Modeling and Control of Quantum Systems

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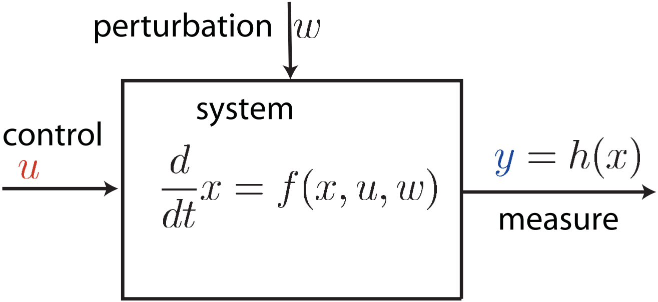
http://cas.ensmp.fr/~rouchon/QuantumSyst/index.html

Lecture 1: October 4, 2010

Outline

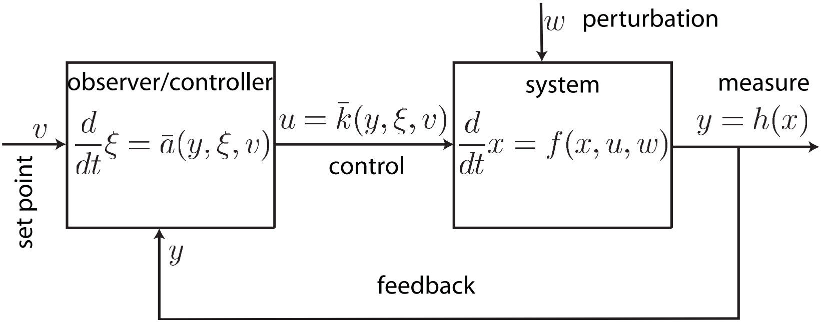


For the harmonic oscillator of pulsation ω with measured position y, controlled by the force u and subject to an additional unknown force w.

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad y = x_1$$

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$

Feedback for classical systems



Proportional Integral Derivative (PID) for $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$ with the set point $v = y^{\text{set point}}$

$$u = -K_p\left(y - y^{\text{set point}}\right) - K_d \frac{d}{dt}\left(y - y^{\text{set point}}\right) - K_{\text{int}} \int \left(y - y^{\text{set point}}\right)$$

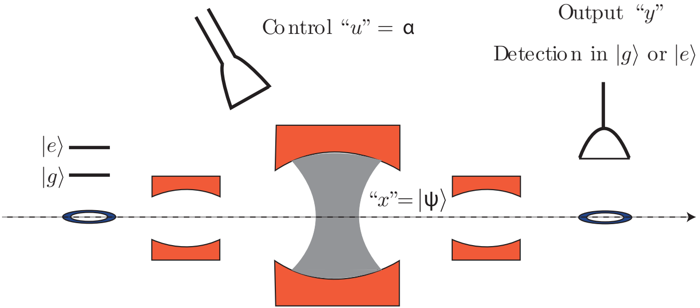
with the positive gains $(K_p, K_d, K_{\text{int}})$ tuned as follows $(0 < \Omega_0 \sim \omega$, $0 < \xi \sim 1$, $0 < \epsilon \ll 1$:

$$K_p = \Omega_0^2, \quad K_d = 2\xi\Omega_0, \quad , K_{\text{int}} = \epsilon\Omega_0^3.$$

Control theory for classical systems

- Controllability: the control u can steer the state x to any location (example: $\frac{d}{dt}x_1 = x_2$, $\frac{d}{dt}x_2 = -\omega^2 x_1 + u$).
- Observability: from the knowledge of u and y one can recover without ambiguity the state x.
- Feed-forward $u = u^r(t)$ associated to reference trajectory $t \mapsto (x^r(t), u^r(t), y^r(t))$ (performance).
- Feed-back $u = u^r(t) + \Delta u$ where Δu depends on the measured output error $\Delta y = y - y^r(t)$ (stability).
- Stability and robustness : asymptotic regime for t large of Δx and Δy, sensitivity to perturbations and errors.

Control of quantum harmonic oscillator: LKB photon-box



Simple schematic of LKB experiment for control of cavity field

The model

$$
|\psi\rangle_{k+1} = \begin{cases} \dfrac{D_\alpha \, \mathcal{M}_g \, |\psi\rangle_k}{\left\| \mathcal{M}_g \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |g\rangle \left(\text{proba. } \left\| \mathcal{M}_g \, |\psi\rangle_k \right\|_{\mathcal{H}}^2 \right) \\ \dfrac{D_\alpha \, \mathcal{M}_e \, |\psi\rangle_k}{\left\| \mathcal{M}_e \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |e\rangle \left(\text{proba. } \left\| \mathcal{M}_e \, |\psi\rangle_k \right\|_{\mathcal{H}}^2 \right) \end{cases}
$$

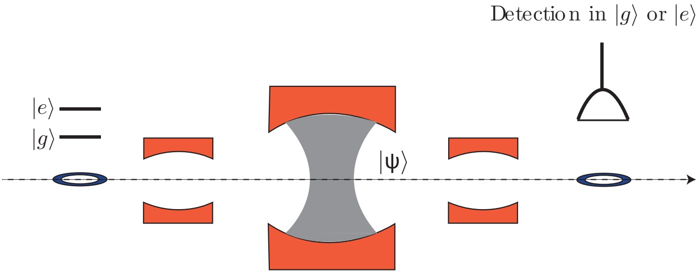
Outline of the 8 lectures

- Lect. 1 (Oct. 4) Introduction on LKB Photon-Box. Quantum harmonic oscillator (creation/annihilation operator, coherent state, non-controllability).
- Lect. 2 (Oct. 11) 2-level system (Pauli matrices, Bloch sphere, RWA, Rabi oscillation, controllability). Jaynes-Cummings model (RWA, resonant and off resonant propagator).
- Lect. 3 (Oct. 25) Controllability and motion planing (RWA, resonant and optimal control)
- Lect. 4 (Nov. 22) Motion planing (adiabatic, Lyapunov, Law-Eberly)
- Lect. 5 (Nov. 29) Quantum trajectories (discrete time, Kraus operators, LKB-photon box)
- Lect. 6 (Dec. 6) Feedback stabilization (Photon-box, quantum filter, Lyapunov, separation principle, delay compensation)
- Lect. 7 (Dec. 13) Quantum trajectories (continuous time with Poisson process, Lindblad operators, time/scale reduction, synchronization loop on a Λ-system)
- Lect. 8 (Dec. 14) Quantum trajectories (continuous time with Wiener process, homodyn detection, Lyapunov feedback stabilization of entangled states).

Main references

- Mathematical system theory and control:
  - H.K. Khalil. *Nonlinear Systems*. MacMillan, 1992.
  - J.M. Coron. *Control and Nonlinearity*. American Mathematical Society, 2007.
  - D. D'Alessandro. *Introduction to Quantum Control and Dynamics*. Chapman & Hall/CRC, 2008.
- Quantum physics and information
  - S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford University Press, 2006.
  - H.M. Wiseman and G.J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2009.
  - M.A. Nielsen and I.L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
  - D. Steck. Quantum and atom optics (notes for a course). http://atomoptics.uoregon.edu/ dsteck/teaching/quantum-optics/,
    2010.

Photon-box (1): measurement process



Simple schematic of LKB experiment for measurement of cavity field

Photon-box (2) : atom-field entanglement

Initial state Atom in $|g\rangle$ and cavity in $|\psi\rangle \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ \sum_{k=n}^{\infty} c_n |n\rangle \;|\; (c_n) \in l^2(\mathbb{C}) \right\}.$$

We can write the initial state as

$$|g\rangle \otimes |\psi\rangle \in \mathbb{C}^2 \otimes \mathcal{H}.$$

State before detection a joint unitary evolution implies an entangled state

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle$$

where $\mathcal{M}_g$ and $\mathcal{M}_e$ are operators acting on $\mathcal{H}$.
The unitarity condition implies:

$$\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$$

Example of non-resonant interaction

$$\mathcal{M}_g = \cos(\vartheta \mathbf{N} + \varphi), \quad \mathcal{M}_e = \sin(\vartheta \mathbf{N} + \varphi), \quad \mathbf{N} = \mathrm{diag}(n)$$

Photon-box (3): entanglement

Final state is inseparable: we can not write

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle = \left(\tilde{\alpha} |g\rangle + \tilde{\beta} |e\rangle\right) \otimes \left(\sum_n \tilde{c}_n |n\rangle\right).$$

We can not associate to the cavity (nor to the atom) a well-defined wavefunction just before the measurement.

However, we can still compute the probability of having the atom in $|g\rangle$ or in $|e\rangle$:

$$P_g = \left\| \mathcal{M}_g |\psi\rangle \right\|^2_{\mathcal{H}}, \qquad P_e = \left\| \mathcal{M}_e |\psi\rangle \right\|^2_{\mathcal{H}}.$$

Photon-box (4): measurement and collapse

Measurement in $|g\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|g\rangle \otimes \mathcal{M}_g |\psi\rangle}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}}},$$

Measurement in $|e\rangle$

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|e\rangle \otimes \mathcal{M}_e |\psi\rangle}{\left\| \mathcal{M}_e |\psi\rangle \right\|_{\mathcal{H}}},$$

Photon-box (5): quantum Monte-Carlo trajectories

Stochastic evolution: ψ_k the wave function after the measurement of atom number $k-1$.

$$|\psi\rangle_{k+1} = \begin{cases} \dfrac{D_\alpha \, \mathcal{M}_g \, |\psi\rangle_k}{\left\| \mathcal{M}_g \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |g\rangle \ \left(\text{proba. } \left\| \mathcal{M}_g \, |\psi\rangle_k \right\|^2_{\mathcal{H}}\right) \\ \dfrac{D_\alpha \, \mathcal{M}_e \, |\psi\rangle_k}{\left\| \mathcal{M}_e \, |\psi\rangle_k \right\|_{\mathcal{H}}} & \text{Detect. in } |e\rangle \ \left(\text{proba. } \left\| \mathcal{M}_e \, |\psi\rangle_k \right\|^2_{\mathcal{H}}\right) \end{cases}$$

We have a Markov chain

Photon-box (6): imperfect measurement

The atom-detector does not always detect the atoms.
Therefore 3 outcomes:
Atom in $|g\rangle$, Atom in $|e\rangle$, No detection

Best estimate for the no-detection case

$$\mathbb{E}\left(|\psi\rangle_{+} \mid |\psi\rangle\right) = \left\|\mathcal{M}_g |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_g |\psi\rangle + \left\|\mathcal{M}_e |\psi\rangle\right\|_{\mathcal{H}} \mathcal{M}_e |\psi\rangle$$

This is not a well-defined wavefunction

Barycenter in the sense of geodesics of $\mathbb{S}(\mathcal{H})$
not invariant with respect to a change of global phase

We need a barycenter in the sense of the projective space
$\mathbb{CP}(\mathcal{H}) \equiv \mathbb{S}(\mathcal{H})/\mathbb{S}^1$

Photon-box (7): density matrix language

Projector over the state $|\psi\rangle$: $P_{|\psi\rangle} = |\psi\rangle\langle\psi|$

Detection in $|g\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger}{\left\|\mathcal{M}_g|\psi\rangle\right\|_{\mathcal{H}}^2} = \frac{\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger}{\left|\left\langle\psi \mid \mathcal{M}_g^\dagger\mathcal{M}_g \mid \psi\right\rangle\right|^2} = \frac{\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger}{\mathrm{Tr}\left(\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger\right)}$$

Detection in $|e\rangle$: the projector is given by

$$P_{|\psi_+\rangle} = \frac{\mathcal{M}_e|\psi\rangle\langle\psi|\mathcal{M}_e^\dagger}{\mathrm{Tr}\left(\mathcal{M}_e|\psi\rangle\langle\psi|\mathcal{M}_e^\dagger\right)}$$

Probabilities:

$$p_g = \mathrm{Tr}\left(\mathcal{M}_g|\psi\rangle\langle\psi|\mathcal{M}_g^\dagger\right) \quad \text{and} \quad p_e = \mathrm{Tr}\left(\mathcal{M}_e|\psi\rangle\langle\psi|\mathcal{M}_e^\dagger\right)$$

Photon-box (8): density matrix language

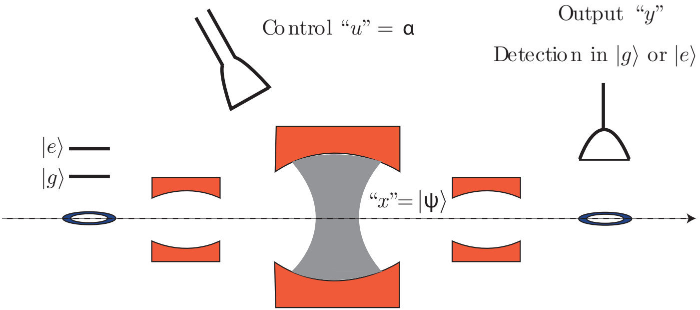
Imperfect detection: barycenter

$$
\begin{aligned}
|\psi\rangle\langle\psi| \longrightarrow\ & p_g \frac{\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger}{\text{Tr}\left(\mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger\right)} + p_e \frac{\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger}{\text{Tr}\left(\mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger\right)} \\
&= \mathcal{M}_g |\psi\rangle\langle\psi| \mathcal{M}_g^\dagger + \mathcal{M}_e |\psi\rangle\langle\psi| \mathcal{M}_e^\dagger.
\end{aligned}
$$

This is not anymore a projector: no well-defined wave function

New state space

$$
\mathcal{X} = \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho^\dagger = \rho, \rho \geq 0, \text{Tr}(\rho) = 1\}
$$



Simple schematic of LKB experiment for control of cavity field

Harmonic oscillator[1] (1): quantization and correspondence principle

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

$$\tfrac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \tfrac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the lectures)

$$i\tfrac{d}{dt}|\psi\rangle = H|\psi\rangle, \quad H = \omega(P^2 + X^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where H results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}X$ and p by impulsion operator $\sqrt{2}P = -i\frac{\partial}{\partial x}$.

PDE model: $i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}$.

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[1] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Harmonic oscillator (2): annihilation and creation operators

Averaged position $\langle X \rangle_t = \langle \psi | X | \psi \rangle$ and impulsion $\langle P \rangle_t = \langle \psi | P | \psi \rangle$ [2]:

$$\langle X \rangle_t = \tfrac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x |\psi|^2 dx, , \quad \langle P \rangle_t = -\tfrac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation a and creation operators $a^\dagger$:

$$a = X + iP = \tfrac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad a^\dagger = X - iP = \tfrac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[X, P] = \tfrac{i}{2}, \quad [a, a^\dagger] = 1, \quad H = \omega(P^2 + X^2) = \omega \left(a^\dagger a + \tfrac{1}{2} \right).$$

Set $X_\lambda = \frac{1}{2} \left(e^{-i\lambda} a + e^{i\lambda} a^\dagger \right)$ for any angle λ:

$$\left[X_\lambda, X_{\lambda + \frac{\pi}{2}} \right] = \tfrac{i}{2}.$$

[2] We assume everywhere that for each t, $x \mapsto \psi(x,t)$ is of the Schwartz class (fast decay at infinity + smooth).

Harmonic oscillator (3): spectral decomposition and Fock states

$[a, a^\dagger] = 1$ implies that the spectrum of $a^\dagger a$ is non-degenerate and is $\mathbb{N}$.

Fock state with n photon(s): the eigen-state of $a^\dagger a$ associated to the eigen-value n:

$$a^\dagger a \, |n\rangle = n \, |n\rangle \, , \quad a|n\rangle = \sqrt{n} \, |n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1} \, |n+1\rangle.$$

The ground state $|0\rangle$ (0 photon state or vacuum state) satisfies $a|0\rangle = 0$ and corresponds to the Gaussian function:

$$|0\rangle \sim \psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2).$$

The operator a (resp. $a^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger).$$

The associated controlled PDE

$$i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \left(\frac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator D_α (unitary) with $\alpha \in \mathbb{C}$:

$$D_\alpha = e^{\alpha a^\dagger - \alpha^* a} = e^{2i\Im\alpha X - 2\imath\Re\alpha P}$$

From Baker-Campbell Hausdorf formula valid for any operators A and B,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

we get the Glauber formula when $[A,[A,B]] = [B,[A,B]] = 0$:

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]}.$$

With $A = \alpha a^\dagger$ and $B = -\alpha^* a$, Glauber formula gives:

$$D_\alpha = e^{-\frac{|\alpha|^2}{2}} \, e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{+\frac{|\alpha|^2}{2}} \, e^{-\alpha^* a} e^{\alpha a^\dagger}$$

$$D_{-\alpha} a D_\alpha = a + \alpha \quad \text{and} \quad D_{-\alpha} a^\dagger D_\alpha = a^\dagger + \alpha^*.$$

With $A = 2i\Im\alpha X \sim i\sqrt{2}\Im\alpha x$ and $B = -2\imath\Re\alpha P \sim -\sqrt{2}\Re\alpha \frac{\partial}{\partial x}$, Glauber formula gives[3]:

$$D_\alpha = e^{-i\Re\alpha\Im\alpha} \, e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha\frac{\partial}{\partial x}}$$

$$(D_\alpha \left|\psi\right\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha} \, e^{i\sqrt{2}\Im\alpha x} \psi(x - \sqrt{2}\Re\alpha, t)$$

For any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have

$$D_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} D_\alpha D_\beta$$

$$D_{\alpha+\epsilon} D_{-\alpha} = \left(1 + \frac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right)\mathbf{1} + \epsilon a^\dagger - \epsilon^* a + O(|\epsilon|^2)$$

$$\left(\frac{d}{dt} D_\alpha\right) D_{-\alpha} = \left(\frac{\alpha\frac{d}{dt}\alpha^* - \alpha^*\frac{d}{dt}\alpha}{2}\right)\mathbf{1} + \left(\frac{d}{dt}\alpha\right) a^\dagger - \left(\frac{d}{dt}\alpha^*\right) a.$$

[3] Remember that a time-delay of r corresponds to the operator $e^{-r\frac{d}{dt}}$.

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(a^\dagger a + \frac{1}{2}\right) + u(a + a^\dagger)\right)|\psi\rangle$. Set $\langle a\rangle = \langle\psi|a\psi\rangle$. Then

$$\frac{d}{dt}\langle a\rangle = -i\omega\langle a\rangle - iu.$$

From $a = X + iP$, we have $\langle a\rangle = \langle X\rangle + i\langle P\rangle$ where $\langle X\rangle = \langle\psi|X|\psi\rangle \in \mathbb{R}$ and $\langle P\rangle = \langle\psi|P|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle X\rangle = \omega\langle P\rangle, \quad \frac{d}{dt}\langle P\rangle = -\omega\langle X\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t} D_{\langle a\rangle_t} |\chi\rangle$ with

$$\theta_t = \int_0^t \left(|\langle a\rangle|^2 + u\Re(\langle a\rangle)\right), \quad D_{\langle a\rangle_t} = e^{\langle a\rangle_t a^\dagger - \langle a\rangle_t^* a},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega a^\dagger a|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle a\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

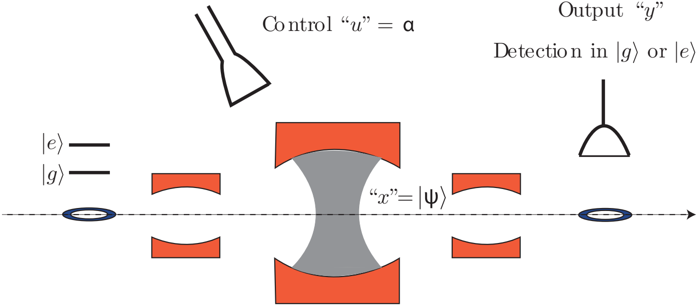
Coherent states

$$|\alpha\rangle = D_\alpha |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigen-state of a: $a|\alpha\rangle = \alpha|\alpha\rangle$.
A widely known result in quantum optics[4]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum[5]

[4] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*.Wiley, 1989.

[5] see also: MM-PR, IEEE Trans. Automatic Control, 2004 and MM-PR, CDC-ECC, 2005.



Simple schematic of LKB experiment for control of cavity field

The LKB photon-box