Modeling and Control of the LKB Photon-Box: [1] From discrete to continuous-time systems

Pierre Rouchon

pierre.rouchon@mines-paristech.fr

Würzburg, July 2011

[1] LKB: Laboratoire Kastler Brossel, ENS, Paris.
Several slides have been used during the IHP course (fall 2010) given with Mazyar Mirrahimi (INRIA) see:
http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

Stochastic models for open-quantum systems

Discrete-time models are Markov chains

$$\rho_{k+1} = \frac{1}{p_\nu(\rho_k)} M_\nu \rho_k M_\nu^\dagger \quad \text{with proba.} \quad p_\nu(\rho_k) = \text{Tr}\left(M_\nu \rho_k M_\nu^\dagger\right)$$

associated to Kraus maps (ensemble average, open quantum channels)

$$\mathbb{E}\left(\rho_{k+1}/\rho_k\right) = K(\rho_k) = \sum_\nu M_\nu \rho_k M_\nu^\dagger \quad \text{with} \quad \sum_\nu M_\nu^\dagger M_\nu = \mathbf{1}$$

Continuous-time models are stochastic differential systems

$$d\rho = \left(-i[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) dt + \left(L\rho + \rho L^\dagger - \text{Tr}\left((L+L^\dagger)\rho\right)\rho\right) dw$$

driven by Wiener processes[2] $dw = dy - \text{Tr}\left((L+L^\dagger)\rho\right) dt$ with measure y and associated to Lindbald master equations:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

[2]Another common possibility not considered here: SDE driven by Poisson processes.

Numerics based on Ito rules and Markov chains

For Monte-Carlo simulations of

$$d\rho = \left(-i[H,\rho] + L\rho L^\dagger - \tfrac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) dt + \left(L\rho + \rho L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\rho\right)\rho\right) dw$$

take a small sampling time dt, generate a random number dw_t according to a Gaussian law of standard deviation $\sqrt{dt}$, and set $\rho_{t+dt} = \mathbb{M}_{dy_t}(\rho_t)$ where the jump operator $\mathbb{M}_{dy_t}$ is labelled by the measurement outcome $dy_t = \mathrm{Tr}\left((L+L^\dagger)\,\rho_t\right) dt + dw_t$:

$$\mathbb{M}_{dy_t}(\rho_t) = \frac{\left(I+(-iH-\frac{1}{2}L^\dagger L)dt+Ldy_t\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt+L^\dagger dy_t\right)}{\mathrm{Tr}\left(\left(I+(-iH-\frac{1}{2}L^\dagger L)dt+Ldy_t\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt+L^\dagger dy_t\right)\right)}.$$

Then ρ_{t+dt} remains always a density operator and using the Ito rules (dw of order $\sqrt{dt}$ and $dw^2 \equiv dt$) we get the good $d\rho = \rho_{t+dt} - \rho_t$ up to $O((dt)^{3/2})$ terms.

Quantum filtering

Assume $t \mapsto y_t$ is the detector signal. Then the quantum filter equation providing an estimation $\widehat{\rho}$ of the state ρ reads [3]

$$
\begin{aligned}
d\widehat{\rho} = & \left(-i[H,\widehat{\rho}] + L\widehat{\rho}L^\dagger - \frac{1}{2}(L^\dagger L\widehat{\rho} + \widehat{\rho}L^\dagger L)\right) dt \\
& + \left(L\widehat{\rho} + \widehat{\rho}L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)\widehat{\rho}\right) \left(dy - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right) dt\right)
\end{aligned}
$$

since $dw = dy - \mathrm{Tr}\left((L+L^\dagger)\rho\right) dt$. To get this filter take a small sampling time dt, use the measure outcome dy_t and the jump operator labelled by dy_t to update ρ according to $\widehat{\rho}_{t+dt} = \mathbb{M}_{dy_t}(\widehat{\rho}_t)$. You recover then up to $O(dt^{3/2})$ terms the above filter that reads also

$$
\begin{aligned}
d\widehat{\rho} = & \left(-i[H,\widehat{\rho}] + L\widehat{\rho}L^\dagger - \frac{1}{2}(L^\dagger L\widehat{\rho} + \widehat{\rho}L^\dagger L)\right) dt \\
& + \left(L\widehat{\rho} + \widehat{\rho}L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)\widehat{\rho}\right) dw \\
& - \left(L\widehat{\rho} + \widehat{\rho}L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\widehat{\rho}\right)\widehat{\rho}\right) \mathrm{Tr}\left((L+L^\dagger)(\widehat{\rho}-\rho)\right) dt.
\end{aligned}
$$

[3] See Belavkin theoretical work

Numerics based on Kraus maps

For the Lindblad equation

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

take a small sampling time *dt* and set

$$\rho_{t+dt} = \frac{\left(I+(-iH-\frac{1}{2}L^\dagger L)dt\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt\right)+dtL\rho_t L^\dagger}{\mathrm{Tr}\left(\left(I+(-iH-\frac{1}{2}L^\dagger L)dt\right)\rho_t\left(I+(iH-\frac{1}{2}L^\dagger L)dt\right)+dtL\rho_t L^\dagger\right)}.$$

Then ρ_{t+dt} remains always a density operator and $\frac{d}{dt}\rho = (\rho_{t+dt} - \rho_t)/dt$ up to $O(dt)$ terms.

The controlled SDE of a two-level system

- Attached to $H = \frac{u}{2}\sigma_x$, $L = \sqrt{\frac{\gamma}{2}}\sigma_z$, the controlled SDE

$$d\rho = \left(-i\frac{u}{2}[\sigma_x, \rho] + \frac{\gamma}{2}\sigma_z\rho\sigma_z - \gamma\rho\right) dt + \sqrt{\frac{\gamma}{2}}\Big(\sigma_z\rho + \rho\sigma_z - 2\text{Tr}\,(\sigma_z\rho)\rho\Big) dw$$

  where $u \in \mathbb{R}$ is the control input and
  $dy = \sqrt{2\gamma}\ \text{Tr}\,(\sigma_z\rho)dt + dw$ is the measured output.
- For open-loop almost-sure convergence towards $|g\rangle\langle g|$ or $|e\rangle\langle e|$ take $V(\rho) = \text{Tr}\,(\sigma_z\rho)^2$ as Lyapunov function.
- For closing the loop take . . . or see M. Mirrahimi and R. Van Handel: Stabilizing feedback controls for quantum systems. SIAM Journal on Control and Optimization, 2007

Few references

- L. Arnold, Stochastic differential equations: theory and applications. Wiley-Interscience, New York, 1974.
- N.G. Van Kampen, Stochastic processes in physics and chemistry. Elsevier, 1992.
- The lectures on quantum circuits given by Michel Devoret at Collège de France: `http://www.physinfo.fr/lectures.html`