Modeling and Control of Quantum Systems

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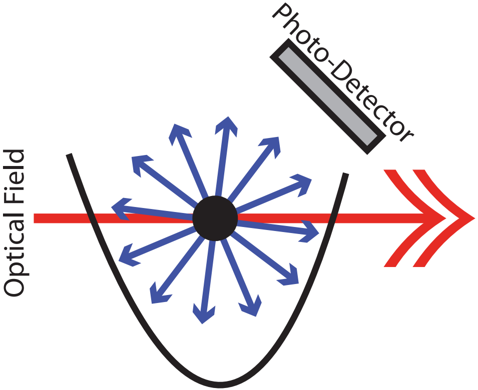
Continuous-time measurement

A single atom within a Paul trap is addressed by an external optical field and the spontaneously emitted photons are detected by surrounding photodetectors.



Outline

1. Spontaneous emission, quantum Monte-Carlo trajectories and Lindblad equation
2. Λ-system
3. Slow/fast dynamics and model reduction
4. Physical interpretation and reduced Monte-Carlo trajectories

Spontaneous emission and its modeling (1)

State space: $\{\rho \in \mathbb{C}^{2\times 2} \mid \rho^\dagger = \rho,\ \rho \geq 0,\ \mathrm{Tr}(\rho) = 1\}$

Probability of having a jump in $[t, t+dt[$:

$$p_{\text{jump}} = \Gamma \langle e \mid \rho(t) \mid e \rangle dt.$$

Γ: decay rate of the system which is equivalent to the inverse of the atomic lifetime of the excited state $|e\rangle$.

Associated measurement operator:

$$\mathcal{M}_{\text{jump}} = \sqrt{\Gamma dt}\, \sigma_-, \qquad \sigma_- = |g\rangle \langle e| \,.$$

As soon as we detect a photon, the density matrix collapses into the ground state:

$$\rho_{t+dt} = \frac{\mathcal{M}_{\text{jump}} \rho(t) \mathcal{M}_{\text{jump}}^\dagger}{\mathrm{Tr}\left(\mathcal{M}_{\text{jump}} \rho(t) \mathcal{M}_{\text{jump}}^\dagger\right)} = |g\rangle \langle g| \,.$$

Spontaneous emission and its modeling (2)

Question

What happens to the density matrix when we do not detect any photon?

Answer: some information is gained on the state; with a larger probability we have been in $|g\rangle$; We call the associated measurement operator $\mathcal{M}_{\text{no-jump}}$; We have $\mathcal{M}_{\text{no-jump}} \neq \mathbf{1}$.

POVM requirement:

$$\mathcal{M}_{\text{jump}}^{\dagger}\mathcal{M}_{\text{jump}} + \mathcal{M}_{\text{no-jump}}^{\dagger}\mathcal{M}_{\text{no-jump}} = \mathbf{1}.$$

How to compute $\mathcal{M}_{\text{no-jump}}$?

Spontaneous emission and its modeling (3)

Generic form of $\mathcal{M}_{\text{no-jump}}$: it must be of the form $\mathbf{1} + O(dt)$,

$$\mathcal{M}_{\text{no-jump}} = \mathbf{1} - \Gamma dt \mathcal{A} - i dt H,$$

where $\mathcal{A}$ and H are Hermitian matrices in $\mathbb{C}^{2\times 2}$.

POVM requirement+ 1st order development:

$$\mathcal{A} = \frac{1}{2}\sigma_+\sigma_-, \qquad \sigma_+ = \sigma_-^\dagger = |e\rangle\langle g|\,.$$

No-jump dynamics:

$$\rho(t+dt) = \frac{\mathcal{M}_{\text{no-jump}}\rho\mathcal{M}_{\text{no-jump}}^\dagger}{\text{Tr}\left(\mathcal{M}_{\text{no-jump}}\rho(t)\mathcal{M}_{\text{no-jump}}^\dagger\right)}$$

$$= \rho(t) - dt\frac{\Gamma}{2}\left(\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_-\right) + dt\Gamma\,\text{Tr}\left(\sigma_-\rho(t)\sigma_+\right)\rho(t)$$

$$-i\, dt\, [\mathcal{B}, \rho(t)],$$

H implies a unitary evolution and can be added to the usual Hamiltonian of the system: a corrective Hamiltonian due to the coupling to the vacuum modes of the free radiation field. This implies a relaxation-induced shift in the energy levels of the atom (Lamb shift).

Quantum Monte-Carlo trajectories

$$\rho(t+dt) = \begin{cases} \dfrac{\sigma_-\rho(t)\sigma_+}{\mathrm{Tr}\left(\sigma_-\rho(t)\sigma_+\right)} = |g\rangle\langle g| & \text{with probability } dt\Gamma\mathrm{Tr}\left(\sigma_-\rho(t)\sigma_+\right), \\ \rho(t) - i\, dt\, [H(t), \rho(t)] - dt\dfrac{\Gamma}{2}\left(\sigma_+\sigma_-\rho(t) + \rho(t)\sigma_+\sigma_-\right) & \\ \quad + dt\Gamma\mathrm{Tr}\left(\sigma_-\rho(t)\sigma_+\right)\rho(t) & \text{with probability } 1 - dt\Gamma\mathrm{Tr}\left(\sigma_-\rho(t)\sigma_+\right), \end{cases}$$

Poisson process: in any given time interval $[t, t+dt[$, we define dN_t such that it is unity with probability $\Gamma\mathrm{Tr}\left(\sigma_-\rho(t)\sigma_+\right)dt$ and zero otherwise. We have

$$\mathbb{E}\left(dN_t \mid \rho(t)\right) = \Gamma\mathrm{Tr}\left(\sigma_-\rho(t)\sigma_+\right)dt.$$

Stochastic master equation:

$$\rho(t+dt) - \rho(t) = d\rho = \left(-i[H(t), \rho] - \frac{\Gamma}{2}\left(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-\right) + \Gamma\mathrm{Tr}\left(\sigma_-\rho\sigma_+\right)\rho\right) dt + \left(\frac{\sigma_-\rho\sigma_+}{\mathrm{Tr}\left(\sigma_-\rho\sigma_+\right)} - \rho\right) dN_t.$$

Lindblad master equation

We consider a statistical ensemble of identical two-level atoms with no mutual interactions. Applying the statistical independence of dN_t and ρ_t, we get the following average dynamics

$$\frac{d\rho}{dt} = -i[H(t), \rho] + \Gamma \left(\sigma_- \rho \sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_- \right),$$

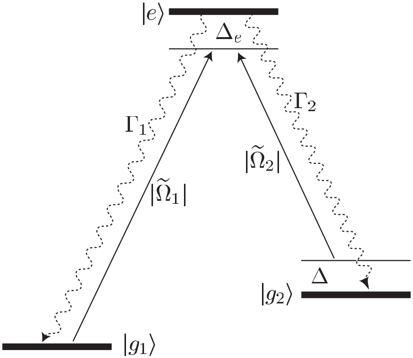
where (by an abuse of notations) ρ actually stands for the expectation value of ρ in the above jump dynamics.

Exercice

When $H = 0$*, show that* $\lim_{t\mapsto+\infty} \rho(t) = |g\rangle\langle g|$.

Λ-system

State space: $\{\rho \in \mathbb{C}^{3\times 3} \mid \rho^\dagger = \rho,\ \rho \geq 0,\ \mathrm{Tr}\,(\rho) = 1\}$.



Relevant energy levels, transitions and decoherence rates for the Λ-system.

Λ-system: stochastic master equation

$$
\begin{aligned}
d\rho = & -i[H_0 + u(t)H_1, \rho]dt \\
& - \frac{1}{2}(Q_1^\dagger Q_1 \rho + \rho Q_1^\dagger Q_1)dt + \mathrm{Tr}\left(Q_1 \rho Q_1^\dagger\right)\rho dt + \left(\frac{Q_1 \rho Q_1^\dagger}{\mathrm{Tr}\left(Q_1 \rho Q_1^\dagger\right)} - \rho\right) dN_t^1 \\
& - \frac{1}{2}(Q_2^\dagger Q_2 \rho + \rho Q_2^\dagger Q_2)dt + \mathrm{Tr}\left(Q_2 \rho Q_2^\dagger\right)\rho dt + \left(\frac{Q_2 \rho Q_2^\dagger}{\mathrm{Tr}\left(Q_2 \rho Q_2^\dagger\right)} - \rho\right) dN_t^2,
\end{aligned}
$$

where

$$
\begin{aligned}
H_0 &= \omega_e \left|e\right\rangle\left\langle e\right| + \omega_{g1}\left|g_1\right\rangle\left\langle g_1\right| + \omega_{g2}\left|g_2\right\rangle\left\langle g_2\right|, \\
H_1 &= \mu_1(\left|g_1\right\rangle\left\langle e\right| + \left|e\right\rangle\left\langle g_1\right|) + \mu_2(\left|g_2\right\rangle\left\langle e\right| + \left|e\right\rangle\left\langle g_2\right|), \\
Q_1 &= \sqrt{\Gamma_1}\left|g_1\right\rangle\left\langle e\right|, \qquad Q_2 = \sqrt{\Gamma_2}\left|g_2\right\rangle\left\langle e\right|,
\end{aligned}
$$

and where dN_t^1 and dN_t^2 are independent Poisson increments with averages

$$
\mathbb{E}\left(dN_t^1\right) = \mathrm{Tr}\left(Q_1 \rho Q_1^\dagger\right)dt, \qquad \mathbb{E}\left(dN_t^2\right) = \mathrm{Tr}\left(Q_2 \rho Q_2^\dagger\right)dt.
$$

Λ-system: time scales

Quasi-resonant field:

$$u(t) = u_1 e^{i(\omega_1+\Delta_e)t} + u_1^* e^{-i(\omega_1+\Delta_e)t} + u_2 e^{i(\omega_2+\Delta_e+\Delta)t} + u_2^* e^{-i(\omega_2+\Delta_e+\Delta)t},$$

where $\omega_1 = \omega_e - \omega_{g1}$ and $\omega_2 = \omega_e - \omega_{g2}$, u_1 and u_2 are slowly varying complex amplitudes and Δ_e and Δ are small detuning terms. We have three time scales here:

- the very fast time-scale associated to the optical frequencies ω_1 and ω_2;
- the fast time-scale associated to the lifetimes of the excited state's transitions, Γ_1 and Γ_2;
- the slow time-scale associated to the laser amplitudes $|\mu_1 u_1|$ and $|\mu_2 u_2|$.

We have

$$|\mu_k u_k| \ll \Gamma_{k'} \ll \omega_{k''} \qquad \text{and} \qquad \left|\frac{d}{dt} u_k\right| \Big/ |u_k| \ll \Gamma_{k'}, \qquad k, k', k'' \in \{1, 2\}.$$

Λ-system: RWA

Lindblad equation:

$$\frac{d\rho}{dt} = -i[H_0 + u(t)H_1, \rho] + \frac{1}{2}\sum_{k=1}^{2}\left(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k\right).$$

Rotating frame: $\rho(t) \to U_t^\dagger \rho(t) U_t$ with

$$U_t = e^{-i\left(\omega_e|e\rangle\langle e| + (\omega_{g1}-\Delta_e)|g_1\rangle\langle g_1| + (\omega_{g2}-\Delta_e-\Delta)|g_2\rangle\langle g_2|\right)t}$$

Removing the highly oscillating terms of frequencies $2\omega_1$ **and** $2\omega_2$**:**

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \frac{1}{2}\sum_{k=1}^{2}(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k).$$

where

$$\tilde{H} = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|) + \left(\Delta_e + \frac{\Delta}{2}\right)(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|) + \Omega_1|g_1\rangle\langle e| + \Omega_1^*|e\rangle\langle g_1| + \Omega_2|g_2\rangle\langle e| + \Omega_2^*|e\rangle\langle g_2|.$$

where $\Omega_k = \mu_k u_k$ are the slowly varying complex Rabi amplitudes.

Slow/fast dynamics

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \frac{1}{2}\sum_{k=1}^{2}(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k).$$

Time-scale separation:

$$|\Delta_e|, |\Delta|, |\Omega_k| \ll \Gamma_{k'} \quad \text{and} \quad \left|\frac{d}{dt}\Omega_k\right| \Big/ |\Omega_k| \ll \Gamma_{k'}, \qquad k, k' \in \{1, 2\}.$$

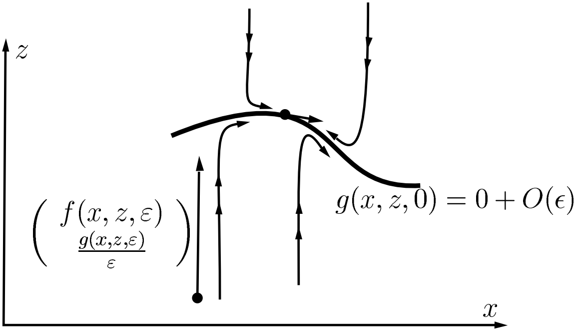
We take $\Gamma_k = \bar{\Gamma}_k/\epsilon$ where ϵ is a small positive parameter and $\bar{\Gamma}_k$'s are of the same order as $\tilde{H}$:

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \sum_{k=1}^{2}\frac{\bar{\Gamma}_k}{2\epsilon}(2\sigma_k\rho\sigma_k^\dagger - \sigma_k^\dagger\sigma_k\rho - \rho\sigma_k^\dagger\sigma_k),$$

where $\sigma_k = |g_k\rangle\langle e|$.

Singular perturbation techniques (1)

$$(\Sigma_\varepsilon): \quad \frac{d}{dt}x = f(x,z,\varepsilon), \qquad \varepsilon\frac{d}{dt}z = g(x,z,\varepsilon)$$



Slow/fast system in Tikhonov normal; under some assumptions, the slow approximation (also called quasi-static or adiabatic elimination), consists in setting directly ε to 0 in the equation defining (Σ_ε); this yields to a differential-algebraic system $\frac{d}{dt}x = f(x,z,0)$ where z is an implicit function of x defined by $0 = g(x,z,0)$.

Singular perturbation techniques (2)

Tikhonov Theorem

Consider the singularly perturbed system :

$$(\Sigma_\varepsilon): \quad \frac{d}{dt}x = f(x,z,\varepsilon), \qquad \varepsilon\frac{d}{dt}z = g(x,z,\varepsilon)$$

where (x,z) belongs to an open subset of $\mathbb{R}^n \times \mathbb{R}^p$, f and g are smooth functions, ε is a small positive parameter. Assume that

- $g(x,z,0) = 0$ admits a solution $z = \Phi(x)$, with Φ smooth function of x and such that $\frac{\partial g}{\partial z}(x,\Phi(x),0)$ is a stable matrix (eigenvalues with strictly negative real parts).
- the reduced slow sub-system $\frac{d}{dt}x = f(x,\Phi(x),0)$, $x(0) = x_0$ admits a unique solution $x^0(t)$ defined for $t \in [0,T]$, $0 < T < +\infty$ for some $T > 0$.

Then, for $\varepsilon > 0$ small enough, (Σ_ε) admits a unique solution $(x^\varepsilon(t), z^\varepsilon(t))$ defined on $[0,T]$ with initial condition $(x^\varepsilon(0), z^\varepsilon(0)) = (x_0, z_0)$ as soon as z_0 belongs to the attraction domain of the equilibrium $\Phi(x_0)$ for the fast sub-system, $\varepsilon\frac{d}{dt}\zeta = g(x_0,\zeta,0)$. Moreover we have, for any $\eta > 0$,

$$\lim_{\varepsilon\to 0^+}\left(\max_{t\in[\eta,T]}\left(\|x^\varepsilon(t) - x^0(t)\| + \|z^\varepsilon(t) - z^0(t)\|\right)\right) = 0.$$

Singular perturbation techniques (3)

Higher-order approximations and center manifold techniques

We consider a slow/fast system of the form

$$(\Sigma_\varepsilon): \quad \frac{d}{dt}x = f(x, z, \varepsilon), \qquad \varepsilon \frac{d}{dt}z = -Az + \varepsilon h(x, z)$$

where all the eigenvalues of the matrix A have strictly positive real parts. The invariant attractive manifold admits for equation

$$z = \varepsilon A^{-1} h(x, 0) + O(\varepsilon^2)$$

and the restriction of the dynamics on this slow invariant manifold reads

$$\frac{d}{dt}x = f(x, \varepsilon A^{-1} h(x, 0)) + O(\varepsilon^2) = f(x, 0) + \varepsilon \left.\frac{\partial f}{\partial z}\right|_{(x,0)} A^{-1} h(x, 0) + O(\varepsilon^2).$$

The second order term is then given by:

$$z = \varepsilon A^{-1} h(x, 0) + \varepsilon^2 A^{-1} \left(\left.\frac{\partial h}{\partial z}\right|_{(x,0)} A^{-1} h(x, 0) - A^{-1} \left.\frac{\partial h}{\partial x}\right|_{(x,0)} f(x, 0) \right) + O(\varepsilon^3),$$

and so on.

Singular perturbation techniques (4)

Roughly speaking, an approximation of order ν in ε of the slow invariant manifold provides an approximation on time intervals of length of order $\frac{1}{\varepsilon^\nu}$ as sketched below:

- $z = 0$ is an approximation of order 0; the slow reduced model $\frac{d}{dt}x = f(x, 0)$ is valid on time intervals of length 1.
- $z = \varepsilon A^{-1} h(x, 0)$ is an approximation of order 1: the slow reduced model $\frac{d}{dt}x = f(x, \varepsilon A^{-1} h(x, 0))$ is valid on time intervals of length $\frac{1}{\varepsilon}$.
- $z = \varepsilon A^{-1} h(x, 0) + \varepsilon^2 A^{-1} \left(\frac{\partial h}{\partial z}|_{(x,0)} A^{-1} h(x, 0) - A^{-1} \frac{\partial h}{\partial x}|_{(x,0)} f(x, 0) \right)$ is an approximation of order 2: the slow reduced model

$$\frac{d}{dt}x = f\left(x,\ \varepsilon A^{-1} h(x, 0) + \varepsilon^2 A^{-1} \left(\frac{\partial h}{\partial z}|_{(x,0)} A^{-1} h(x, 0) - A^{-1} \frac{\partial h}{\partial x}|_{(x,0)} f(x, 0) \right) \right)$$

  is valid on time intervals of length $\frac{1}{\varepsilon^2}$.

Singular perturbation for slow/fast Λ-system

Slow/fast system in non-standard form:

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \sum_{k=1}^{2} \frac{\overline{\Gamma}_k}{2\epsilon}(2\sigma_k \rho \sigma_k^\dagger - \sigma_k^\dagger \sigma_k \rho - \rho \sigma_k^\dagger \sigma_k), \qquad \sigma_k = |g_k\rangle \langle e| .$$

Define, with $P = |e\rangle \langle e|$,

$$\rho_f = P\rho + \rho P - P\rho P \quad , \quad \rho_s = (1-P)\rho(1-P) + \frac{1}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \sum_{k=1}^{2} \overline{\Gamma}_k \, \sigma_k \rho \sigma_k^\dagger .$$

ρ_s remains a density matrix but not ρ_f. We have

$$\rho = \rho_s + \rho_f - \frac{1}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \sum_{k=1}^{2} \overline{\Gamma}_k \, \sigma_k \rho_f \sigma_k^\dagger$$

and therefore $\rho \mapsto (\rho_f, \rho_s)$ is a bijective map (change of variables).
Slow/fast system in standard form:

$$\frac{d}{dt}\rho_f = -\frac{(\overline{\Gamma}_1 + \overline{\Gamma}_2)}{2\epsilon}(\rho_f + P\rho_f P) - i(P[\tilde{H}, \rho] + [\tilde{H}, \rho]P - P[\tilde{H}, \rho]P),$$

$$i\frac{d}{dt}\rho_s = (1-P)[\tilde{H}, \rho](1-P) + \frac{1}{\overline{\Gamma}_1 + \overline{\Gamma}_2} \sum_{k=1}^{2} \overline{\Gamma}_k \sigma_k [\tilde{H}, \rho] \sigma_k^\dagger .$$

1st order slow/fast approximation for Λ-system

The system is of the form ($x \sim \rho_s$, $z \sim \rho_f$)

$$(\Sigma_\varepsilon): \quad \tfrac{d}{dt}x = f(x,z,\varepsilon), \qquad \varepsilon \tfrac{d}{dt}z = -Az + \varepsilon h(x,z)$$

where A is a positive definite super-operator sending ρ_f to $\rho_f + P\rho_f P$. Its inverse A^{-1} is given by

$$\rho_f \mapsto \rho_f - \frac{1}{2}P\rho_f P.$$

First order approximation for ρ_f:

$$\rho_f = \frac{-2i\epsilon}{\overline{\Gamma}_1 + \overline{\Gamma}_2}\,(P\tilde{H}\rho_s - \rho_s\tilde{H}P) + O(\epsilon^2).$$

First order dynamics for ρ_s:

$$\frac{d}{dt}\rho_s = -i[\overline{H},\rho_s] + \frac{\epsilon}{2}\sum_{k=1}^{2}\left(2\overline{Q}_k\rho_s\overline{Q}_k^\dagger - \overline{Q}_k^\dagger\overline{Q}_k\rho_s - \rho_s\overline{Q}_k^\dagger\overline{Q}_k\right)$$

where we have defined

$$\overline{H} = (1-P)\tilde{H}(1-P) \qquad \text{and} \qquad \overline{Q}_k = \frac{2\sqrt{\overline{\Gamma}_k}}{\overline{\Gamma}_1+\overline{\Gamma}_2}(1-P)\sigma_k\tilde{H}(1-P).$$

Slow/fast approximation for Λ-system

Theorem

Consider ρ the solution of the Lindblad master equation

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \sum_{k=1}^{2} \frac{\overline{\Gamma}_k}{2\epsilon}(2\sigma_k \rho \sigma_k^\dagger - \sigma_k^\dagger \sigma_k \rho - \rho \sigma_k^\dagger \sigma_k),$$

with $0 < \epsilon \ll 1$ and ρ_s the solution of the slow master equation

$$\frac{d}{dt}\rho_s = -i[\overline{H}, \rho_s] + \frac{\epsilon}{2}\sum_{k=1}^{2}\left(2\overline{Q}_k \rho_s \overline{Q}_k^\dagger - \overline{Q}_k^\dagger \overline{Q}_k \rho_s - \rho_s \overline{Q}_k^\dagger \overline{Q}_k\right)$$

with

$$\overline{H} = (1-P)\tilde{H}(1-P) \qquad \text{and} \qquad \overline{Q}_k = \frac{2\sqrt{\overline{\Gamma}_k}}{\overline{\Gamma}_1 + \overline{\Gamma}_2}(1-P)\sigma_k \tilde{H}(1-P).$$

Assume for the initial states
$\|\rho(0) - \rho_s(0)\| = \sqrt{\text{Tr}\left((\rho(0) - \rho_s(0))(\rho(0) - \rho_s(0))\right)} = O(\epsilon)$. Then

$$\|\rho(t) - \rho_s(t)\| = \sqrt{\text{Tr}\left((\rho(t) - \rho_s(t))(\rho(t) - \rho_s(t))\right)} = O(\epsilon)$$

on a time scale $t \sim 1/\epsilon$.

Slow/fast approximation: summary

The slow approximation (also called by physicists adiabatic approximation) of the system described by

$$\frac{d}{dt}\rho = -i[\tilde{H}, \rho] + \frac{1}{2}\sum_{k=1}^{2}\left(2Q_k\rho Q_k^\dagger - Q_k^\dagger Q_k\rho - \rho Q_k^\dagger Q_k\right)$$

with $Q_k = \sqrt{\Gamma_k}\,|g_k\rangle\langle e|$ and where the Γ_k's are much larger than $\tilde{H}$, is given by

$$\frac{d}{dt}\rho_s = -i[H_s, \rho_s] + \frac{1}{2}\sum_{k=1}^{2}\left(2Q_{s,k}\rho_s Q_{s,k}^\dagger - Q_{s,k}^\dagger Q_{s,k}\rho_s - \rho_s Q_{s,k}^\dagger Q_{s,k}\right)$$

where ρ_s is the density operator associated with the space spanned by the $|g_1\rangle$ and $|g_2\rangle$, and where the slow Hamiltonian and the slow jump operators are ($P = |e\rangle\langle e|$)

$$H_s = (1-P)\tilde{H}(1-P) \quad \text{and} \quad Q_{s,k} = \frac{2}{\Gamma_1+\Gamma_2}Q_k\tilde{H}(1-P), \qquad k \in \{1,2\}.$$

Reduced Monte-Carlo trajectories (1)

We have

$$H_s = \frac{\Delta}{2}(|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|) + (\Delta_e + \frac{\Delta}{2})(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|).$$

and

$$Q_{s,k} = 2\sqrt{\Gamma_k}\frac{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}{\Gamma_1 + \Gamma_2}|g_k\rangle\langle b_\Omega| \quad \text{with} \quad |b_\Omega\rangle = \frac{\Omega_1|g_1\rangle + \Omega_2|g_2\rangle}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}.$$

The slow master equation lives on the Hilbert space spanned by $|g_1\rangle$ and $|g_2\rangle$.

Reduced stochastic master equation:

$$d\rho_s = -i\frac{\Delta}{2}\left[|g_2\rangle\langle g_2| - |g_1\rangle\langle g_1|, \rho_s\right]dt$$

$$-\frac{1}{2}\left(Q_{s,1}^\dagger Q_{s,1}\rho_s + \rho_s Q_{s,1}^\dagger Q_{s,1}\right)dt + \text{Tr}\left(Q_{s,1}\rho_s Q_{s,1}^\dagger\right)\rho_s dt + \left(\frac{Q_{s,1}\rho_s Q_{s,1}^\dagger}{\text{Tr}\left(Q_{s,1}\rho_s Q_{s,1}^\dagger\right)} - \rho_s\right)dN_t^{s,1}$$

$$-\frac{1}{2}\left(Q_{s,2}^\dagger Q_{s,2}\rho_s + \rho_s Q_{s,2}^\dagger Q_{s,2}\right)dt + \text{Tr}\left(Q_{s,2}\rho_s Q_{s,2}^\dagger\right)\rho_s dt + \left(\frac{Q_{s,2}\rho_s Q_{s,2}^\dagger}{\text{Tr}\left(Q_{s,2}\rho_s Q_{s,2}^\dagger\right)} - \rho_s\right)dN_t^{s,2}.$$

Reduced Monte-Carlo trajectories (2)

Here $dN_t^{s,1}$ and $dN_t^{s,2}$ are independent Poisson increments with averages

$$\mathbb{E}\left(dN_t^{s,1}\right) = \text{Tr}\left(Q_{s,1}\rho_s Q_{s,1}^\dagger\right)dt = 4\Gamma_1 \frac{|\Omega_1|^2 + |\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\text{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s\right)dt$$

$$\mathbb{E}\left(dN_t^{s,2}\right) = \text{Tr}\left(Q_{s,2}\rho_s Q_{s,2}^\dagger\right)dt = 4\Gamma_2 \frac{|\Omega_1|^2 + |\Omega_2|^2}{(\Gamma_1+\Gamma_2)^2}\text{Tr}\left(|b_\Omega\rangle\langle b_\Omega|\,\rho_s\right)dt.$$

Reduced Monte-Carlo trajectories (3)

We define

$$\gamma_k = 4\Gamma_k \frac{|\Omega_1|^2 + |\Omega_2|^2}{(\Gamma_1 + \Gamma_2)^2}, \qquad k \in \{1, 2\},$$

the evolution through the time interval $(t, t + dt)$ can be interpreted as below:

- ρ_s jumps into the ground state $|g_1\rangle \langle g_1|$ with probability $dt\gamma_1 \mathrm{Tr}\left(|b_\Omega\rangle \langle b_\Omega| \rho_s(t)\right)$;
- or it jumps into the ground state $|g_2\rangle \langle g_2|$ with probability $dt\gamma_2 \mathrm{Tr}\left(|b_\Omega\rangle \langle b_\Omega| \rho_s(t)\right)$;
- or finally, it evolves through the dynamics

$$\frac{d}{dt}\rho_s = -i\frac{\Delta}{2}\left[|g_2\rangle \langle g_2| - |g_1\rangle \langle g_1|, \rho_s\right]$$
$$- \frac{(\gamma_1 + \gamma_2)}{2}\Big(|b_\Omega\rangle \langle b_\Omega| \rho_s + \rho_s |b_\Omega\rangle \langle b_\Omega| - 2\mathrm{Tr}\left(|b_\Omega\rangle \langle b_\Omega| \rho_s\right)\rho_s \Big),$$

with probability $1 - dt(\gamma_1 + \gamma_2)\mathrm{Tr}\left(|b_\Omega\rangle \langle b_\Omega| \rho_s(t)\right)$.

Physical interpretation

the state $|b_\Omega\rangle$ is often called the bright state and the orthogonal state

$$|d_\Omega\rangle = \frac{\Omega_2^*}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}} |g_1\rangle - \frac{\Omega_1^*}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}} |g_2\rangle$$

is called the dark state. Indeed, the probability of jumping towards one of the ground states by emitting a photon is proportional to the population of the bright state $|b_\Omega\rangle$. Therefore, whenever the system is in the state $|d_\Omega\rangle$, no photon will be emitted: hence the name of the dark state.

Theorem

Whenever $\Delta = 0$, the density matrix ρ_s, solution of the reduced slow stochastic master equation converges almost surely towards the dark state $|d_\Omega\rangle \langle d_\Omega|$.

Remark

The phenomenon of converging towards the dark state is often referred as the coherent population trapping in the physics literature. The target state can be controlled via the ratio Ω_1/Ω_2. The case $\Omega_2 = 0$ ($|d_\Omega\rangle = |g_2\rangle$) corresponds to the optical pumping phenomena.

Proof of coherent population trapping

We consider the Markov process:

$$f_t = \mathrm{Tr}\left(\left|d_\Omega\right\rangle\left\langle d_\Omega\right|\rho(t)\right).$$

We can easily compute the evolution of the expectation value of f_t:

$$\frac{d}{dt}\mathbb{E}\left(f_t\right) = \frac{\gamma_1|\Omega_2|^2 + \gamma_2|\Omega_1|^2}{|\Omega_1|^2 + |\Omega_2|^2}\left(1 - \mathbb{E}\left(f_t\right)\right).$$

This, together with the fact that $f_t \in [0, 1]$, implies that

$$\lim_{t\to\infty}\mathbb{E}\left(f_t\right) = 1.$$

Finally, this together with the dominated convergence theorem implies the almost sure convergence.