Quantum Systems: Dynamics and Control[1]

Mazyar Mirrahimi[2], Pierre Rouchon[3], Alain Sarlette[4]

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[1] See the web page: http://cas.ensmp.fr/~rouchon/MasterUPMC/index.html

[2] INRIA Paris, QUANTIC research team
[3] Mines ParisTech, QUANTIC research team
[4] INRIA Paris, QUANTIC research team

Outline

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Controlled Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle,$$

- $|\psi\rangle \in \mathcal{H}$ the system's wavefunction with $\left\| |\psi\rangle \right\|_{\mathcal{H}} = 1$;
- the free Hamiltonian, $\boldsymbol{H}_0$, and the control Hamiltonian, $\boldsymbol{H}_1$, are Hermitian operators on $\mathcal{H}$;
- the control $u(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ is a scalar control.

Two key examples:

- Qubit: $\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1 = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x}$.
- Quantum harmonic oscillator:
  $\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1 = \omega_c(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}) + u(t)(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.

Almost periodic control of small amplitudes

We consider the controls of the form

$$u(t) = \epsilon \left(\sum_{j=1}^{r} u_j e^{i\omega_j t} + u_j^* e^{-i\omega_j t} \right)$$

- $\epsilon > 0$ is a small parameter;
- ϵu_j is the constant complex amplitude associated to the frequency $\omega_j \geq 0$;
- r stands for the number of independent frequencies ($\omega_j \neq \omega_k$ for $j \neq k$).

We are interested in approximations, for ϵ tending to 0^+, of trajectories $t \mapsto |\psi_\epsilon\rangle_t$ of

$$\frac{d}{dt}|\psi_\epsilon\rangle = \left(\boldsymbol{A}_0 + \epsilon \left(\sum_{j=1}^{r} u_j e^{i\omega_j t} + u_j^* e^{-i\omega_j t} \right) \boldsymbol{A}_1 \right) |\psi_\epsilon\rangle$$

where $\boldsymbol{A}_0 = -i\boldsymbol{H}_0$ and $\boldsymbol{A}_1 = -i\boldsymbol{H}_1$ are skew-Hermitian.

Outline

Time-periodic non-linear systems

We consider a non-linear ODE of the form:

$$\frac{d}{dt}x = \epsilon f(x, t), \qquad x \in \mathbb{R}^n, \qquad \epsilon \ll 1,$$

where f is T-periodic in t and depends smoothly on x.

We will see how its solution is well-approximated by the solution of the time-independent system, the averaged system:

$$\frac{d}{dt}z = \epsilon \bar{f}(z)$$

where $\bar{f}(z) = \frac{1}{T}\int_0^T f(z, t)dt$.

Consider $\frac{d}{dt}x = \epsilon f(x, t)$ with $x \in U \subset \mathbb{R}^n$, $0 \leq \epsilon \ll 1$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ smooth and period $T > 0$ in t. Also assume U to be bounded.

- If z is the solution of $\frac{d}{dt}z = \epsilon \bar{f}(z)$ with the initial condition z_0, and assuming $|x_0 - z_0| = \mathcal{O}(\epsilon)$, we have $|x(t) - z(t)| = \mathcal{O}(\epsilon)$ on a time-scale $t \sim 1/\epsilon$.
- If $\bar{z}$ is a hyperbolic fixed point of the averaged system then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$, the main system possesses a unique hyperbolic periodic orbit $\gamma_\epsilon(t) = \bar{z} + \mathcal{O}(\epsilon)$ of the same stability type as $\bar{z}$.

J. Guckenheimer and P. Holmes, Nonlinear oscillations, Dynamical systems and Bifurcation of Vector Fields, Springer, 1983.

Fixed suspension point:

$$\frac{d^2}{dt^2}\theta = \frac{g}{l}\sin\theta$$

g: free fall acceleration, l: pendulum's length, θ: angle to the vertical; $\theta = \pi$ stable and $\theta = 0$ unstable equilibrium.

Suspension point in vertical oscillation:

Dynamics of the suspension point: $z = \frac{v}{\Omega}\cos(\Omega t)$ ($a = v/\Omega > 0$ amplitude and Ω frequency).

Pendulum's dynamics: replace acceleration g by $g + \ddot{z} = g - v\Omega\cos(\Omega t)$,

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = \frac{g - v\Omega\cos(\Omega t)}{l}\sin\theta.$$

Replacing the velocity ω by the momentum $p_\theta = \omega + \frac{v\sin(\Omega t)}{l}\sin\theta$:

$$\frac{d}{dt}\theta = p_\theta - \frac{v\sin(\Omega t)}{l}\sin\theta,$$
$$\frac{d}{dt}p_\theta = \left(\frac{g}{l} - \frac{v^2\sin^2(\Omega t)}{l^2}\cos\theta\right)\sin\theta + \frac{v\sin(\Omega t)}{l}p_\theta\cos\theta.$$

For large enough Ω, we can average these time-periodic dynamics over $[t - \pi/\Omega, t + \pi/\Omega]$:

$$\frac{d}{dt}\theta = p_\theta, \quad \frac{d}{dt}p_\theta = \left(\frac{g}{l} - \frac{v^2}{2l^2}\cos\theta\right)\sin\theta.$$

Around $\theta = 0$ the approximation of small angles gives $\frac{d^2}{dt^2}\theta = \frac{g - v^2/2l}{l}\theta$.
If $v^2/2l > g$ then the system becomes stable around $\theta = 0$.

Outline

Bilinear Schrödinger equation

Un-measured quantum system $\rightarrow$ **Bilinear Schrödinger equation**

$$i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle,$$

- $|\psi\rangle \in \mathcal{H}$ the system's wavefunction with $\left\| |\psi\rangle \right\|_{\mathcal{H}} = 1$;
- the free Hamiltonian, $\boldsymbol{H}_0$, is a Hermitian operator defined on $\mathcal{H}$;
- the control Hamiltonian, $\boldsymbol{H}_1$, is a Hermitian operator defined on $\mathcal{H}$;
- the control $u(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ is a scalar control.

Here we consider the case of finite dimensional $\mathcal{H}$

Almost periodic control

We consider the controls of the form

$$u(t) = \epsilon \left(\sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} + \boldsymbol{u}_j^* e^{-i\omega_j t} \right)$$

- $\epsilon > 0$ is a small parameter;
- $\epsilon \boldsymbol{u}_j$ is the constant complex amplitude associated to the pulsation $\omega_j \geq 0$;
- r stands for the number of independent frequencies ($\omega_j \neq \omega_k$ for $j \neq k$).

We are interested in approximations, for ϵ tending to 0^+, of trajectories $t \mapsto |\psi_\epsilon\rangle_t$ of

$$\frac{d}{dt}|\psi_\epsilon\rangle = \left(\boldsymbol{A}_0 + \epsilon \left(\sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} + \boldsymbol{u}_j^* e^{-i\omega_j t} \right) \boldsymbol{A}_1 \right) |\psi_\epsilon\rangle$$

where $\boldsymbol{A}_0 = -i\boldsymbol{H}_0$ and $\boldsymbol{A}_1 = -i\boldsymbol{H}_1$ are skew-Hermitian.

Rotating frame

Consider the following change of variables

$$|\psi_\epsilon\rangle_t = e^{\boldsymbol{A}_0 t}|\phi_\epsilon\rangle_t.$$

The resulting system is said to be in the "interaction frame"

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon \boldsymbol{B}(t)|\phi_\epsilon\rangle$$

where $\boldsymbol{B}(t)$ is a skew-Hermitian operator whose time-dependence is almost periodic:

$$B(t) = \sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} e^{-\boldsymbol{A}_0 t} \boldsymbol{A}_1 e^{\boldsymbol{A}_0 t} + \boldsymbol{u}_j^* e^{-i\omega_j t} e^{-\boldsymbol{A}_0 t} \boldsymbol{A}_1 e^{\boldsymbol{A}_0 t}.$$

Main idea

We can write

$$\boldsymbol{B}(t) = \bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t),$$

where $\bar{\boldsymbol{B}}$ is a constant skew-Hermitian matrix and $\widetilde{\boldsymbol{B}}(t)$ is a bounded almost periodic skew-Hermitian matrix.

Multi-frequency averaging: first order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t)\right)|\phi_\epsilon\rangle,$$

and

$$\frac{d}{dt}|\phi_\epsilon^{1^{\text{st}}}\rangle = \epsilon\bar{\boldsymbol{B}}|\phi_\epsilon^{1^{\text{st}}}\rangle,$$

initialized at the same state $|\phi_\epsilon^{1^{\text{st}}}\rangle_0 = |\phi_\epsilon\rangle_0$.

Theorem: first order approximation (Rotating Wave Approximation)

Consider the functions $|\phi_\epsilon\rangle$ and $|\phi_\epsilon^{1^{\text{st}}}\rangle$ initialized at the same state and following the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi_\epsilon^{1^{\text{st}}}\rangle_t \right\| \leq M\epsilon$$

Multi-frequency averaging: first order

Proof's idea

Almost periodic change of variables:

$$|\chi_\epsilon\rangle = (1 - \epsilon \widetilde{\boldsymbol{B}}(t))|\phi_\epsilon\rangle$$

well-defined for $\epsilon > 0$ sufficiently small.
The dynamics can be written as

$$\frac{d}{dt}|\chi_\epsilon\rangle = (\epsilon \bar{\boldsymbol{B}} + \epsilon^2 \boldsymbol{F}(\epsilon, t))|\chi_\epsilon\rangle$$

where $\boldsymbol{F}(\epsilon, t)$ is uniformly bounded in time.

Multi-frequency averaging: second order

More precisely, the dynamics of $|\chi_\epsilon\rangle$ is given by

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} + \epsilon^2[\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{B}}(t)] - \epsilon^2\widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t) + \epsilon^3\boldsymbol{E}(\epsilon, t)\right)|\chi_\epsilon\rangle$$

- $\boldsymbol{E}(\epsilon, t)$ is still almost periodic but its entries are no more linear combinations of time-exponentials;
- $\widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t)$ is an almost periodic operator whose entries are linear combinations of oscillating time-exponentials.

We can write

$$\widetilde{\boldsymbol{B}}(t) = \frac{d}{dt}\widetilde{\boldsymbol{C}}(t) \qquad \text{and} \qquad \widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t) = \bar{\boldsymbol{D}} + \frac{d}{dt}\widetilde{\boldsymbol{D}}(t)$$

where $\widetilde{\boldsymbol{C}}(t)$ and $\widetilde{\boldsymbol{D}}(t)$ are almost periodic. We have

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}} + \epsilon^2\frac{d}{dt}\left([\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}(t)] - \widetilde{\boldsymbol{D}}(t)\right) + \epsilon^3\boldsymbol{E}(\epsilon, t)\right)|\chi_\epsilon\rangle$$

where the skew-Hermitian operators $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{D}}$ are constants and the other ones $\widetilde{\boldsymbol{C}}$, $\widetilde{\boldsymbol{D}}$, and $\boldsymbol{E}$ are almost periodic.

Multi-frequency averaging: second order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t)\right)|\phi_\epsilon\rangle,$$

and

$$\frac{d}{dt}|\phi_\epsilon^{2^{\text{nd}}}\rangle = (\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}})|\phi_\epsilon^{2^{\text{nd}}}\rangle,$$

initialized at $|\phi_\epsilon\rangle_0$ and $|\phi_\epsilon^{2^{\text{nd}}}\rangle_0 = (I - \epsilon\widetilde{\boldsymbol{B}}(0))|\phi_\epsilon\rangle_0$.

Theorem: second order approximation

Consider $|\phi_\epsilon\rangle_t$ and $|\phi_\epsilon^{2^{\text{nd}}}\rangle_t$ solutions of the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta]$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - (I + \epsilon\widetilde{\boldsymbol{B}}(t))|\phi_\epsilon^{2^{\text{nd}}}\rangle_t \right\| \leq M\boldsymbol{\epsilon}^{\boldsymbol{2}}$$

$$\max_{t\in\left[0,\frac{1}{\epsilon^2}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi_\epsilon^{2^{\text{nd}}}\rangle_t \right\| \leq M\boldsymbol{\epsilon}$$

Multi-frequency averaging: second order

Proof's idea

Another almost periodic change of variables

$$|\xi_\epsilon\rangle = \left(\boldsymbol{I} - \epsilon^2\left([\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}(t)] - \widetilde{\boldsymbol{D}}(t)\right)\right)|\chi_\epsilon\rangle.$$

The dynamics can be written as

$$\frac{d}{dt}|\xi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}} + \epsilon^3\boldsymbol{F}(\epsilon, t)\right)|\xi_\epsilon\rangle$$

where $\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}}$ is skew Hermitian and $\boldsymbol{F}$ is almost periodic and therefore uniformly bounded in time.

The Rotating Wave Approximation (RWA) recipes

Schrödinger dynamics $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}(t)|\psi\rangle$, with

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 + \sum_{k=1}^{m} u_k(t)\boldsymbol{H}_k, \qquad u_k(t) = \sum_{j=1}^{r} \boldsymbol{u}_{k,j} e^{i\omega_j t} + \boldsymbol{u}_{k,j}^* e^{-i\omega_j t}.$$

The Hamiltonian in interaction frame

$$\boldsymbol{H}_{\text{int}}(t) = \sum_{k,j} \left(\boldsymbol{u}_{k,j} e^{i\omega_j t} + \boldsymbol{u}_{k,j}^* e^{-i\omega_j t}\right) e^{i\boldsymbol{H}_0 t} \boldsymbol{H}_k e^{-i\boldsymbol{H}_0 t}$$

We define the first order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \boldsymbol{H}_{\text{int}}(t)dt,$$

and the second order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t \left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\right)}$$

Choose the amplitudes $\boldsymbol{u}_{k,j}$ and the frequencies ω_j such that the propagators of $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}}$ or $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ admit simple explicit forms that are used to find $t \mapsto u(t)$ steering $|\psi\rangle$ from one location to another one.

Outline

In $i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x}\right)|\psi\rangle$, take a resonant control $u(t) = \boldsymbol{u}e^{i\omega_{eg}t} + \boldsymbol{u}^* e^{-i\omega_{eg}t}$ with $\boldsymbol{u}$ slowly varying complex amplitude $\left|\frac{d}{dt}\boldsymbol{u}\right| \ll \omega_{eg}|\boldsymbol{u}|$. Set $H_0 = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\epsilon H_1 = \frac{u}{2}\boldsymbol{\sigma_x}$ and consider $|\psi\rangle = e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ to eliminate the drift H_0 and to get the Hamiltonian in the interaction frame:

$$i\frac{d}{dt}|\phi\rangle = \frac{u(t)}{2}e^{\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}\boldsymbol{\sigma_x}e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$$

with $\boldsymbol{H}_{\text{int}} = \frac{u}{2}e^{i\omega_{eg}t}\ \overbrace{\frac{\boldsymbol{\sigma_x} + i\boldsymbol{\sigma_y}}{2}}^{\boldsymbol{\sigma_+} = |e\rangle\langle g|} + \frac{u}{2}e^{-i\omega_{eg}t}\ \overbrace{\frac{\boldsymbol{\sigma_x} - i\boldsymbol{\sigma_y}}{2}}^{\boldsymbol{\sigma_-} = |g\rangle\langle e|}$

The RWA consists in neglecting the oscillating terms at frequency $2\omega_{eg}$ when $|\boldsymbol{u}| \ll \omega_{eg}$:

$$H_{int} = \left(\frac{\boldsymbol{u}e^{2i\omega_{eg}t} + \boldsymbol{u}^*}{2}\right)\boldsymbol{\sigma_+} + \left(\frac{\boldsymbol{u} + \boldsymbol{u}^* e^{-2i\omega_{eg}t}}{2}\right)\boldsymbol{\sigma_-}.$$

Thus

$$\overline{H_{int}} = \frac{\boldsymbol{u}^*\boldsymbol{\sigma_+} + \boldsymbol{u}\boldsymbol{\sigma_-}}{2}.$$

The decomposition of $\boldsymbol{H}_{\text{int}}$,

$$\boldsymbol{H}_{\text{int}} = \underbrace{\frac{\boldsymbol{u}^*}{2}\boldsymbol{\sigma_+} + \frac{\boldsymbol{u}}{2}\boldsymbol{\sigma_-}}_{\overline{\boldsymbol{H}_{\text{int}}}} + \underbrace{\frac{\boldsymbol{u}e^{2i\omega_{\text{eg}}t}}{2}\boldsymbol{\sigma_+} + \frac{\boldsymbol{u}^*e^{-2i\omega_{\text{eg}}t}}{2}\boldsymbol{\sigma_-}}_{\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}},$$

provides the first order approximation (RWA) $\boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt$, and also the second order approximation $\boldsymbol{H}^{2^{\text{nd}}}_{\text{rwa}} = \boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)}$. Since $\int_t \boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}} = \frac{\boldsymbol{u}e^{2i\omega_{\text{eg}}t}}{4i\omega_{\text{eg}}}\boldsymbol{\sigma_+} - \frac{\boldsymbol{u}^*e^{-2i\omega_{\text{eg}}t}}{4i\omega_{\text{eg}}}\boldsymbol{\sigma_-}$, we have

$$\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)} = -\frac{|\boldsymbol{u}|^2}{8i\omega_{\text{eg}}}\boldsymbol{\sigma_z}$$

(use $\boldsymbol{\sigma_+}^2 = \boldsymbol{\sigma_-}^2 = 0$ and $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+}$).
The second order approximation reads:

$$\boldsymbol{H}^{2^{\text{nd}}}_{\text{rwa}} = \boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}} + \left(\frac{|\boldsymbol{u}|^2}{8\omega_{\text{eg}}}\right)\boldsymbol{\sigma_z} = \frac{\boldsymbol{u}^*}{2}\boldsymbol{\sigma_+} + \frac{\boldsymbol{u}}{2}\boldsymbol{\sigma_-} + \left(\frac{|\boldsymbol{u}|^2}{8\omega_{\text{eg}}}\right)\boldsymbol{\sigma_z}.$$

The 2nd order correction $\frac{|\boldsymbol{u}|^2}{4\omega_{\text{eg}}}(\boldsymbol{\sigma_z}/2)$ is called the Bloch-Siegert shift.

Exercise: controllability of the 2-level systems and Rabi oscillation

Take the first order approximation

$$(\Sigma) \quad i\frac{d}{dt}|\phi\rangle = \frac{(\boldsymbol{u}^*\boldsymbol{\sigma_+} + \boldsymbol{u}\boldsymbol{\sigma_-})}{2}|\phi\rangle = \frac{(\boldsymbol{u}^*|e\rangle\langle g| + \boldsymbol{u}|g\rangle\langle e|)}{2}|\phi\rangle$$

with control $\boldsymbol{u} \in \mathbb{C}$.

1. Take constant control $\boldsymbol{u}(t) = \Omega_r e^{i\theta}$ for $t \in [0, T]$, $T > 0$. Show that $i\frac{d}{dt}|\phi\rangle = \frac{\Omega_r(\cos\theta\boldsymbol{\sigma_x}+\sin\theta\boldsymbol{\sigma_y})}{2}|\phi\rangle$.
2. Set $\Theta_r = \frac{\Omega_r}{2}T$. Show that the solution at T of the propagator $\boldsymbol{U}_t \in SU(2)$, $i\frac{d}{dt}\boldsymbol{U} = \frac{\Omega_r(\cos\theta\boldsymbol{\sigma_x}+\sin\theta\boldsymbol{\sigma_y})}{2}\boldsymbol{U}$, $\boldsymbol{U}_0 = \boldsymbol{I}$ is given by

$$\boldsymbol{U}_T = \cos\Theta_r\boldsymbol{I} - i\sin\Theta_r\left(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y}\right),$$

3. Take a wave function $|\bar{\phi}\rangle$. Show that exist Ω_r and θ such that $U_T|g\rangle = e^{i\alpha}|\bar{\phi}\rangle$, where α is some global phase.
4. Prove that for any given two wave functions $|\phi_a\rangle$ and $|\phi_b\rangle$ exists a piece-wise constant control $[0, 2T] \ni t \mapsto \boldsymbol{u}(t) \in \mathbb{C}$ such that the solution of (Σ) with $|\phi\rangle_0 = |\phi_a\rangle$ satisfies $|\phi\rangle_T = e^{i\beta}|\phi_b\rangle$ for some global phase β.