\Big(e^{Nf(0)} N^{-n/2-1}\Big).$$

If $g(0) = 0$, with $\left.\frac{\partial g}{\partial z}\right|_0 = 0$, then we have:

$$\mathcal{I}_g(N) = \left(\frac{\mathrm{Tr}\left(- \left.\frac{\partial^2 g}{\partial z^2}\right|_0 \left(\left.\frac{\partial^2 f}{\partial z^2}\right|_0\right)^{-1}\right)\ (2\pi)^{n/2}}{2\sqrt{\left|\det\left(\left.\frac{\partial^2 f}{\partial z^2}\right|_0\right)\right|}} \right) e^{Nf(0)} N^{-n/2-1} + O\left(e^{Nf(0)} N^{-n/2-2}\right).$$

Corollary (interior max): Assume that f admits a unique maximum on $\overline{\mathcal{D}}$ at $z = 0$ with $\left.\frac{\partial^2 f}{\partial z^2}\right|_0$ negative definite. Then we have the following asymptotic for any analytic function $g(z)$:

$$\mathcal{M}_g(N) \triangleq \frac{\int_{z\in(-1,1)^n} g(z) \exp\left(Nf(z)\right) \mathrm{d}z}{\int_{z\in(-1,1)^n} \exp\left(Nf(z)\right) \mathrm{d}z} = g(0) + O(N^{-1})$$

We have also:

$$\mathcal{V}_g(N) \triangleq \frac{\int_{z\in(-1,1)^n} \left(g(z) - \mathcal{M}_g(N)\right)^2 \exp\left(Nf(z)\right)\ \mathrm{d}z}{\int_{z\in(-1,1)^n} \exp\left(Nf(z)\right)\ \mathrm{d}z}$$

$$= \frac{\mathrm{Tr}\left(-\left.\frac{\partial^2 g}{\partial z^2}\right|_0 \left(\left.\frac{\partial^2 f}{\partial z^2}\right|_0\right)^{-1}\right)}{2N} + O\big(N^{-2}\big).$$

Theorem (boundary max):
$\mathcal{I}_g(N) = \int_{x\in(0,1)} \int_{z\in(-1,1)^n} x^m g(x,z) \exp\left(Nf(x,z)\right) \,\mathrm{d}x\, \mathrm{d}z$ with f and g analytic functions of (x,z) on a compact neighbourhood of $\overline{\mathcal{D}}$, the closure of $\mathcal{D}$. Assume that f admits a unique maximum on $\overline{\mathcal{D}}$ at $(x,z)=(0,0)$, with $\frac{\partial^2 f}{\partial z^2}\Big|_{(0,0)}$ negative definite and $\frac{\partial f}{\partial x}\Big|_{(0,0)} < 0$. If $g(0,0)\neq 0$, we have the following dominant term in the asymptotic expansion of $\mathcal{I}_g(N)$ for large N:

$$\mathcal{I}_g(N) = \left(\frac{g(0,0)\; m!\; (2\pi)^{n/2}\; e^{Nf(0,0)} N^{-m-n/2-1}}{\sqrt{\left|\det\left(\frac{\partial^2 f}{\partial z^2}\Big|_{(0,0)}\right)\right|}\;\left(-\frac{\partial f}{\partial x}\Big|_{(0,0)}\right)^{m+1}}\right) + O\Big(e^{Nf(0,0)}N^{-m-n/2-2}\Big).$$

If $g(0,0)=0$, with $\frac{\partial g}{\partial x}\Big|_{(0,0)} = 0$ and $\frac{\partial g}{\partial z}\Big|_{(0,0)} = 0$, then we have:

$$\mathcal{I}_g(N) = \left(\frac{\mathrm{Tr}\left(-\frac{\partial^2 g}{\partial z^2}\Big|_{(0,0)}\left(\frac{\partial^2 f}{\partial z^2}\Big|_{(0,0)}\right)^{-1}\right)\; m!\; (2\pi)^{n/2}}{2\sqrt{\left|\det\left(\frac{\partial^2 f}{\partial z^2}\Big|_{(0,0)}\right)\right|}\;\left(-\frac{\partial f}{\partial x}\Big|_{(0,0)}\right)^{m+1}}\right) e^{Nf(0,0))} N^{-m-n/2-2}$$

$$+\, O\left(e^{Nf(0,0))}N^{-m-n/2-3}\right).$$

Corollary (boundary max): Assume that f admits a unique maximum on $\overline{\mathcal{D}}$ at $(x,z)=(0,0)$, with $\left.\frac{\partial^2 f}{\partial z^2}\right|_{(0,0)}$ negative definite and $\left.\frac{\partial f}{\partial x}\right|_{(0,0)}<0$. Then, we have the following asymptotic for any analytic function $g(x,z)$:

$$\mathcal{M}_g(N) \triangleq \frac{\int_{x\in(0,1)}\int_{z\in(-1,1)^n} x^m g(x,z)\exp\left(Nf(x,z)\right)\,\mathrm{d}x\,\mathrm{d}z}{\int_{x\in(0,1)}\int_{z\in(-1,1)^n} x^m \exp\left(Nf(x,z)\right)\,\mathrm{d}x\,\mathrm{d}z} = g(0,0)+O(N^{-1})$$

We have also:

$$\mathcal{V}_g(N) \triangleq \frac{\int_{x\in(0,1)}\int_{z\in(-1,1)^n} x^m \Big(g(x,z)-\mathcal{M}_g(N)\Big)^2 \exp\left(Nf(x,z)\right)\,\mathrm{d}x\,\mathrm{d}z}{\int_{x\in(0,1)}\int_{z\in(-1,1)^n} x^m \exp\left(Nf(x,z)\right)\,\mathrm{d}x\,\mathrm{d}z}$$

$$= \frac{\mathrm{Tr}\left(-\left.\frac{\partial^2 g}{\partial z^2}\right|_{(0,0)}\left(\left.\frac{\partial^2 f}{\partial z^2}\right|_{(0,0)}\right)^{-1}\right)}{2N} + O(N^{-2}).$$