Singular perturbations and Lindblad-Kossakowski differential equations

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Outline

Open quantum systems

The Lindbald-Kossakowski equation

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H,\rho] + \sum_{k=1}^{N} \frac{1}{2}\left(2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k\right)$$

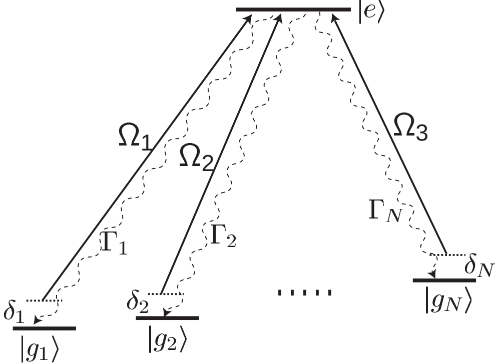
is the master equation associated to an ensemble average of quantum trajectories (stochastic jump dynamics of a single quantum system where the "environment is watching"[1]).
Contribution: when the Lindblad operators L_k are associated to highly unstable excited states, we propose a systematic method to eliminate the resulting fast and asymptotically stable dynamics. The obtained slow dynamics

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s,\rho_s] + \sum_{k=1}^{N} \frac{1}{2}\left(2L_{s,k}\rho_s L_{s,k}^\dagger - L_{s,k}^\dagger L_{s,k}\rho_s - \rho_s L_{s,k}^\dagger L_{s,k}\right)$$

is still of Lindbald-Kossakowski form $((\rho_s, H_s, L_{s,k}) = \text{fnct}(\rho, H, L_k))$.

[1] H.-P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Clarendon-Press, Oxford, 2006. S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons*. Oxford University Press, 2006.

Prototype of open quantum system: Λ-systems.



N stable states $|g_k\rangle$, $k = 1, \ldots, N$.
Unstable state $|e\rangle$
Quasi resonant laser transition, $|g_k\rangle \leftrightarrows |e\rangle$ with de-tuning δ_k and Rabi pulsations $\Omega_k \in \mathbb{C}$.
Spontaneous emission rate $|e\rangle \mapsto |g_k\rangle$: Γ_k.

Lindbald-Kossakowski master-equation for the density matrix ρ

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H, \rho] + \sum_{k=1}^{N} \frac{1}{2}\left(2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k\right)$$

$\frac{H}{\hbar} = \sum_{k=1}^{N} \delta_k |g_k\rangle\langle g_k| + \Omega_k |g_k\rangle\langle e| + \Omega_k^* |e\rangle\langle g_k|$,
$L_k = \sqrt{\Gamma_k}\, |g_k\rangle\langle e|$. Photon flux (measure): $y = \sum_{k=1}^{N} \mathrm{tr}\left(L_k^\dagger L_k \rho\right)$.
Two time-scales: $|\delta_k|, |\Omega_k| \ll \Gamma_k$.

Main result: adiabatic elimination of the unstable state $|e\rangle$[2]

The slow/fast dynamics

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H,\rho] + \sum_{k=1}^{N} \frac{1}{2}\left(2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k\right)$$

with $L_k = \sqrt{\Gamma_k}\,|g_k\rangle\langle e|$, $\Gamma = \left(\sum_k \Gamma_k\right)$ much larger than $\frac{H}{\hbar}$, $y = \sum_k \operatorname{tr}\left(L_k^\dagger L_k \rho\right)$, is approximated by the slow dynamics

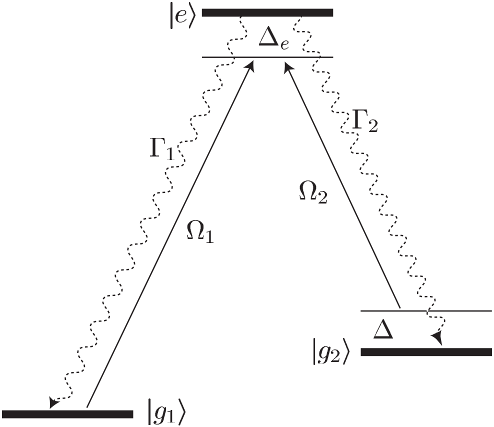
$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s,\rho_s] + \sum_{k=1}^{N} \frac{1}{2}\left(2L_{s,k}\rho_s L_{s,k}^\dagger - L_{s,k}^\dagger L_{s,k}\rho_s - \rho_s L_{s,k}^\dagger L_{s,k}\right)$$

with $\rho_s = (1-P)\rho(1-P)$ the slow density operator, $H_s = (1-P)H(1-P)$ the slow Hamiltonian and $L_{s,k} = 2\frac{L_k}{\Gamma}\frac{H}{\hbar}(1-P)$ the slow jump operators ($P = |e\rangle\langle e|$). The slow approximation of y is still given by the standard formula

$$y_s = \sum_{k=1}^{n} \operatorname{tr}\left(L_{s,k}^\dagger L_{s,k}\rho_s\right).$$

[2]See, Mirrahimi-R, CDC 2006 and IEEE AC to appear in 2009.

Application to the 3-level system (coherence population trapping[3])



Input: $\Omega_1, \Omega_2 \in \mathbb{C}$ and $\frac{d}{dt}\Delta$

Output: photo-detector click times corresponding to jumps from $|e\rangle$ to $|g_1\rangle$ or $|g_2\rangle$.

Two time-scales: $|\Omega_1|, |\Omega_2|, |\Delta_e|, |\Delta| \ll \Gamma_1, \Gamma_2$

[3] See, e.g., Arimondo: *Progr. Optics*, 35:257, 1996.

The slow/fast master equation

Master equation of the Λ-system

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H,\rho] + \frac{1}{2}\sum_{k=1}^{2}(2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k),$$

with jump operators $L_k = \sqrt{\Gamma_k}\,|g_k\rangle\langle e|$ and Hamiltonian

$$\frac{H}{\hbar} = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|) - \left(\Delta_e + \frac{\Delta}{2}\right)|e\rangle\langle e|$$
$$+ \Omega_1|g_1\rangle\langle e| + \Omega_1^*|e\rangle\langle g_1| + \Omega_2|g_2\rangle\langle e| + \Omega_2^*|e\rangle\langle g_2|.$$

Since $|\Omega_1|, |\Omega_2|, |\Delta_e|, |\Delta| \ll \Gamma_1, \Gamma_2$ we have two time-scales: a fast exponential decay for "$|e\rangle$" and a slow evolution for "$(|g_1\rangle, |g_2\rangle)$".

The slow master equation with bright and dark states.

The above general result leads to a reduced master equation that is still of Lindblad type with a slow Hamiltonian H_s and slow jump operators $L_{s,k}$:

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s, \rho_s] + \frac{1}{2}\sum_{k=1}^{2}(2L_{s,k}\rho_s L_{s,k}^\dagger - L_{s,k}^\dagger L_{s,k}\rho_s - \rho_s L_{s,k}^\dagger L_{s,k}),$$

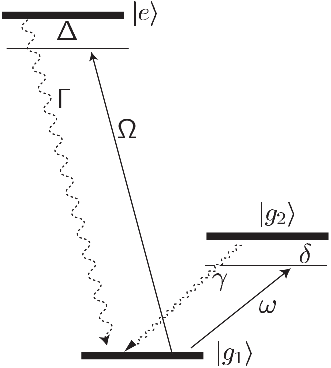
with $H_s = \frac{\Delta}{2}\sigma_z = \frac{\Delta(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|)}{2}$, $L_{s,k} = \sqrt{\gamma_k}\,|g_k\rangle\,\langle b_\Omega|$ where $\gamma_k = 4\frac{|\Omega_1|^2+|\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\Gamma_k$ and $|b_\Omega\rangle$ is the bright state:

$$|b_\Omega\rangle = \frac{\Omega_1}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}\,|g_1\rangle + \frac{\Omega_2}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}\,|g_2\rangle$$

For $\Delta = 0$, ρ_s converges towards the dark state $|d_\Omega\rangle$:

$$|d_\Omega\rangle = -\frac{\Omega_2^*}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}\,|g_1\rangle + \frac{\Omega_1^*}{\sqrt{|\Omega_1|^2+|\Omega_2|^2}}\,|g_2\rangle\,.$$

Extension to V-systems: Dehmelt's electron shelving scheme[4]



A stable state $|g_1\rangle$. A quasi-stable state: $|g_2\rangle$ with a long life time $1/\gamma$.
An unstable state: $|e\rangle$ with a short life time $1/\Gamma$.
Quasi resonant transitions:

- $|g_1\rangle \leftrightarrows |e\rangle$ with de-tuning Δ and Rabi pulsation $\Omega \in \mathbb{C}$.
- $|g_1\rangle \leftrightarrows |g_2\rangle$ with de-tuning δ and Rabi pulsation $\omega \in \mathbb{C}$.

Lindbald-Kossakowski master-equation for the density matrix ρ

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H,\rho] + \frac{1}{2}(2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L) + \frac{1}{2}(2l\rho l^\dagger - l^\dagger l\rho - \rho l^\dagger l)$$

$$\frac{H}{\hbar} = \Delta\,|e\rangle\langle e| + \Omega\,|g_1\rangle\langle e| + \Omega^*\,|e\rangle\langle g_1| + \delta\,|g_2\rangle\langle g_2| + \omega\,|g_1\rangle\langle g_2| + \omega^*\,|g_2\rangle\langle g_1|$$

$$L = \sqrt{\Gamma}\,|g_1\rangle\langle e|, \quad l = \sqrt{\gamma}\,|g_1\rangle\langle g_2|$$

Photon flux (measure): $y = \mathrm{tr}\left(L^\dagger L\rho\right) + \mathrm{tr}\left(l^\dagger l\rho\right)$. $(|\delta|, |\omega|, |\Omega|, \gamma \ll \Gamma)$.

[4] See, e.g., Cohen-Tannoudji-Dalibard: Europhys. Lett., 1986.

The slow master equation

The slow dynamics is still of Lindblad type with a slow Hamiltonian H_s, slow jump operators L_s and $l_s = l$:

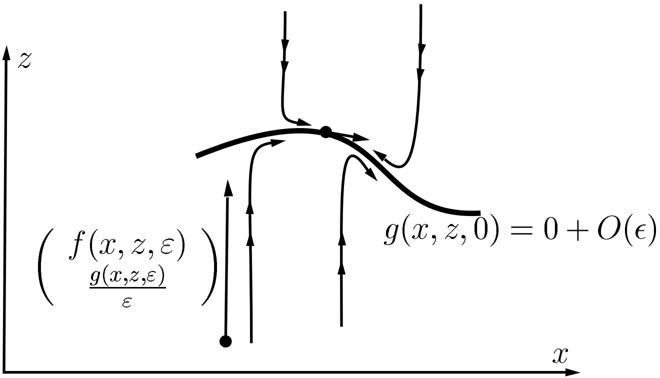
$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s, \rho_s] + \frac{1}{2}(2L_s\rho_s L_s^\dagger - L_s^\dagger L_s\rho_s - \rho_s L_s^\dagger L_s) + \frac{1}{2}(2l_s\rho_s l_s^\dagger - l_s^\dagger l_s\rho_s - \rho_s l_s^\dagger l_s)$$

$$\frac{H_s}{\hbar} = \delta \left|g_2\right\rangle \left\langle g_2\right| + \omega \left|g_1\right\rangle \left\langle g_2\right| + \omega^* \left|g_2\right\rangle \left\langle g_1\right|$$

$$L_s = 2\sqrt{\frac{|\Omega|^2}{\Gamma}} \left|g_1\right\rangle \left\langle g_1\right|, \quad l_s = l = \sqrt{\gamma} \left|g_1\right\rangle \left\langle g_2\right|$$

Photon flux (measure): $y = \mathrm{tr}\left(L_s^\dagger L_s \rho_s\right) + \mathrm{tr}\left(l_s^\dagger l_s \rho_s\right)$.

Slow/fast systems in Tikhonov normal form



$$(\Sigma^\varepsilon)\begin{cases} \frac{dx}{dt} = f(x,z,\varepsilon) \\ \varepsilon\frac{dz}{dt} = g(x,z,\varepsilon) \end{cases}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$,
$0 < \varepsilon \ll 1$ a small parameter ,
f and g regular functions.

Slow approximation (zero order in ε)

As soon as $g(x, z, 0) = 0$ admits a solution, $z = \rho(x)$, with ρ smooth function of x and $\frac{\partial g}{\partial z}(x, \rho(x), 0)$ is a stable matrix, the approximation of

$$(\Sigma^\varepsilon) \begin{cases} \dfrac{dx}{dt} = f(x, z, \varepsilon) \\ \varepsilon \dfrac{dz}{dt} = g(x, z, \varepsilon) \end{cases} \quad \text{by} \quad (\Sigma^0) \begin{cases} \dfrac{dx}{dt} = f(x, z, 0) \\ 0 = g(x, z, 0) \end{cases}$$

is valid for time intervals of length 1.
For longer intervals of length $1/\varepsilon$, correction terms of order 1 in ε should be included in Σ^0. They can be computed via center manifold techniques and Carr's approximation lemma[5].

[5] See, e.g., J. Carr: Application of Center Manifold Theory. Springer, 1981.

Proof based on matrix computations only [6]

With $Q_k = |g_k\rangle\langle e|$, $\Gamma_k = \frac{\overline{\Gamma}_k}{\varepsilon}$ and $0 < \varepsilon \ll 1$ the slow/fast master equation reads

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_{k=1}^{N} \frac{\overline{\Gamma}_k}{2\varepsilon}(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k \rho - \rho Q_k^\dagger Q_k).$$

Change of variables $\rho \mapsto (\rho_f, \rho_s)$ to put the system in Tikhonov normal form ($P = |e\rangle\langle e|$): $\rho_f = P\rho + \rho P - P\rho P$,
$\rho_s = (1-P)\rho(1-P) + \frac{1}{\left(\sum_{k=1}^N \overline{\Gamma}_k\right)} \sum_{k=1}^N \overline{\Gamma}_k \, Q_k \rho Q_k^\dagger$, with inverse
$\rho = \rho_s + \rho_f - \frac{1}{\left(\sum_{k=1}^N \overline{\Gamma}_k\right)} \sum_{k=1}^N \overline{\Gamma}_k \, Q_k \rho_f Q_k^\dagger$.

The dynamics in (ρ_s, ρ_f) "Tikhonov coordinates":

$$\frac{d}{dt}\rho_s = (1-P)\left[\frac{-\imath H}{\hbar}, \rho\right](1-P) + \frac{1}{\left(\sum_{k=1}^N \overline{\Gamma}_k\right)} \sum_{k=1}^N \overline{\Gamma}_k Q_k \left[\frac{-\imath H}{\hbar}, \rho\right] Q_k^\dagger$$

$$\varepsilon\frac{d}{dt}\rho_f = -\frac{\left(\sum_{k=1}^N \overline{\Gamma}_k\right)}{2}(\rho_f + P\rho_f P) - \frac{\varepsilon\imath}{\hbar}(P[H,\rho] + [H,\rho]P - P[H,\rho]P).$$

[6] See, Mirrahimi-R, CDC 2006 and IEEE AC to appear in 2009.

Order zero approximation in ε

- Setting ε to 0 in

$$\frac{d}{dt}\rho_s = (1-P)\left[\frac{-\imath H}{\hbar},\rho\right](1-P) + \frac{1}{\left(\sum_{k=1}^{N}\overline{\Gamma}_k\right)}\sum_{k=1}^{N}\overline{\Gamma}_k Q_k\left[\frac{-\imath H}{\hbar},\rho\right]Q_k^{\dagger}$$

$$\varepsilon\frac{d}{dt}\rho_f = -\frac{\left(\sum_{k=1}^{N}\overline{\Gamma}_k\right)}{2}(\rho_f + P\rho_f P) - \frac{\varepsilon\imath}{\hbar}(P[H,\rho] + [H,\rho]P - P[H,\rho]P).$$

yields to the coherent dynamics

$$\imath\hbar\frac{d}{dt}\rho_s = [(1-P)H(1-P),\rho_s]$$
$$\rho_f = 0$$

with $y = 0$.

- Need for higher order corrections terms in ε

High order approximation via center manifold techniques [7]

Consider the slow/fast system (f and g are regular functions)

$$\frac{d}{dt}x = f(x,z), \qquad \varepsilon \frac{d}{dt}z = -Az + \varepsilon g(x,z)$$

where all the eigenvalues of the matrix A have strictly positive real parts, and $0 < \varepsilon \ll 1$. The slow invariant attractive manifold admits for equation (boundary layer)

$$z = \varepsilon A^{-1} g(x,0) + O(\varepsilon^2)$$

and the restriction of the dynamics on this slow invariant manifold reads

$$\frac{d}{dt}x = f(x, \varepsilon A^{-1} g(x,0)) + O(\varepsilon^2) = f(x,0) + \varepsilon \frac{\partial f}{\partial z}|_{(x,0)} A^{-1} g(x,0) + O(\varepsilon^2).$$

Center-manifold approximations yield to second order terms in the expansion of z:

$$z = \varepsilon A^{-1} g(x,0) + \varepsilon^2 A^{-1} \left(\frac{\partial g}{\partial z}|_{(x,0)} A^{-1} g(x,0) - A^{-1} \frac{\partial g}{\partial x}|_{(x,0)} f(x,0) \right) + O(\varepsilon^3).$$

[7] See, e.g., Fenichel J. Diff. Eq. 1979 or Duchêne-R Chem. Eng. Sci. 1996.

Order one approximation in ε

Addition of first order correction terms in ε are related to decoherence and thus to Lindblad terms:

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s, \rho_s] + 2\varepsilon \sum_{k=1}^{N} \overline{\Gamma}_k \left(2Q_{s,k}\rho_s Q_{s,k}^\dagger - Q_{s,k}^\dagger Q_{s,k}\rho_s - \rho_s Q_{s,k}^\dagger Q_{s,k} \right)$$

where

$$H_s = (1-P)H(1-P) \qquad \text{and} \qquad Q_{s,k} = \frac{1}{\hbar \left(\sum_{l=1}^{N} \overline{\Gamma}_l \right)} (1-P)Q_k H(1-P).$$

The boundary layer reads

$$\rho_f = \frac{-2\imath\, \varepsilon}{\hbar \left(\sum_{k=1}^{N} \overline{\Gamma}_k \right)} \left(PH\rho_s - \rho_s HP \right) + O(\varepsilon^2).$$

and the output (measure)

$$y(t) = 4\varepsilon \left(\sum_{k=1}^{N} \overline{\Gamma}_k \right) \mathrm{tr}\left(\overline{P}\rho_s \right) + O(\varepsilon^2),$$

Concluding remarks

- The proposed adiabatic reduction mixing non commutative computations with operators and dynamical systems theory (geometric singular perturbations theory, invariant manifold) preserves the "physics" (CPT slow dynamics).
- In the slow master equation, the decoherence terms depend on the control input Ω_k: influence on controllability and optimal control?[8]
- Straightforward extensions to: several unstable states $|e_r\rangle$ with fast relaxation to stable states $|g_k\rangle$; slow decoherence between the "stable" states $|g_k\rangle$.
- A method to approximate slow/fast quantum trajectories by slow quantum trajectories where the jumps from $|e\rangle$ to $|g_k\rangle$ are replaced by jumps inside the "slow space"[9]

[8]See, e.g., Altafini and Bonnard-Chyba-Sugny for the recent results on controllability and optimal control of such dissipative systems.

[9]For mathematical justifications see: Bouten-Silberfarb: Commun. Math. Phys., 2008; Bouten-vanHandel-Silberfarb: Journal of Functional Analysis, 2008; Gough-vanHandel: J. Stat. Phys., 2007.

Quantum trajectories[10] associated to the slow master equation

Set $\gamma_k = 4\frac{|\Omega_1|^2+|\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\Gamma_k$ for $k = 1, 2$.
At each infinitesimal time step dt,

- ρ_s jumps
  - towards the state $|g_1\rangle\langle g_1|$ with a jump probability given by: $\langle b_\Omega|\rho_s|b_\Omega\rangle\ \gamma_1\ dt$.
  - or towards the state $|g_2\rangle\langle g_2|$ with a jump probability given by: $\langle b_\Omega|\rho_s|b_\Omega\rangle\ \gamma_2\ dt$.
- or ρ_s does not jump with probability

$$1 - \langle b_\Omega|\rho_s|b_\Omega\rangle\ (\gamma_1+\gamma_2)\ dt$$

and then evolves on the Bloch sphere according to

$$\frac{1}{\gamma}\frac{d}{dt}\rho_s = -\imath\frac{\Delta}{2\gamma}[\sigma_z,\rho_s] - \frac{|b_\Omega\rangle\langle b_\Omega|\rho_s + \rho_s|b_\Omega\rangle\langle b_\Omega|}{2} + \langle b_\Omega|\rho_s|b_\Omega\rangle\rho_s.$$

with $\gamma = \gamma_1 + \gamma_2$.

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[10] See, e.g., Haroche-Raimond book, 2006.

Quantum trajectories in Bloch-sphere coordinates

Set $\beta = 2\arg(\Omega_1 + \imath\Omega_2)$ and

$$\rho_s = \frac{1 + X(|b\rangle\langle d| + |d\rangle\langle b|) + Y(\imath|b\rangle\langle d| - \imath|d\rangle\langle b|) + Z(|d\rangle\langle d| - |b\rangle\langle b|)}{2}$$

At each infinitesimal time step dt, the point $(X, Y, Z) \in \mathbb{S}^2$,

- jumps
  - towards the state $(\sin\beta, 0, \cos\beta)$ with a jump probability given by: $\frac{(1-Z)}{2}\,\gamma_1\,dt$.
  - or towards the state $(-\sin\beta, 0, -\cos\beta)$ with a jump probability given by: $\frac{(1-Z)}{2}\,\gamma_2\,dt$.
- or does not jump with probability $1 - \frac{(1-Z)}{2}(\gamma_1 + \gamma_2)\,dt$ and then evolves according to

$$\frac{d}{dt}X = -\Delta\cos\beta\, Y - \gamma\frac{XZ}{2}$$
$$\frac{d}{dt}Y = \Delta\cos\beta X - \Delta\sin\beta Z - \gamma\frac{YZ}{2}$$
$$\frac{d}{dt}Z = \Delta\sin\beta\, Y + \frac{\gamma(1 - Z^2)}{2}$$

Convergence of the no-jump dynamics

For $|\Delta| < \frac{\gamma}{2}$, the nonlinear system in $\mathbb{S}^2$

$$\begin{aligned}
\frac{d}{dt}X &= -\Delta \cos\beta Y - \gamma\frac{XZ}{2} \\
\frac{d}{dt}Y &= \Delta \cos\beta X - \Delta \sin\beta Z - \gamma\frac{YZ}{2} \\
\frac{d}{dt}Z &= \Delta \sin\beta Y + \frac{\gamma(1-Z^2)}{2}
\end{aligned}$$

admits a two equilibirum points, one is unstable and the other one is quasi-global asymptotically stable.
Proof: based on Poincaré-Bendixon on the sphere.[11]

[11] See, Mirrahimi-R, 2008, arxiv:0806.1392v1