Mathematical methods for modeling and control of open quantum systems[1]

Mazyar Mirrahimi[2], Pierre Rouchon[3], Alain Sarlette [4]

December 2, 2021

[1] Lecture-notes, slides and Matlab simulation scripts available at:
http://cas.ensmp.fr/~rouchon/LIASFMA/index.html

[2] Inria-Paris
[3] Mines ParisTech
[4] Inria-Paris

Outline

Outline

General structure of Markov model in discrete time

- Any open model of quantum system in discrete time is governed by a Markov chain of the form

$$\boldsymbol{\rho}_{k+1} = \frac{\mathbb{K}_{y_k}(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\mathbb{K}_{y_k}(\boldsymbol{\rho}_k)\right)},$$

  with the probability $\mathrm{Tr}\left(\mathbb{K}_{y_k}(\boldsymbol{\rho}_k)\right)$ to have the measurement outcome y_k knowing ρ_k.

- The structure of the super-operators $\mathbb{K}_y$ is as follows. Each $\mathbb{K}_y$ is a linear completely positive map (a quantum operation, a partial Kraus map[5]) and $\sum_y \mathbb{K}_y(\boldsymbol{\rho}) = \mathbb{K}(\boldsymbol{\rho})$ is a Kraus map, i.e. $\mathbb{K}(\boldsymbol{\rho}) = \sum_\mu \boldsymbol{K}_\mu \boldsymbol{\rho} \boldsymbol{K}_\mu^\dagger$ with $\sum_\mu \boldsymbol{K}_\mu^\dagger \boldsymbol{K}_\mu = \boldsymbol{I}$.

[5] Each $\mathbb{K}_y$ admits the expression

$$\mathbb{K}_y(\boldsymbol{\rho}) = \sum_\nu \boldsymbol{M}_{y,\nu} \boldsymbol{\rho} \boldsymbol{M}_{y,\nu}^\dagger$$

where $(\boldsymbol{M}_{y,\nu})$ are bounded operators on $\mathcal{H}$.

Schrödinger view point of ensemble average dynamics

- Without measurement record, the quantum state ρ_k obeys to the master equation
$$\boldsymbol{\rho}_{k+1} = \mathbb{K}(\boldsymbol{\rho}_k).$$
since $\mathbb{E}\left(\rho_{k+1} \mid \rho_k\right) = \mathbb{K}(\boldsymbol{\rho}_k)$ (ensemble average).
- In finite dimension, $\mathbb{K}$ is always a contraction (not strict in general) for many metrics such as the following ones: for any density operators $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ we have
$$\|\mathbb{K}(\boldsymbol{\rho}) - \mathbb{K}(\boldsymbol{\rho}')\|_1 \leq \|\boldsymbol{\rho} - \boldsymbol{\rho}'\|_1 \text{ and } F(\mathbb{K}(\boldsymbol{\rho}), \mathbb{K}(\boldsymbol{\rho}')) \geq F(\boldsymbol{\rho}, \boldsymbol{\rho}')$$
where the trace norm $\| \bullet \|_1$ and fidelity F are given by
$$\|\boldsymbol{\rho} - \boldsymbol{\rho}'\|_1 \triangleq \text{Tr}\left(|\boldsymbol{\rho} - \boldsymbol{\rho}'|\right) \text{ and } F(\boldsymbol{\rho}, \boldsymbol{\rho}') \triangleq \text{Tr}\left(\sqrt{\sqrt{\boldsymbol{\rho}}\boldsymbol{\rho}'\sqrt{\boldsymbol{\rho}}}\right).$$

Heisenberg view point of ensemble average dynamics

- The "Heisenberg description" is given by iterates $\boldsymbol{A}_{k+1} = \mathbb{K}^*(\boldsymbol{A}_k)$ from an initial bounded Hermitian operator $\boldsymbol{A}_0$ of the dual map $\mathbb{K}^*$ characterized as follows: $\text{Tr}\,(\boldsymbol{A}\mathbb{K}(\boldsymbol{\rho})) = \text{Tr}\,(\mathbb{K}^*(\boldsymbol{A})\boldsymbol{\rho})$ for any bounded operator $\boldsymbol{A}$ on $\mathcal{H}$. Thus

  $$\mathbb{K}^*(\boldsymbol{A}) = \sum_\mu \boldsymbol{K}_\mu^\dagger \boldsymbol{A} \boldsymbol{K}_\mu \;\; \text{when } \mathbb{K}(\boldsymbol{\rho}) = \sum_\mu \boldsymbol{K}_\mu \boldsymbol{\rho} \boldsymbol{K}_\mu^\dagger .$$

  $\mathbb{K}^*$ is an unital map, i.e., $\mathbb{K}^*(\boldsymbol{I}) = \boldsymbol{I}$, and the image via $\mathbb{K}^*$ of any bounded operator is a bounded operator.

- When $\mathcal{H}$ is of finite dimension, we have, for any Hermitian operator $\boldsymbol{A}$:

  $$\lambda_{min}(\boldsymbol{A}) \leq \lambda_{min}(\mathbb{K}^*(\boldsymbol{A})) \leq \lambda_{max}(\mathbb{K}^*(\boldsymbol{A})) \leq \lambda_{max}(\boldsymbol{A})$$

  where λ_{min} and λ_{max} correspond to the smallest and largest eigenvalues.

- If $\overline{\boldsymbol{A}} = \mathbb{K}^*(\overline{\boldsymbol{A}})$, then $\text{Tr}\left(\boldsymbol{\rho}_k \overline{\boldsymbol{A}}\right) = \text{Tr}\left(\boldsymbol{\rho}_0 \overline{\boldsymbol{A}}\right)$ is a constant of motion of $\boldsymbol{\rho}$.

Convergence in Schrödinger and Heisenberg pictures

Take a Kraus map $\mathbb{K}$ and its adjoint unital map $\mathbb{K}^*$. When $\mathcal{H}$ is of finite dimension, the following two statements are equivalent :

- Global convergence towards the fixed point $\overline{\boldsymbol{\rho}} = \mathbb{K}(\overline{\boldsymbol{\rho}})$ of $\boldsymbol{\rho}_{k+1} = \mathbb{K}(\boldsymbol{\rho}_k)$: for any initial density operator $\boldsymbol{\rho}_0$, $\lim_{k \mapsto +\infty} \boldsymbol{\rho}_k = \overline{\boldsymbol{\rho}}$.
- Global convergence of $\boldsymbol{A}_{k+1} = \mathbb{K}^*(\boldsymbol{A}_k)$: there exists a unique density operator $\overline{\boldsymbol{\rho}}$ such that, for any initial bounded operator $\boldsymbol{A}_0$, $\lim_{k \mapsto +\infty} A_k = \mathrm{Tr}\left(A_0 \overline{\boldsymbol{\rho}}\right) \boldsymbol{I}$.

Discrete-time Stochastic Master Equations (SME)

Trace preserving Kraus map $\boldsymbol{K}_u$ depending on the classical control input u:

$$\boldsymbol{K}_u(\boldsymbol{\rho}) = \sum_{\xi} \boldsymbol{M}_{u,\xi} \boldsymbol{\rho} \boldsymbol{M}_{u,\xi}^\dagger \quad \text{with} \quad \sum_{\xi} \boldsymbol{M}_{u,\xi}^\dagger \boldsymbol{M}_{u,\xi} = \boldsymbol{I}.$$

Take a left stochastic matrix $\left[\eta_{y,\xi}\right]$ ($\eta_{y,\xi} \geq 0$ and $\sum_y \eta_{y,\xi} \equiv 1, \ \forall \xi$) and set $\boldsymbol{K}_{u,y}(\boldsymbol{\rho}) = \sum_\xi \eta_{y,\xi} \boldsymbol{M}_{u,\xi} \boldsymbol{\rho} \boldsymbol{M}_{u,\xi}^\dagger$. The associated Markov chain reads:

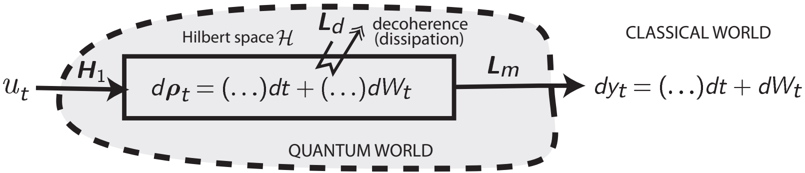
$$\boldsymbol{\rho}_{k+1} = \frac{\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)}{\text{Tr}\left(\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)\right)} \quad \text{measurement } y_k \text{ with probability } \text{Tr}\left(\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)\right).$$

Classical input u, hidden state $\boldsymbol{\rho}$, measured output y.
Ensemble average given by $\boldsymbol{K}_u$ since $\mathbb{E}\left(\boldsymbol{\rho}_{k+1} \mid \boldsymbol{\rho}_k, u_k\right) = \boldsymbol{K}_{u_k}(\boldsymbol{\rho}_k)$.
Markov model useful for:

1. Monte-Carlo simulations of quantum trajectories (decoherence, measurement back-action).
2. quantum filtering and parameter estimation: e.g. to get the quantum state $\boldsymbol{\rho}_k$ from $\boldsymbol{\rho}_0$ and $(y_0, \ldots, y_{k-1})$ (Belavkin quantum filter developed for diffusive models).
3. feedback design and Monte-Carlo closed-loop simulations

Outline

Classical I/O dynamics for diffusive Stochastic Master Equation [6]



Continuous-time models: stochastic differential systems (Itō formulation)
density operator ρ ($\rho^\dagger = \rho$, $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$) as state ($\hbar \equiv 1$ here):

$$d\rho_t = \Big(-i[\boldsymbol{H}_0 + u_t \boldsymbol{H}_1, \rho_t] + \sum_{\nu=d,m} \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{\mathbf{I}}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) \Big) dt$$
$$+ \sqrt{\eta_m} \Big(\boldsymbol{L}_m \rho_t + \rho_t \boldsymbol{L}_m^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_m + \boldsymbol{L}_m^\dagger) \rho_t \right) \rho_t \Big) dW_t$$

driven by the Wiener process W_t, with measurement y_t,

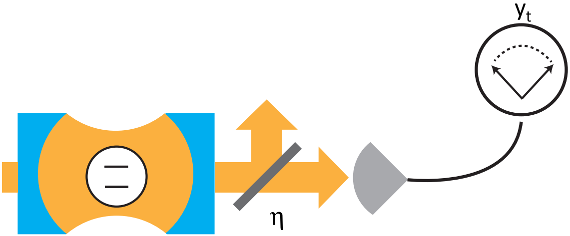
$$dy_t = \sqrt{\eta_m}\, \mathrm{Tr}\left((\boldsymbol{L}_m + \boldsymbol{L}_m^\dagger)\, \rho_t \right) dt + dW_t$$ detection efficiencies $\eta_m \in [0, 1]$.

Measurement backaction: **$d\rho$ and dy share the same noises dW**. Very different from the Kalman I/O state-space description widely used in control engineering.

---

[6] A. Barchielli, M. Gregoratti (2009): Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag.

Markov process under continuous measurement



Inverse setup of photon-box: photons read out a qubit.

Two major differences

- measurement output taking values from a continuum of possible outcomes
$$dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right)dt + dW_t.$$
- Time continuous dynamics.

Stochastic master equation: Markov process under continuous measurement

$$d\boldsymbol{\rho}_t = \left(-i[\boldsymbol{H},\boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right) dW_{\nu,t},$$

where $W_{\nu,t}$ are independent Wiener processes, associated to measured signals with efficiencies $\eta_\nu \in [0,1]$:

$$dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right) dt.$$

Wiener process W_t:

- $W_0 = 0$;
- $t \to W_t$ is almost surely everywhere continuous;
- For $0 \leq s_1 < t_1 \leq s_2 < t_2$, $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables satisfying $W_t - W_s \sim N(0, t-s)$.

Average dynamics: Lindblad master equation

$$d\mathbb{E}(\boldsymbol{\rho}_t) = \left(-i[\boldsymbol{H},\mathbb{E}(\boldsymbol{\rho}_t)] + \textstyle\sum_\nu \boldsymbol{L}_\nu \mathbb{E}(\boldsymbol{\rho}_t) \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \mathbb{E}(\boldsymbol{\rho}_t) + \mathbb{E}(\boldsymbol{\rho}_t) \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt.$$

Ito stochastic calculus

Given a diffusive Stochastic Differential Equation (SDE)

$$dX_t = F(X_t, t)dt + \sum_{\nu} G_\nu(X_t, t)dW_{\nu,t},$$

we have the following chain rule:

Ito's rule

Defining $f_t = f(X_t)$ a C^2 function of X, we have

$$df_t = \left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_{\nu} \frac{\partial^2 f}{\partial X^2}\Big|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right) dt + \sum_{\nu} \frac{\partial f}{\partial X}\Big|_{X_t} G_\nu(X_t, t)dW_{\nu,t}.$$

Furthermore

$$\frac{d}{dt}\mathbb{E}(f_t) = \mathbb{E}\left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_{\nu} \frac{\partial^2 f}{\partial X^2}\Big|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right).$$

Link to partial Kraus maps (1)

$$d\boldsymbol{\rho}_t = \left(-i[\boldsymbol{H}, \boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right) \boldsymbol{\rho}_t\right) dW_{\nu,t},$$

equivalent to

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t} \boldsymbol{\rho}_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t} \boldsymbol{\rho}_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt\right)}$$

with

$$\boldsymbol{M}_{dy_t} = \boldsymbol{I} + \left(-i\boldsymbol{H} - \frac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu\right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \boldsymbol{L}_\nu.$$

Moreover, defining $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$:

$$\mathbb{P}\left(\left(s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu]\right)_\nu \mid \boldsymbol{\rho}_t\right) = \mathrm{Tr}\left(\boldsymbol{M}_{s\sqrt{dt}} \boldsymbol{\rho}_t \boldsymbol{M}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt\right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

Example of Ito calculations

With $dy_t = \mathrm{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right) dt + dW_t$

$$d\boldsymbol{\rho}_t = \left(\boldsymbol{L}\boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger\boldsymbol{L})\right) dt + \left(\boldsymbol{L}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}^\dagger - \mathrm{Tr}\left((\boldsymbol{L} + \boldsymbol{L}^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right) dW_t,$$

reads

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger\right)}$$

where $\boldsymbol{M}_{dy_t} = \boldsymbol{I} - \frac{dt}{2}\boldsymbol{L}^\dagger\boldsymbol{L} + dy_t\boldsymbol{L}$ and where one uses expansion including first order terms in dt and Ito rules

$$d\rho_t = \boldsymbol{\rho}_{t+dt} - \boldsymbol{\rho}_t, \quad dW_t = O(\sqrt{dt}), \quad dW_t^2 = dt, \quad dt\, dW_t = 0, \ldots$$

Link to partial Kraus maps (2)

- $\mathbb{P}$ defines a probability density up to a correction of order dt^2:

$$\int \mathbb{P}(s_t \in [s, s+ds] \mid \boldsymbol{\rho}_t) = 1 + O(dt^2).$$

- Mean value of measured signal

$$\int s_\nu \mathbb{P}(s_t \in [s, s+ds] \mid \boldsymbol{\rho}_t) = \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)\sqrt{dt} + O(dt^{3/2}).$$

- Variance of measured signal

$$\int s_\nu^2\, \mathbb{P}(s_t \in [s, s+ds] \mid \boldsymbol{\rho}_t) = 1 + O(dt).$$

Compatible with $dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right) dt$.

Link to partial Kraus maps (3)

$$d\boldsymbol{\rho}_t = \left(-i[\boldsymbol{H},\boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu)\right)dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)\boldsymbol{\rho}_t\right)dW_{\nu,t},$$

equivalent to

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt\right)}$$

- Indicates that the solution remains in the space of semi-definite positive Hermitian matrices;
- Provides a time-discretized numerical scheme preserving non-negativity of $\boldsymbol{\rho}$.

Theorem

The above master equation admits a unique solution remaining for all $t \geq 0$ in $\{\boldsymbol{\rho} \in \mathbb{C}^{N\times N} \ : \ \boldsymbol{\rho} = \boldsymbol{\rho}^\dagger, \ \boldsymbol{\rho} \geq 0, \ \mathrm{Tr}(\boldsymbol{\rho}) = 1\}$.

Outline

Jump SME

With Poisson process $N(t)$, $\langle dN(t) \rangle = \left(\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V \rho_t V^\dagger \right) \right) dt$, and detection imperfections modeled by $\overline{\theta} \geq 0$ (shot-noise rate) and $\overline{\eta} \in [0, 1]$ (detection efficiency), the quantum state ρ_t is usually mixed and obeys to

$$
\begin{aligned}
d\rho_t = & \left(-i[H, \rho_t] + V \rho_t V^\dagger - \frac{1}{2}(V^\dagger V \rho_t + \rho_t V^\dagger V) \right) dt \\
& + \left(\frac{\overline{\theta} \rho_t + \overline{\eta} V \rho_t V^\dagger}{\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V \rho_t V^\dagger \right)} - \rho_t \right) \left(dN(t) - \left(\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V \rho_t V^\dagger \right) \right) dt \right)
\end{aligned}
$$

With proba. $1 - \left(\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V \rho_t V^\dagger \right) \right) dt$, $dN(t) = 0$ and

$$
\rho_{t+dt} = \frac{M_0 \rho_t M_0^\dagger + (1 - \overline{\eta}) V \rho_t V^\dagger dt}{\operatorname{Tr}\left(M_0 \rho_t M_0^\dagger + (1 - \overline{\eta}) V \rho_t V^\dagger dt \right)}
$$

with $M_0 = I - \left(iH + \frac{1}{2} V^\dagger V \right) dt$.
With proba. $\left(\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V \rho_t V^\dagger \right) \right) dt$, $N(t + dt) - N(t) = 1$ and

$$
\rho_{t+dt} = \frac{M_0 \tilde{\rho}_t M_0^\dagger + (1 - \overline{\eta}) V \tilde{\rho}_t V^\dagger dt}{\operatorname{Tr}\left(M_0 \tilde{\rho}_t M_0^\dagger + (1 - \overline{\eta}) V \tilde{\rho}_t V^\dagger dt \right)} \text{ with } \tilde{\rho}_t = \frac{\overline{\theta} \rho_t + \overline{\eta} V \rho_t V^\dagger}{\overline{\theta} + \overline{\eta} \operatorname{Tr}\left(V \rho_t V^\dagger \right)}.
$$

Outline

Diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\right) dt$$
$$+ \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right) dW_t$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)} - \rho_t\right)\left(dN(t) - \left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

With $dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((L+L^\dagger)\rho_t\right) dt + dW_t$ and $dN(t) = 0$ with proba $1 - \left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt}{\mathrm{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt\right)}$$

with $M_{dy_t} = I - \left(iH + \frac{1}{2}L^\dagger L + \frac{1}{2}V^\dagger V\right) dt + \sqrt{\eta}\, dy_t L$.
For $N(t+dt) - N(t) = 1$ of proba. $\left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$ we have

$$\rho_{t+dt} = \frac{M_{dy_t}\tilde{\rho}_t M_{dy_t}^\dagger + (1-\eta)L\tilde{\rho}_t L^\dagger dt + (1-\overline{\eta})V\tilde{\rho}_t V^\dagger dt}{\mathrm{Tr}\left(M_{dy_t}\tilde{\rho}_t M_{dy_t}^\dagger + (1-\eta)L\tilde{\rho}_t L^\dagger dt + (1-\overline{\eta})V\tilde{\rho}_t V^\dagger dt\right)} \text{ with } \tilde{\rho}_t = \frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)}$$

General diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu)\right) dt$$

$$+ \sum_\nu \sqrt{\eta_\nu}\left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right)\rho_t\right) \boldsymbol{dW_{\nu,t}}$$

$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN_\mu(t)} - \left(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger\right)\right) dt\right)$$

where $\eta_\nu \in [0,1]$, $\overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.
When $\forall \mu$, $\boldsymbol{dN_\mu(t) = 0}$, we have

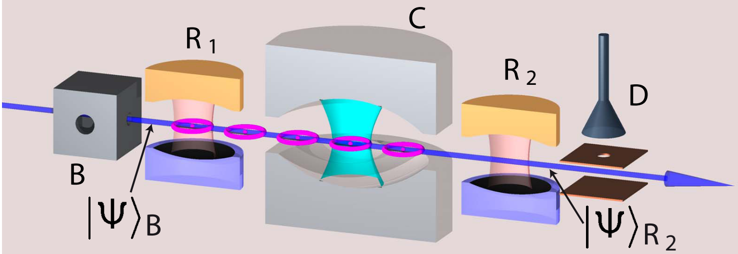
$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\mathrm{Tr}\left(M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt\right)}$$

with $M_{\boldsymbol{dy_t}} = I - \left(iH + \frac{1}{2}\sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2}\sum_\mu V_\mu^\dagger V_\mu\right) dt + \sum_\nu \sqrt{\eta_\nu}\, \boldsymbol{dy_{\nu,t}} L_\nu$ and where $\boldsymbol{dy_{\nu,t}} = \sqrt{\eta_\nu}\, \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right) dt + \boldsymbol{dW_{\nu,t}}$.

If, for some μ, $\boldsymbol{N_\mu(t+dt) - N_\mu(t) = 1}$, we have a similar transition rule

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}} \tilde{\rho}_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \tilde{\rho}_t L_\nu^\dagger dt + \sum_{\mu'} (1-\overline{\eta}_{\mu'}) V_{\mu'} \tilde{\rho}_t V_{\mu'}^\dagger dt}{\mathrm{Tr}\left(M_{\boldsymbol{dy_t}} \tilde{\rho}_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \tilde{\rho}_t L_\nu^\dagger dt + \sum_{\mu'} (1-\overline{\eta}_{\mu'}) V_{\mu'} \tilde{\rho}_t V_{\mu'}^\dagger dt\right)} \text{ with } \tilde{\rho}_t = \frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger\right)}.$$

Outline



Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \dfrac{\boldsymbol{M}_g|\psi_k\rangle}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle}} & \text{with } y_k = g\text{, probability } \mathbb{P}_g = \left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\right\rangle; \\ \dfrac{\boldsymbol{M}_e|\psi_k\rangle}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle}} & \text{with } y_k = e\text{, probability } \mathbb{P}_e = \left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\right\rangle, \end{cases}$$

with

$$\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta).$$

QND measurement of photons

Markov process: density operator $\rho_k = |\psi_k\rangle\langle\psi_k|$ as state.

$$\rho_{k+1} = \begin{cases} \frac{\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right), \end{cases}$$

with

$$\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta), \quad \boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta).$$

Quantum Monte Carlo simulations (2 Matlab scripts):

`IdealQNDphoton.m` `RealisticQNDphoton.m`

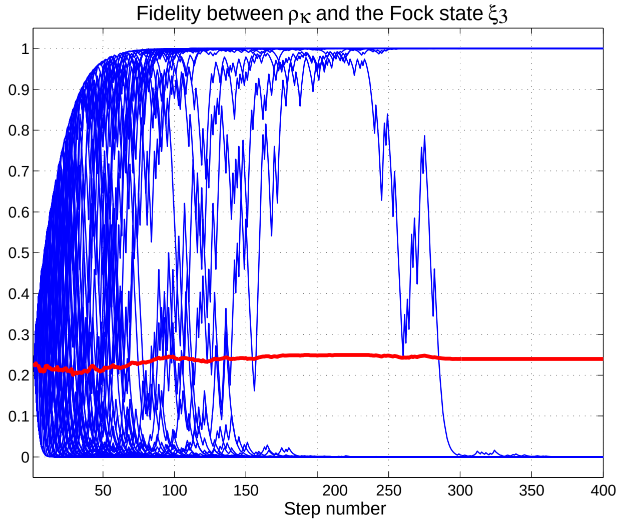
Experimental data

Quantum Non-Demolition (QND) measurement

The measurement operators $\boldsymbol{M}_{g,e}$ commute with the photon-number observable $\boldsymbol{N}$: photon-number states $|n\rangle\langle n|$ are fixed points of the measurement process. We say that the measurement is QND for the observable $\boldsymbol{N}$.

Asymptotic behavior: numerical simulations

100 Monte-Carlo simulations of $\text{Tr}\left(\rho_k |3\rangle\langle 3|\right)$ versus k



Some definitions (see e.g. C.W. Gardiner: Handbook of stochastic methods ... [3rd ed], Springer, 2004)

Convergence of a random process

Consider (X_k) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a metric space $\mathcal{X}$. The random process X_k is said to,

1. converge in probability towards the random variable X if for all $\epsilon > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left(|X_k - X| > \epsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\omega \in \Omega \mid |X_k(\omega) - X(\omega)| > \epsilon\right) = 0;$$

2. converge almost surely towards the random variable X if

$$\mathbb{P}\left(\lim_{k\to\infty} X_k = X\right) = \mathbb{P}\left(\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = X(\omega)\right) = 1;$$

3. converge in mean towards the random variable X if $\lim_{k\to\infty} \mathbb{E}\left(|X_k - X|\right) = 0$.

Some definitions

Markov process

The sequence $(X_k)_{k=1}^{\infty}$ is called a Markov process, if for all k and ℓ satisfying $k > \ell$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$,

$$\mathbb{E}\left(f(X_k) \mid X_1, \ldots, X_\ell\right) = \mathbb{E}\left(f(X_k) \mid X_\ell\right).$$

Martingales

The sequence $(X_k)_{k=1}^{\infty}$ is called respectively a *supermartingale*, a *submartingale* or a martingale, if $\mathbb{E}\left(|X_k|\right) < \infty$ for $k = 1, 2, \cdots$, and

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) \leq X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \geq \ell$$

or

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) \geq X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \geq \ell,$$

or finally,

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) = X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \geq \ell.$$

Martingales asymptotic behavior

H.J. Kushner invariance Theorem

Let $\{X_k\}$ be a Markov chain on the compact state space S. Suppose that there exists a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_{k+1}) \mid X_k = x\right) - V(x) = -\sigma(x)$, where $\sigma(x) \geq 0$ is a positive continuous function of x. Then the ω-limit set (in the sense of almost sure convergence) of X_k is included in the following set

$$I = \{X \mid \sigma(X) = 0\}.$$

Trivially, the same result holds true for the case where $\mathbb{E}\left(V(X_{k+1}) \mid X_k = x\right) - V(x) = \sigma(x)$ with $\sigma(x) \geq 0$ and $V(x)$ bounded from above ($V(X_k)$ is a submartingale),.

Stochastic version of Lasalle invariance principle for Lyapunov function of deterministic dynamics.

Asymptotic behavior

Theorem

Consider for $\boldsymbol{M}_g = \cos(\varphi_0 + \boldsymbol{N}\vartheta)$ and $\boldsymbol{M}_e = \sin(\varphi_0 + \boldsymbol{N}\vartheta)$

$$\rho_{k+1} = \begin{cases} \frac{\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right), \end{cases}$$

with an initial density matrix ρ_0 defined on the subspace $\mathrm{span}\{|n\rangle \mid n = 0, 1, \cdots, n^{\max}\}$. Also, assume the non-degeneracy assumption $\forall n \neq m \in \{0, 1, \cdots, n^{\max}\}$, $\cos^2(\varphi_m) \neq \cos^2(\varphi_n)$ where $\varphi_n = \varphi_0 + n\vartheta$.
Then

- for any $n \in \{0, \ldots, n^{\max}\}$, $\mathrm{Tr}\left(\rho_k |n\rangle\langle n|\right) = \langle n|\rho_k|n\rangle$ is a martingale
- ρ_k converges with probability 1 to one of the $n^{\max} + 1$ Fock state $|n\rangle\langle n|$ with $n \in \{0, \ldots, n^{\max}\}$.
- the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $\mathrm{Tr}\left(\rho_0 |n\rangle\langle n|\right) = \langle n|\rho_0|n\rangle$.

Proof based on QND super-martingales

- For any function f, $V_f(\rho) = \mathrm{Tr}\left(f(\boldsymbol{N})\rho\right)$ is a martingale: $\mathbb{E}\left(V_f(\rho_{k+1}) \mid \rho_k\right) = V_f(\rho_k)$.
- $V(\rho) = \sum_{n \neq m} \sqrt{\langle n|\rho|n\rangle \langle m|\rho|m\rangle}$ is a strict super-martingale:

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) = \sum_{n \neq m} \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right) \sqrt{\langle n|\rho|n\rangle \langle m|\rho|m\rangle} \leq rV(\rho_k)$$

  with $r = \max_{n \neq m} \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right)$ and $r < 1$.
- $V(\rho) \geq 0$ and $V(\rho) = 0$ means that exists n such that $\rho = |n\rangle\langle n|$.

Interpretation: for large k, $V(\rho_k)$ is very close to 0, thus very close to $|n\rangle\langle n|$ ("pure state" = maximal information state) for an a priori random n.
Information extracted by measurement makes state "less uncertain" *a posteriori* but not more predictable *a priori*.

Exercice

Consider the Markov chain $\boldsymbol{\rho}_{k+1} = \boldsymbol{M}_{y_k}(\boldsymbol{\rho}_k)\boldsymbol{M}_{y_k}^\dagger / \operatorname{Tr}\left(\boldsymbol{M}_{y_k}(\boldsymbol{\rho}_k)\boldsymbol{M}_{y_k}^\dagger\right)$ where $y_k = g$ (resp. $y_k = e$) with probability $p_{g,k} = \operatorname{Tr}\left(\boldsymbol{M}_g \boldsymbol{\rho}_k \boldsymbol{M}_g^\dagger\right)$ (resp. $p_{e,k} = \operatorname{Tr}\left(\boldsymbol{M}_e \boldsymbol{\rho}_k \boldsymbol{M}_e^\dagger\right)$). The Kraus operators are given by

$$\boldsymbol{M}_g = \cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger$$

$$\boldsymbol{M}_e = -\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) - \cos\left(\tfrac{\theta_1}{2}\right)\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

with $\theta_1 = 0$. Assume the initial state to be defined on the subspace $\{|n\rangle\}_{n=0}^{n^{\max}}$ and that the cavity state at step k is described by the density operator $\boldsymbol{\rho}_k$.

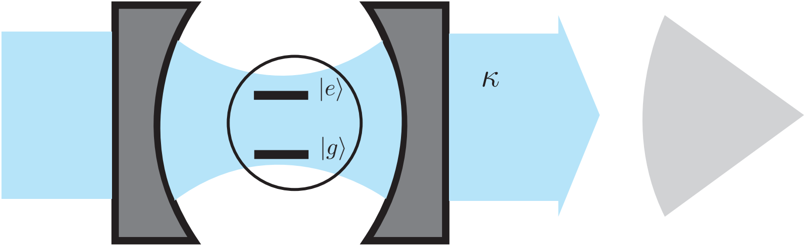
1. Show that

$$\mathbb{E}\left(\operatorname{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}_{k+1}\right) \mid \boldsymbol{\rho}_k\right) = \operatorname{Tr}\left(\boldsymbol{N}\boldsymbol{\rho}_k\right) - \operatorname{Tr}\left(\sin^2\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)\boldsymbol{\rho}_k\right).$$

2. Assume that for any integer n, $\Theta\sqrt{n}/\pi$ is irrational. Then prove that almost surely $\boldsymbol{\rho}_k$ tends to the vacuum state $|0\rangle\langle 0|$ whatever its initial condition is.
3. When $\Theta\sqrt{n}/\pi$ is rational for some integer n, describe the possible ω-limit sets for $\boldsymbol{\rho}_k$.

Outline

Dispersive measurement of a qubit



Inverse setup of photon-box: photons read out a qubit.

Approximate model

Cavity's dynamics are removed (singular perturbation techniques) to achieve a qubit SME:

$$
\begin{aligned}
d\rho_t &= -i[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt \\
&\quad + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t, \\
dy_t &= dW_t + \sqrt{\eta\Gamma_m}\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,dt.
\end{aligned}
$$

Quantum Non-Demolition measurement

$$
\begin{aligned}
d\rho_t &= -i[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt \\
&\qquad + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t, \\
dy_t &= dW_t + \sqrt{\eta\Gamma_m}\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,dt.
\end{aligned}
$$

Uncontrolled case: $\boldsymbol{H} = \omega_{eg}\boldsymbol{\sigma_z}/2$.

Interpretation as a Markov process with Kraus operators

$$
\boldsymbol{M}_{dy_t} = \boldsymbol{I} - \left(i\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{\Gamma_m}{8}\boldsymbol{I}\right)dt + \frac{\sqrt{\eta\Gamma_m}}{2}\boldsymbol{\sigma_z}dy_t,
$$

$$
\sqrt{(1-\eta)dt}\boldsymbol{L} = \frac{\sqrt{(1-\eta)\Gamma_m dt}}{2}\boldsymbol{\sigma_z}.
$$

QND measurement

Kraus operators $\boldsymbol{M}_{dy_t}$ and $\sqrt{(1-\eta)dt}\boldsymbol{L}$ commute with observable $\boldsymbol{\sigma_z}$: qubit states $|g\rangle\langle g|$ and $|e\rangle\langle e|$ are fixed points of the measurement process. The measurement is QND for the observable $\boldsymbol{\sigma_z}$.

QND measurement: asymptotic behavior

Theorem

Consider the SME

$$d\rho_t = -i[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt \\ + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t,$$

with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\eta > 0$.

- For any initial state ρ_0, the solution ρ_t converges almost surely as $t \to \infty$ to one of the states $|g\rangle\langle g|$ or $|e\rangle\langle e|$.
- The probability of convergence to $|g\rangle\langle g|$ (respectively $|e\rangle\langle e|$) is given by $p_g = \mathrm{Tr}\,(|g\rangle\langle g|\rho_0)$ (respectively $\mathrm{Tr}\,(|e\rangle\langle e|\rho_0)$).
- The convergence rate is given by $\eta\Gamma_M/2$.

Proof based on the Lyapunov function $V(\rho) = \sqrt{1 - \mathrm{Tr}^2\,(\boldsymbol{\sigma_z}\rho)}$ with

$$\frac{d}{dt}\mathbb{E}\,(V(\rho)) = -\frac{\eta\Gamma_M}{2}\mathbb{E}\,(V(\rho))$$

Monte Carlo simulations: `IdealQNDqubit.m` `RealisticQNDqubit.m`