Quantum Control[1]
International Graduate School on Control

www.eeci-igsc.eu

Pierre Rouchon[2]

Lecture 1
Chengdu, July 8, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

[2] Mines ParisTech, INRIA Paris

Outline

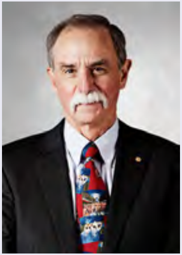
Controlling quantum degrees of freedom

Some applications

- Nuclear Magnetic Resonance (NMR) applications;
- Quantum chemical synthesis;
- High resolution measurement devices (e.g. atomic/optic clocks);
- Quantum communication;
- Quantum computation .

Physics Nobel prize 2012

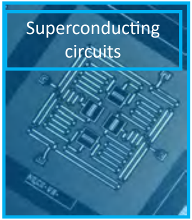
Serge Haroche



David J. Wineland

Nobel prize: ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems.

Technologies for quantum simulation and computation[3]



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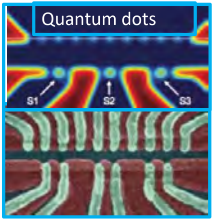

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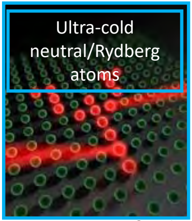

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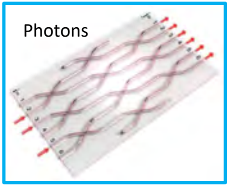

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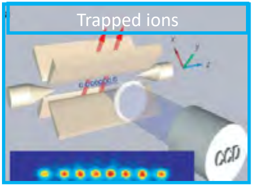

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Requirement:

Scalable modular architecture

Control software from the very beginning.

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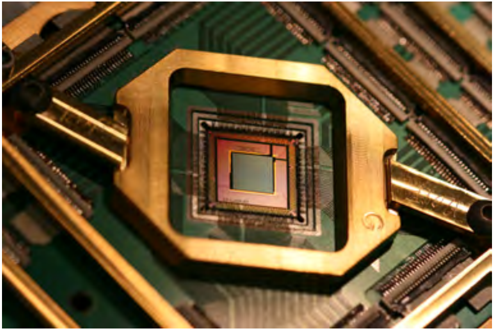
[3] Courtesy of Walter Riess, IBM Research - Zurich.

Quantum computation: towards quantum electronics

D-Wave machine: machines to solve certain huge-dimensional optimization problems (state space of dimension 2^{100}).



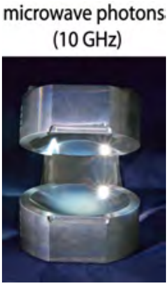
Major challenge: Fragility of quantum information versus external noise.

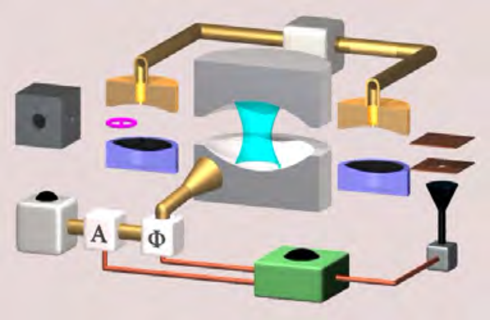
Quantum error correction

We protect quantum information by stabilizing a manifold of quantum states.

Outline

The first experimental realization of a quantum-state feedback:





Theory: I. Dotsenko, . . . : Quantum feedback by discrete quantum non-demolition measurements: towards on-demand generation of photon-number states. Physical Review A, **2009**, 80: 013805-013813.
Experiment: C. Sayrin, . . . , **S. Haroche**:
Real-time quantum feedback prepares and stabilizes photon number states. Nature, **2011**, 477, 73-77.

[4] Laboratoire Kastler-Brossel (LKB), http://www.lkb.upmc.fr/cqed/

Three quantum features emphasized by the LKB photon box [5]

1. Schrödinger ($\hbar = 1$): wave function $|\psi\rangle$ in Hilbert space $\mathcal{H}$,

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1.$$

   Unitary propagator $\boldsymbol{U}$ solution of $\frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U}(0) = I$.

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle$ depending on $|\psi\rangle$, just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[5] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

Composite system (S, M): harmonic oscillator $\otimes$ qubit.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n \, |n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

where $|n\rangle$ is the photon-number state with n photons ($\langle n_1|n_2\rangle = \delta_{n_1,n_2}$).

- Meter M is a qubit, a 2-level system:

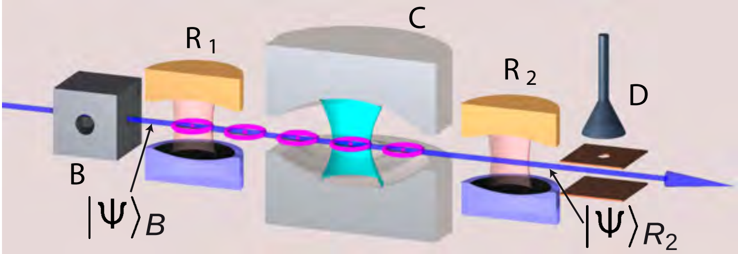
$$\mathcal{H}_M = \left\{ \psi_g \, |g\rangle + \psi_e |e\rangle \;\middle|\; \psi_g, \psi_e \in \mathbb{C} \right\},$$

where $|g\rangle$ (resp. $|e\rangle$) is the ground (resp. excited) state ($\langle g|g\rangle = \langle e|e\rangle = 1$ and $\langle g|e\rangle = 0$)

- **State of the composite system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M$:**

$$\begin{aligned}|\boldsymbol{\Psi}\rangle &= \sum_{n\geq 0} \Big(\Psi_{ng} \, |n\rangle \otimes |g\rangle + \Psi_{ne} \, |n\rangle \otimes |e\rangle \Big) \\ &= \left(\sum_{n\geq 0} \Psi_{ng} \, |n\rangle \right) \otimes |g\rangle + \left(\sum_{n\geq 0} \Psi_{ne} \, |n\rangle \right) \otimes |e\rangle, \quad \Psi_{ne}, \Psi_{ng} \in \mathbb{C}.\end{aligned}$$

Ortho-normal basis: $\big(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle\big)_{n\in\mathbb{N}}$.



- When atom comes out B, the quantum state $|\boldsymbol{\Psi}\rangle_B$ of the composite system is separable: $|\boldsymbol{\Psi}\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}$ is a unitary transformation (Schrödinger propagator) defining the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

Quantum trajectories (2)

Just before detector D the quantum state is **entangled**:

$$|\boldsymbol{\Psi}\rangle_{R_2} = (\boldsymbol{M}_g|\psi\rangle) \otimes |g\rangle + (\boldsymbol{M}_e|\psi\rangle) \otimes |e\rangle$$

Just after outcome y, the state becomes **separable** [6]:

$$|\boldsymbol{\Psi}\rangle_D = \left(\frac{\boldsymbol{M}_y}{\sqrt{\left\langle \psi | \boldsymbol{M}_y^\dagger \boldsymbol{M}_y | \psi \right\rangle}} |\psi\rangle \right) \otimes |y\rangle.$$

Outcome y obtained with probability $\mathbb{P}_y = \left\langle \psi | \boldsymbol{M}_y^\dagger \boldsymbol{M}_y | \psi \right\rangle$..

Quantum trajectories (Markov chain, stochastic dynamics):

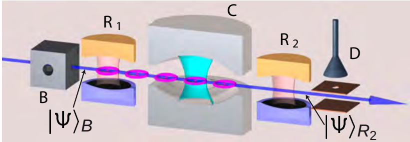
$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle}} |\psi_k\rangle, & y_k = g \text{ with probability } \left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle; \\ \frac{\boldsymbol{M}_e}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle}} |\psi_k\rangle, & y_k = e \text{ with probability } \left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle; \end{cases}$$

with state $|\psi_k\rangle$ and measurement outcome $y_k \in \{g, e\}$ at time-step k:

[6] Measurement operator $\boldsymbol{O} = \boldsymbol{I}_S \otimes (|e\rangle\langle e| - |g\rangle\langle g|)$.

Exercise: Quantum Non Demolition (QND) measurement of photons [7]

Goal $|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}\left(|\psi\rangle \otimes |g\rangle\right) = ?$



$$\boldsymbol{U}_{R_1} = \boldsymbol{I}_S \otimes \left(\left(\frac{|g\rangle + |e\rangle}{\sqrt{2}} \right) \langle g| + \left(\frac{|g\rangle - |e\rangle}{\sqrt{2}} \right) \langle e| \right)$$

$$\boldsymbol{U}_C = e^{-i\frac{\phi_0}{2}\boldsymbol{N}} \otimes |g\rangle\langle g| + e^{i\frac{\phi_0}{2}\boldsymbol{N}} \otimes |e\rangle\langle e|$$

where $\boldsymbol{N}|n\rangle = n|n\rangle, \forall n \in \mathbb{N}$ and $\phi_0 \in \mathbb{R}$.

$$\boldsymbol{U}_{R_2} = \boldsymbol{U}_{R_1}$$

1. Show that $\boldsymbol{U}_{R_1}\left(|\psi\rangle \otimes |g\rangle\right) = \frac{1}{\sqrt{2}}\left(|\psi\rangle \otimes |g\rangle + |\psi\rangle \otimes |e\rangle\right)$ and
$$\boldsymbol{U}_C\boldsymbol{U}_{R_1}\left(|\psi\rangle \otimes |g\rangle\right) = \frac{1}{\sqrt{2}}\left(\left(e^{-i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle \right) \otimes |g\rangle + \left(e^{i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle \right) \otimes |e\rangle \right).$$
2. Show that $|\boldsymbol{\Psi}\rangle_{R_2} = \left(\cos(\frac{\phi_0}{2}\boldsymbol{N})|\psi\rangle \right) \otimes |g\rangle + \left(i\sin(\frac{\phi_0}{2}\boldsymbol{N})|\psi\rangle \right) \otimes |e\rangle$
3. Deduce that $\boldsymbol{M}_g = \cos(\frac{\phi_0}{2}\boldsymbol{N})$ and $\boldsymbol{M}_e = -i\sin(\frac{\phi_0}{2}\boldsymbol{N})$.
4. Question for Wednesday: write a computer program (e.g. a Scilab or Matlab script) to simulate over 20 sampling steps the attached Markov chain starting from $|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ with parameter $\phi_0 = \pi/3$ (Quantum Monte-Carlo trajectories).

[7] M. Brune, ...: Manipulation of photons in a cavity by dispersive atom-field coupling: quantum non-demolition measurements and generation of "Schrödinger cat" states . Physical Review A, 45:5193-5214, 1992.

Outline

Outline of the lectures

Monday **1-** Introduction (motivating applications; LKB photon-box as prototype of open quantum system). **2-** Spring system (harmonic oscillator, spectral decomposition, annihilation/creation operators, coherent state and displacement). **3-** Spin system (qubit, Pauli matrices). **4-** Composite spin/spring system (tensor product, resonant/dispersive interaction, underlying PDE's).

Tuesday **5-** Averaging and rotating waves approximation (first/second order perturbation expansion,) **6-**Open-loop control via averaging techniques (resonant control for qubit and Jaynes-Cummings systems)

Wednesday **7-** Discrete-time dynamics of the LKB photon box (density operators, measurement imperfection, decoherence, quantum filter) **8-** Discrete-time Stochastic Master Equation (SME) (Positive Operator Value Measurement (POVM), Kraus maps and quantum channels, stability and contractions, Schrödinger and Heisenberg points of view). **9-** Discrete-time Quantum Non Demolition (QND) measurement (martingales, convergence of Markov processes, Kushner invariance Theorem) **10-** Measurement-based feedback and Lyapunov stabilization of photons (LKB photon box with dispersive/resonnant probe atoms, closed-loop Monte-Carlo simulations).

Thursday **11-** Continuous-time Stochastic Master Equation (SME) (Wiener processes and Ito calculus, continuous-time measurement, quantum filtering) **12-** Measurement-based feedback stabilization of a qubit (Lyapunov feedback, closed-loop Monte-Carlo simulations)

Friday **13-** Lindblad master equation (decoherence models for a qubit and an oscillator) **14-** Coherent-feedback stabilization (principle, cat-qubit and multi-photon pumping)

Reference books

1. Cohen-Tannoudji, C.; Diu, B. & Laloë, F.: Mécanique Quantique Hermann, Paris, 1977, I& II (*quantum physics: a well known and tutorial textbook*)
2. S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006. (*quantum physics: spin/spring systems, decoherence, Schrödinger cats, entanglement.*)
3. C. Gardiner, P. Zoller: The Quantum World of Ultra-Cold Atoms and Light I& II. Imperial College Press, 2009. (*quantum physics, measurement and control*)
4. Barnett, S. M. & Radmore, P. M.: Methods in Theoretical Quantum Optics Oxford University Press, 2003. (*mathematical physics: many useful operator formulae for spin/spring systems*)
5. E. Davies: Quantum Theory of Open Systems. Academic Press, 1976. (*mathematical physics: functional analysis aspects when the Hilbert space is of infinite dimension*)
6. Gardiner, C. W.: Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences [3rd ed], Springer, 2004. (*tutorial introduction to probability, Markov processes, stochastic differential equations and Ito calculus.*)
7. M. Nielsen, I. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000. (*tutorial introduction with a computer science and communication view point*)

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Lecture 2
Chengdu, July 8, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
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Outline

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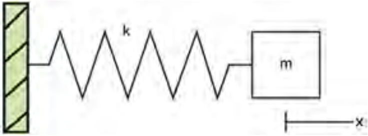
Harmonic oscillator

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

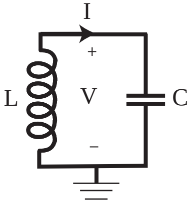
$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Mechanical oscillator



Frictionless spring: $\frac{d^2}{dt^2}x = -\frac{k}{m}x.$

Electrical oscillator:



LC oscillator:

$$\frac{d}{dt}I = \frac{V}{L}, \quad \frac{d}{dt}V = -\frac{I}{C}, \quad \left(\frac{d^2}{dt^2}I = -\frac{1}{LC}I\right).$$

Quantum regime

$k_B T \ll \hbar\omega$: typically for the photon box experiment in these lectures, $\omega = 51 GHz$ and $T = 0.8K$.

Harmonic oscillator[3]: quantization and correspondence principle

$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the lectures)

$$i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where $\boldsymbol{H}$ results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}\boldsymbol{X}$ and p by momentum operator $\sqrt{2}\boldsymbol{P} = -i\frac{\partial}{\partial x}$. $\boldsymbol{H}$ is a Hermitian operator on $L^2(\mathbb{R},\mathbb{C})$, with its domain to be given.

PDE model: $i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}.$

[3] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Average position $\langle \boldsymbol{X} \rangle_t = \langle \psi | \boldsymbol{X} | \psi \rangle$ and momentum $\langle \boldsymbol{P} \rangle_t = \langle \psi | \boldsymbol{P} | \psi \rangle$:

$$\langle \boldsymbol{X} \rangle_t = \tfrac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x |\psi|^2 dx, , \quad \langle \boldsymbol{P} \rangle_t = -\tfrac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation $\boldsymbol{a}$ and creation operators $\boldsymbol{a}^\dagger$ (domains to be given):

$$\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad \boldsymbol{a}^\dagger = \boldsymbol{X} - i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[\boldsymbol{X}, \boldsymbol{P}] = \tfrac{i}{2} \boldsymbol{I}, \quad [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = \omega \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right).$$

Spectrum of Hamiltonian $\boldsymbol{H} = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$:

$$E_n = \omega(n+\frac{1}{2}),\ \ \psi_n(x) = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x),\ \ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Spectral decomposition of $a^\dagger a$ using $[a, a^\dagger] = 1$:

- If $|\psi\rangle$ is an eigenstate associated to eigenvalue λ, $\boldsymbol{a}|\psi\rangle$ and $\boldsymbol{a}^\dagger|\psi\rangle$ are also eigenstates associated to $\lambda - 1$ and $\lambda + 1$.
- $\boldsymbol{a}^\dagger \boldsymbol{a}$ is semi-definite positive.
- The ground state $|\psi_0\rangle$ is necessarily associated to eigenvalue 0 and is given by the Gaussian function $\psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2)$.

Harmonic oscillator: spectral decomposition and Fock states

$[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$: spectrum of $\boldsymbol{a}^\dagger \boldsymbol{a}$ is non-degenerate and is $\mathbb{N}$.

Fock state with n photons (phonons): the eigenstate of $\boldsymbol{a}^\dagger \boldsymbol{a}$ associated to the eigenvalue n ($|n\rangle \sim \psi_n(x)$):

$$\boldsymbol{a}^\dagger \boldsymbol{a}|n\rangle = n|n\rangle, \quad \boldsymbol{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ is called 0-photon state or vacuum state.

The operator $\boldsymbol{a}$ (resp. $\boldsymbol{a}^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Hilbert space of quantum system: $\mathcal{H} = \{\sum_n c_n|n\rangle \mid (c_n) \in l^2(\mathbb{C})\} \sim L^2(\mathbb{R}, \mathbb{C})$.
Domain of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^1(\mathbb{C})\}$.
Domain of $\boldsymbol{H}$ ot $\boldsymbol{a}^\dagger\boldsymbol{a}$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^2(\mathbb{C})\}$.

$$h^k(\mathbb{C}) = \{(c_n) \in l^2(\mathbb{C}) \mid \sum n^k |c_n|^2 < \infty\}, \qquad k = 1, 2.$$

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$, ($\mathbb{H} = \frac{\omega}{2}(p^2 + x^2) + \sqrt{2}ux$)

$$\boldsymbol{H} = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger).$$

The associated controlled PDE

$$i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \left(\frac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator $\boldsymbol{D}_\alpha$ (unitary) with $\alpha \in \mathbb{C}$:

$$\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^*\boldsymbol{a}} = e^{2i\Im\alpha\boldsymbol{X} - 2i\Re\alpha\boldsymbol{P}}$$

From Baker-Campbell Hausdorf formula, for all operators $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$e^{\boldsymbol{A}}\boldsymbol{B}e^{-\boldsymbol{A}} = \boldsymbol{B} + [\boldsymbol{A},\boldsymbol{B}] + \frac{1}{2!}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] + \frac{1}{3!}[\boldsymbol{A},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]] + \ldots$$

we get the Glauber formula[4] when $[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] = [\boldsymbol{B},[\boldsymbol{A},\boldsymbol{B}]] = 0$:

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}\ e^{\boldsymbol{B}}\ e^{-\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]}.$$

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[4] Take s derivative of $e^{s(\boldsymbol{A}+\boldsymbol{B})}$ and of $e^{s\boldsymbol{A}}\ e^{s\boldsymbol{B}}\ e^{-\frac{s^2}{2}[\boldsymbol{A},\boldsymbol{B}]}$.

With $\boldsymbol{A} = \alpha \boldsymbol{a}^\dagger$ and $\boldsymbol{B} = -\alpha^* \boldsymbol{a}$, Glauber formula gives:

$$\boldsymbol{D}_\alpha = e^{-\frac{|\alpha|^2}{2}}\; e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} = e^{+\frac{|\alpha|^2}{2}}\; e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger}$$

$$\boldsymbol{D}_{-\alpha} \boldsymbol{a} \boldsymbol{D}_\alpha = \boldsymbol{a} + \alpha \boldsymbol{I} \quad \text{and} \quad \boldsymbol{D}_{-\alpha} \boldsymbol{a}^\dagger \boldsymbol{D}_\alpha = \boldsymbol{a}^\dagger + \alpha^* \boldsymbol{I}.$$

With $\boldsymbol{A} = 2i\Im\alpha \boldsymbol{X} \sim i\sqrt{2}\Im\alpha x$ and $\boldsymbol{B} = -2\imath\Re\alpha \boldsymbol{P} \sim -\sqrt{2}\Re\alpha \frac{\partial}{\partial x}$, Glauber formula gives[5]:

$$\boldsymbol{D}_\alpha = e^{-i\Re\alpha\Im\alpha}\; e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha \frac{\partial}{\partial x}}$$

$$(\boldsymbol{D}_\alpha |\psi\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha}\; e^{i\sqrt{2}\Im\alpha x} \psi(x - \sqrt{2}\Re\alpha, t)$$

[5]Note that the operator $e^{-r\partial/\partial x}$ corresponds to a translation of x by r.

Harmonic oscillator: lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation
$i\frac{d}{dt}|\psi\rangle = \left(\omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)\right)|\psi\rangle$. Set $\langle\boldsymbol{a}\rangle = \langle\psi|\boldsymbol{a}|\psi\rangle$. Then

$$\frac{d}{dt}\langle\boldsymbol{a}\rangle = -i\omega\langle\boldsymbol{a}\rangle - iu.$$

From $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P}$, we have $\langle\boldsymbol{a}\rangle = \langle\boldsymbol{X}\rangle + i\langle\boldsymbol{P}\rangle$ where
$\langle\boldsymbol{X}\rangle = \langle\psi|\boldsymbol{X}|\psi\rangle \in \mathbb{R}$ and $\langle\boldsymbol{P}\rangle = \langle\psi|\boldsymbol{P}|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle\boldsymbol{X}\rangle = \omega\langle\boldsymbol{P}\rangle, \quad \frac{d}{dt}\langle\boldsymbol{P}\rangle = -\omega\langle\boldsymbol{X}\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t}\boldsymbol{D}_{\langle\boldsymbol{a}\rangle_t}|\chi\rangle$ with

$$\theta_t = \int_0^t \left(\omega|\langle\boldsymbol{a}\rangle|^2 + u\Re(\langle\boldsymbol{a}\rangle)\right), \quad D_{\langle\boldsymbol{a}\rangle_t} = e^{\langle\boldsymbol{a}\rangle_t\boldsymbol{a}^\dagger - \langle\boldsymbol{a}\rangle_t^*\boldsymbol{a}},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle\boldsymbol{a}\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

Coherent states

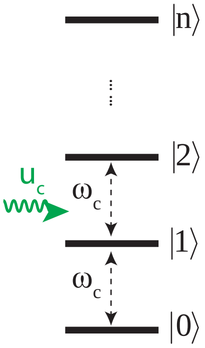
$$|\alpha\rangle = \boldsymbol{D}_\alpha|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigenstate of $\boldsymbol{a}$: $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$.
A widely known result in quantum optics[6]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum.

[6] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1989.

Summary for the quantum harmonic oscillator

- Hilbert space:
  $\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathbb{D} = \{\rho \in \mathcal{L}(\mathcal{H}), \rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\,|n\text{-}1\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^\dagger\boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath\boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}/\hbar = \omega_c\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a}+\boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c$).
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}: \ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}}e^{\imath\sqrt{2}x\Im\alpha}e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.



Exercise: useful operator identities

1. Set $\boldsymbol{X}_\lambda = \frac{1}{2}\left(e^{-i\lambda}\boldsymbol{a} + e^{i\lambda}\boldsymbol{a}^\dagger\right)$ for any angle λ. Show that

$$\left[\boldsymbol{X}_\lambda, \boldsymbol{X}_{\lambda+\frac{\pi}{2}}\right] = \frac{i}{2}\boldsymbol{I}.$$

2. Prove that, for any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have
   1. $\boldsymbol{D}_{\alpha+\beta} = e^{\frac{\alpha^*\beta-\alpha\beta^*}{2}}\boldsymbol{D}_\alpha\boldsymbol{D}_\beta$
   2. $\boldsymbol{D}_{\alpha+\epsilon}\boldsymbol{D}_{-\alpha} = \left(1 + \frac{\alpha\epsilon^*-\alpha^*\epsilon}{2}\right)\boldsymbol{I} + \epsilon\boldsymbol{a}^\dagger - \epsilon^*\boldsymbol{a} + \boldsymbol{O}(|\epsilon|^2)$
   3. $\left(\frac{d}{dt}\boldsymbol{D}_\alpha\right)\boldsymbol{D}_{-\alpha} = \left(\frac{\alpha\frac{d}{dt}\alpha^* - \alpha^*\frac{d}{dt}\alpha}{2}\right)\boldsymbol{I} + \left(\frac{d}{dt}\alpha\right)\boldsymbol{a}^\dagger - \left(\frac{d}{dt}\alpha^*\right)\boldsymbol{a}.$
3. Show formally that for any operators $\boldsymbol{A}$ and $\boldsymbol{B}$ on an Hilbert-space $\mathcal{H}$:

$$e^{\boldsymbol{A}+\epsilon\boldsymbol{B}} = e^{\boldsymbol{A}} + \epsilon\int_0^1 e^{s\boldsymbol{A}}\boldsymbol{B}e^{(1-s)\boldsymbol{A}}ds + O(\epsilon^2).$$

   Deduced that for any C^1 time-varying operator $\boldsymbol{A}(t)$, one has

$$\frac{d}{dt}e^{\boldsymbol{A}(t)} = \int_0^1 e^{s\boldsymbol{A}(t)}\left(\frac{d\boldsymbol{A}}{dt}(t)\right)e^{(1-s)\boldsymbol{A}(t)}ds.$$

Quantum Control[1]
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Lecture 3
Chengdu, July 8, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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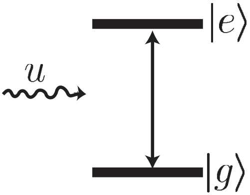
Outline



The simplest quantum system: a ground state $|g\rangle$ of energy ω_g; an excited state $|e\rangle$ of energy ω_e. The quantum state $|\psi\rangle \in \mathbb{C}^2$ is a linear superposition $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and obey to the Schrödinger equation (ψ_g and ψ_e depend on t).

Schrödinger equation for the uncontrolled 2-level system ($\hbar = 1$) :

$$\imath \frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle = \left(\omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|\right)|\psi\rangle$$

where $\boldsymbol{H}_0$ is the Hamiltonian, a Hermitian operator $\boldsymbol{H}_0^\dagger = \boldsymbol{H}_0$.
Energy is defined up to a constant: $\boldsymbol{H}_0$ and $\boldsymbol{H}_0 + \varpi(t)\boldsymbol{I}$ ($\varpi(t) \in \mathbb{R}$ arbitrary) are attached to the same physical system. If $|\psi\rangle$ satisfies $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle$ then $|\chi\rangle = e^{-i\vartheta(t)}|\psi\rangle$ with $\frac{d}{dt}\vartheta = \varpi$ obeys to $i\frac{d}{dt}|\chi\rangle = (\boldsymbol{H}_0 + \varpi\boldsymbol{I})|\chi\rangle$. Thus for any ϑ, $|\psi\rangle$ and $e^{-i\vartheta}|\psi\rangle$ represent the same physical system: The global phase of a quantum system $|\psi\rangle$ can be chosen arbitrarily at any time.

Take origin of energy such that ω_g (resp. ω_e) becomes $-\frac{\omega_e-\omega_g}{2}$ (resp. $\frac{\omega_e-\omega_g}{2}$) and set $\omega_{\text{eg}} = \omega_e - \omega_g$

The solution of $i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle = \frac{\omega_{\text{eg}}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle$ is

$$|\psi\rangle_t = \psi_{g0} e^{\frac{i\omega_{\text{eg}} t}{2}}|g\rangle + \psi_{e0} e^{\frac{-i\omega_{\text{eg}} t}{2}}|e\rangle.$$

With a classical electromagnetic field described by $u(t) \in \mathbb{R}$, the coherent evolution the controlled Hamiltonian

$$\boldsymbol{H}(t) = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x} = \frac{\omega_{\text{eg}}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u(t)}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

The controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle$ reads:

$$i\frac{d}{dt}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} = \frac{\omega_{\text{eg}}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix}.$$

The 3 Pauli Matrices[3]

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|,\ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|,\ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

[3] They correspond, up to multiplication by i, to the 3 imaginary quaternions.

Pauli matrices and some formula

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\boldsymbol{\sigma_x}^2 = \boldsymbol{I}, \quad \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}, \quad [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}, \ \text{circular permutation} \ \ldots$$

- Since for any $\theta \in \mathbb{R}$, $e^{i\theta\boldsymbol{\sigma_x}} = \cos\theta + i\sin\theta\boldsymbol{\sigma_x}$ (idem for $\boldsymbol{\sigma_y}$ and $\boldsymbol{\sigma_z}$), the solution of $i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}|\psi\rangle$ is

$$|\psi\rangle_t = e^{\frac{-i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle_0 = \left(\cos\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{I} - i\sin\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{\sigma_z}\right)|\psi\rangle_0$$

- For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have

$$\boldsymbol{\sigma_\alpha} e^{i\theta\boldsymbol{\sigma_\beta}} = e^{-i\theta\boldsymbol{\sigma_\beta}}\boldsymbol{\sigma_\alpha}, \qquad \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{-1} = \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{\dagger} = e^{-i\theta\boldsymbol{\sigma_\alpha}}.$$

and also

$$e^{-\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta}e^{\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}} = e^{-i\theta\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} = \boldsymbol{\sigma_\beta}e^{i\theta\boldsymbol{\sigma_\alpha}}$$

We start from $|\psi\rangle$ that obeys $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$. We consider the orthogonal projector on $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$, called density operator. Then ρ is an Hermitian operator ≥ 0, that satisfies $\text{Tr}(\rho) = 1$, $\rho^2 = \rho$ and obeys to the Liouville equation:

$$\frac{d}{dt}\rho = -i[\boldsymbol{H}, \rho].$$

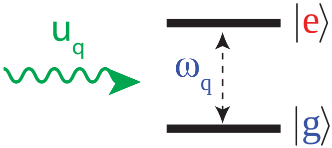
For a two level system $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and

$$\rho = \frac{\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}}{2}$$

where $(x, y, z) = (2\Re(\psi_g\psi_e^*), 2\Im(\psi_g\psi_e^*), |\psi_e|^2 - |\psi_g|^2) \in \mathbb{R}^3$ represent a vector $\vec{M} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, the Bloch vector, that evolves on the unite sphere of $\mathbb{R}^3$, $\mathbb{S}^2$ called the the Bloch Sphere since $\text{Tr}\left(\rho^2\right) = x^2 + y^2 + z^2 = 1$. The Liouville equation with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x}$ reads

$$\frac{d}{dt}\vec{M} = (u\vec{\imath} + \omega_{eg}\vec{k}) \times \vec{M}.$$

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{\psi_g|g\rangle + \psi_e|e\rangle,\ \psi_g, \psi_e \in \mathbb{C}\right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...



- Hamiltonian: $\boldsymbol{H_M} = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{\frac{1}{2}(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}) \mid (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1\right\}$

Exercise: propagator for a qubit

Consider $\boldsymbol{H} = (u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + w\boldsymbol{\sigma_z})/2$ with $(u, v, w) \in \mathbb{R}^3$.

1. For (u, v, w) constant and non zero, compute the solutions of

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}\boldsymbol{U} \text{ with } \boldsymbol{U}_0 = \boldsymbol{I}$$

   in term of $|\psi\rangle_0$, $\boldsymbol{\sigma} = (u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + w\boldsymbol{\sigma_z})/\sqrt{u^2 + v^2 + w^2}$ and $\omega = \sqrt{u^2 + v^2 + w^2}$. Indication: use the fact that $\boldsymbol{\sigma}^2 = \boldsymbol{I}$.

2. Assume that, (u, v, w) depends on t according to $(u, v, w)(t) = \omega(t)(\bar{u}, \bar{v}, \bar{w})$ with $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^3/\{0\}$ constant of length 1. Compute the solutions of

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}(t)|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}(t)\boldsymbol{U} \text{ with } \boldsymbol{U}_0 = \boldsymbol{I}$$

   in term of $|\psi\rangle_0$, $\overline{\boldsymbol{\sigma}} = \bar{u}\boldsymbol{\sigma_x} + \bar{v}\boldsymbol{\sigma_y} + \bar{w}\boldsymbol{\sigma_z}$ and $\theta(t) = \int_0^t \omega$.

3. Explain why (u, v, w) colinear to the constant vector $(\bar{u}, \bar{v}, \bar{w})$ is crucial, for the computations in previous question.

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Lecture 4
Chengdu, July 8, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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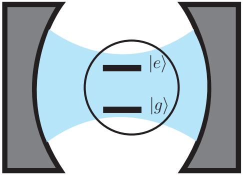
Outline

Outline

Composite system: 2-level and harmonic oscillator



2-level system lives on $\mathbb{C}^2$ with $\boldsymbol{H_q} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$
oscillator lives on $L^2(\mathbb{R},\mathbb{C}) \sim l^2(\mathbb{C})$ with

$$\boldsymbol{H_c} = -\frac{\omega_c}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega_c}{2}x^2 \sim \omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right)$$

$$\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a} \text{ and } \boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} \sim \tfrac{1}{\sqrt{2}}\left(x + \tfrac{\partial}{\partial x}\right)$$

The composite system lives on the tensor product
$\mathbb{C}^2 \otimes L^2(\mathbb{R},\mathbb{C}) \sim \mathbb{C}^2 \otimes l^2(\mathbb{C})$ with spin-spring Hamiltonian

$$\boldsymbol{H} = \tfrac{\omega_{eg}}{2}\,\boldsymbol{\sigma_z} \otimes \boldsymbol{I}_c + \omega_c\,\boldsymbol{I}_q \otimes \left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x} \otimes (\boldsymbol{a}^\dagger - \boldsymbol{a})$$

with the typical scales $\Omega \ll \omega_c, \omega_{eg}$ and $|\omega_c - \omega_{eg}| \ll \omega_c, \omega_{eg}$.
Shortcut notations:

$$\boldsymbol{H} = \underbrace{\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z}}_{\boldsymbol{H_q}} + \underbrace{\omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right)}_{\boldsymbol{H_c}} + \underbrace{i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})}_{\boldsymbol{H_{int}}}$$

The spin-spring PDE

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\mathbf{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_e}{\partial t} = +\tfrac{\omega_{eg}}{2}\psi_e + \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\tfrac{\omega_{eg}}{2}\psi_g + \tfrac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

since $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle = (\psi_e(x,t), \psi_g(x,t))$, $\psi_g(.,t), \psi_e(.,t) \in L^2(\mathbb{R},\mathbb{C})$ and $\|\psi_g\|^2 + \|\psi_e\|^2 = 1$.

Exercise: write the PDE for the controlled Hamiltonian

$$\tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) + i\tfrac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u_c(\boldsymbol{a} + \boldsymbol{a}^\dagger) + u_q\boldsymbol{\sigma_x}$$

where $u_c, u_q \in \mathbb{R}$ are local control inputs associated to the oscillator and qubit, respectively.

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds also to an infinite set of ODE's

$$i\frac{d}{dt}\psi_{e,n} = ((n+1/2)\omega_c + \omega_{\text{eg}}/2)\psi_{e,n} + i\frac{\Omega}{2}\left(\sqrt{n}\,\psi_{g,n-1} - \sqrt{n+1}\,\psi_{g,n+1}\right)$$

$$i\frac{d}{dt}\psi_{g,n} = ((n+1/2)\omega_c - \omega_{\text{eg}}/2)\psi_{g,n} + i\frac{\Omega}{2}\left(\sqrt{n}\,\psi_{e,n-1} - \sqrt{n+1}\,\psi_{e,n+1}\right)$$

where $|\psi\rangle = \sum_{n=0}^{+\infty}\psi_{g,n}|g,n\rangle + \psi_{e,n}|e,n\rangle$, $\psi_{g,n}, \psi_{e,n} \in \mathbb{C}$.

Exercise: write the infinite set of ODE's for

$$\frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u_c(\boldsymbol{a} + \boldsymbol{a}^\dagger) + u_q\boldsymbol{\sigma_x}$$

where $u_c, u_q \in \mathbb{R}$ are local control inputs associated to the oscillator and qubit, respectively.

Dispersive case: approximate Hamiltonian for $\Omega \ll |\omega_c - \omega_{eg}|$.

$$\boldsymbol{H} \approx \boldsymbol{H}_{\text{disp}} = \tfrac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) - \frac{\chi}{2}\,\boldsymbol{\sigma_z}\left(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2}\right) \quad \text{with } \chi = \tfrac{\Omega^2}{2(\omega_c - \omega_{eg})}$$

The corresponding PDE is :

$$i\frac{\partial \psi_e}{\partial t} = +\frac{\omega_{eg}}{2}\psi_e + \frac{1}{2}(\omega_c - \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_e$$
$$i\frac{\partial \psi_g}{\partial t} = -\frac{\omega_{eg}}{2}\psi_g + \frac{1}{2}(\omega_c + \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_g$$

The propagator, the t-dependant unitary operator $\boldsymbol{U}$ solution of $i\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$, reads:

$$\boldsymbol{U}(t) = e^{i\omega_{eg}t/2}\exp\left(-i(\omega_c + \chi/2)t(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2})\right) \otimes |g\rangle\langle g| \\ + e^{-i\omega_{eg}t/2}\exp\left(-i(\omega_c - \chi/2)t(\boldsymbol{N} + \tfrac{\boldsymbol{I}}{2})\right) \otimes |e\rangle\langle e|$$

Exercise: write the infinite set of ODE's attached to the dispersive Hamiltonian $\boldsymbol{H}_{\text{disp}}$.

Resonant case: approximate Hamiltonian for $\omega_c = \omega_{eg} = \omega$.

The Hamiltonian becomes (Jaynes-Cummings Hamiltonian):

$$\boldsymbol{H} \approx \boldsymbol{H}_{JC} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}).$$

The corresponding PDE is :

$$i\frac{\partial \psi_e}{\partial t} = +\frac{\omega}{2}\psi_e + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e - i\frac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi_g$$

$$i\frac{\partial \psi_g}{\partial t} = -\frac{\omega}{2}\psi_g + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + i\frac{\Omega}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\psi_e$$

Exercise: Write the infinite set of ODE's attached to the Jaynes-Cummings Hamiltonian $\boldsymbol{H}$.

Exercise: the Jaynes-Cummings propagator

For $\boldsymbol{H}_{JC} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{N} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})$ show that the propagator, the t-dependant unitary operator $\boldsymbol{U}$ solution of $i\frac{d}{dt}\boldsymbol{U} = \boldsymbol{H}_{JC}\boldsymbol{U}$ with $\boldsymbol{U}(0) = \boldsymbol{I}$, reads
$\boldsymbol{U}(t) = e^{-i\omega t\left(\frac{\boldsymbol{\sigma_z}}{2}+\boldsymbol{N}+\frac{1}{2}\right)} e^{\frac{\Omega t}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})}$ where for any angle θ,

$$e^{\theta(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a})} = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\boldsymbol{N}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}}) - \boldsymbol{\sigma_+} \otimes \boldsymbol{a}\,\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}} + \boldsymbol{\sigma_-} \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger$$

Hint: show that

$$\left[\frac{\boldsymbol{\sigma_z}}{2} + \boldsymbol{N}\,,\ \boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right] = 0$$

$$\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)^{2k} = (-1)^k\left(|g\rangle\langle g| \otimes \boldsymbol{N}^k + |e\rangle\langle e| \otimes (\boldsymbol{N}+\boldsymbol{I})^k\right)$$

$$\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)^{2k+1} = (-1)^k\left(\boldsymbol{\sigma_-} \otimes \boldsymbol{N}^k\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+} \otimes \boldsymbol{a}\boldsymbol{N}^k\right)$$

and compute de series defining the exponential of an operator.

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Lecture 5
Chengdu, July 9, 2019

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Outline

Outline

Bilinear Schrödinger equation

Un-measured quantum system $\rightarrow$ **Bilinear Schrödinger equation**

$$i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle,$$

- $|\psi\rangle \in \mathcal{H}$ the system's wavefunction with $\left\| |\psi\rangle \right\|_{\mathcal{H}} = 1$;
- the free Hamiltonian, $\boldsymbol{H}_0$, is a Hermitian operator defined on $\mathcal{H}$;
- the control Hamiltonian, $\boldsymbol{H}_1$, is a Hermitian operator defined on $\mathcal{H}$;
- the control $u(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ is a scalar control.

Here we consider the case of finite dimensional $\mathcal{H}$

Almost periodic control

We consider the controls of the form

$$u(t) = \epsilon \left(\sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} + \boldsymbol{u}_j^* e^{-i\omega_j t} \right)$$

- $\epsilon > 0$ is a small parameter;
- $\epsilon \boldsymbol{u}_j$ is the constant complex amplitude associated to the pulsation $\omega_j \geq 0$;
- r stands for the number of independent frequencies ($\omega_j \neq \omega_k$ for $j \neq k$).

We are interested in approximations, for ϵ tending to 0^+, of trajectories $t \mapsto |\psi_\epsilon\rangle_t$ of

$$\frac{d}{dt}|\psi_\epsilon\rangle = \left(\boldsymbol{A}_0 + \epsilon \left(\sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} + \boldsymbol{u}_j^* e^{-i\omega_j t} \right) \boldsymbol{A}_1 \right) |\psi_\epsilon\rangle$$

where $\boldsymbol{A}_0 = -i\boldsymbol{H}_0$ and $\boldsymbol{A}_1 = -i\boldsymbol{H}_1$ are skew-Hermitian.

Rotating frame

Consider the following change of variables

$$|\psi_\epsilon\rangle_t = e^{\boldsymbol{A}_0 t}|\phi_\epsilon\rangle_t.$$

The resulting system is said to be in the "interaction frame"

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon \boldsymbol{B}(t)|\phi_\epsilon\rangle$$

where $\boldsymbol{B}(t)$ is a skew-Hermitian operator whose time-dependence is almost periodic:

$$B(t) = \sum_{j=1}^{r} \boldsymbol{u}_j e^{i\omega_j t} e^{-\boldsymbol{A}_0 t} \boldsymbol{A}_1 e^{\boldsymbol{A}_0 t} + \boldsymbol{u}_j^* e^{-i\omega_j t} e^{-\boldsymbol{A}_0 t} \boldsymbol{A}_1 e^{\boldsymbol{A}_0 t}.$$

Main idea

We can write

$$\boldsymbol{B}(t) = \bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t),$$

where $\bar{\boldsymbol{B}}$ is a constant skew-Hermitian matrix and $\widetilde{\boldsymbol{B}}(t)$ is a bounded almost periodic skew-Hermitian matrix.

Multi-frequency averaging: first order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t)\right)|\phi_\epsilon\rangle,$$

and

$$\frac{d}{dt}|\phi_\epsilon^{1^{\text{st}}}\rangle = \epsilon\bar{\boldsymbol{B}}|\phi_\epsilon^{1^{\text{st}}}\rangle,$$

initialized at the same state $|\phi_\epsilon^{1^{\text{st}}}\rangle_0 = |\phi_\epsilon\rangle_0$.

Theorem: first order approximation (Rotating Wave Approximation)

Consider the functions $|\phi_\epsilon\rangle$ and $|\phi_\epsilon^{1^{\text{st}}}\rangle$ initialized at the same state and following the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi_\epsilon^{1^{\text{st}}}\rangle_t \right\| \leq M\epsilon$$

Multi-frequency averaging: first order

Proof's idea

Almost periodic change of variables:

$$|\chi_\epsilon\rangle = (1 - \epsilon \widetilde{\boldsymbol{B}}(t))|\phi_\epsilon\rangle$$

well-defined for $\epsilon > 0$ sufficiently small.
The dynamics can be written as

$$\frac{d}{dt}|\chi_\epsilon\rangle = (\epsilon \bar{\boldsymbol{B}} + \epsilon^2 \boldsymbol{F}(\epsilon, t))|\chi_\epsilon\rangle$$

where $\boldsymbol{F}(\epsilon, t)$ is uniformly bounded in time.

Multi-frequency averaging: second order

More precisely, the dynamics of $|\chi_\epsilon\rangle$ is given by

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} + \epsilon^2[\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{B}}(t)] - \epsilon^2\widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t) + \epsilon^3\boldsymbol{E}(\epsilon, t)\right)|\chi_\epsilon\rangle$$

- $\boldsymbol{E}(\epsilon, t)$ is still almost periodic but its entries are no more linear combinations of time-exponentials;
- $\widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t)$ is an almost periodic operator whose entries are linear combinations of oscillating time-exponentials.

We can write

$$\widetilde{\boldsymbol{B}}(t) = \frac{d}{dt}\widetilde{\boldsymbol{C}}(t) \qquad \text{and} \qquad \widetilde{\boldsymbol{B}}(t)\frac{d}{dt}\widetilde{\boldsymbol{B}}(t) = \bar{\boldsymbol{D}} + \frac{d}{dt}\widetilde{\boldsymbol{D}}(t)$$

where $\widetilde{\boldsymbol{C}}(t)$ and $\widetilde{\boldsymbol{D}}(t)$ are almost periodic. We have

$$\frac{d}{dt}|\chi_\epsilon\rangle = \left(\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}} + \epsilon^2\frac{d}{dt}\left([\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}(t)] - \widetilde{\boldsymbol{D}}(t)\right) + \epsilon^3\boldsymbol{E}(\epsilon, t)\right)|\chi_\epsilon\rangle$$

where the skew-Hermitian operators $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{D}}$ are constants and the other ones $\widetilde{\boldsymbol{C}}$, $\widetilde{\boldsymbol{D}}$, and $\boldsymbol{E}$ are almost periodic.

Multi-frequency averaging: second order

Consider the two systems

$$\frac{d}{dt}|\phi_\epsilon\rangle = \epsilon\left(\bar{\boldsymbol{B}} + \frac{d}{dt}\widetilde{\boldsymbol{B}}(t)\right)|\phi_\epsilon\rangle,$$

and

$$\frac{d}{dt}|\phi_\epsilon^{2^{\text{nd}}}\rangle = (\epsilon\bar{\boldsymbol{B}} - \epsilon^2\bar{\boldsymbol{D}})|\phi_\epsilon^{2^{\text{nd}}}\rangle,$$

initialized at the same state $|\phi_\epsilon^{2^{\text{nd}}}\rangle_0 = |\phi_\epsilon\rangle_0$.

Theorem: second order approximation

Consider the functions $|\phi_\epsilon\rangle$ and $|\phi_\epsilon^{2^{\text{nd}}}\rangle$ initialized at the same state and following the above dynamics. Then, there exist $M > 0$ and $\eta > 0$ such that for all $\epsilon \in]0, \eta[$ we have

$$\max_{t\in\left[0,\frac{1}{\epsilon}\right]} \left\| |\phi_\epsilon\rangle_t - |\phi_\epsilon^{2^{\text{nd}}}\rangle_t \right\| \leq M\epsilon^2$$

Multi-frequency averaging: second order

Proof's idea

Another almost periodic change of variables

$$|\xi_\epsilon\rangle = \left(\boldsymbol{I} - \epsilon^2 \left([\bar{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}(t)] - \widetilde{\boldsymbol{D}}(t) \right) \right) |\chi_\epsilon\rangle.$$

The dynamics can be written as

$$\frac{d}{dt}|\xi_\epsilon\rangle = \left(\epsilon \bar{\boldsymbol{B}} - \epsilon^2 \bar{\boldsymbol{D}} + \epsilon^3 \boldsymbol{F}(\epsilon, t) \right) |\xi_\epsilon\rangle$$

where $\epsilon \bar{\boldsymbol{B}} - \epsilon^2 \bar{\boldsymbol{D}}$ is skew Hermitian and $\boldsymbol{F}$ is almost periodic and therefore uniformly bounded in time.

Outline

The Rotating Wave Approximation (RWA) recipes

Schrödinger dynamics $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}(t)|\psi\rangle$, with

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 + \sum_{k=1}^{m} u_k(t)\boldsymbol{H}_k, \qquad u_k(t) = \sum_{j=1}^{r} \boldsymbol{u}_{k,j}e^{i\omega_j t} + \boldsymbol{u}_{k,j}^{*}e^{-i\omega_j t}.$$

The Hamiltonian in interaction frame

$$\boldsymbol{H}_{\text{int}}(t) = \sum_{k,j} \left(\boldsymbol{u}_{k,j}e^{i\omega_j t} + \boldsymbol{u}_{k,j}^{*}e^{-i\omega_j t}\right) e^{i\boldsymbol{H}_0 t}\boldsymbol{H}_k e^{-i\boldsymbol{H}_0 t}$$

We define the first order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt,$$

and the second order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t \left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\right)}$$

Choose the amplitudes $\boldsymbol{u}_{k,j}$ and the frequencies ω_j such that the propagators of $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}}$ or $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ admit simple explicit forms that are used to find $t \mapsto u(t)$ steering $|\psi\rangle$ from one location to another one.

Exercise: resonant control of a qubit

In $i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x}\right)|\psi\rangle$, take a resonant control $u(t) = \boldsymbol{u}e^{i\omega_{eg}t} + \boldsymbol{u}^* e^{-i\omega_{eg}t}$ with $\boldsymbol{u}$ slowly varying complex amplitude $\left|\frac{d}{dt}\boldsymbol{u}\right| \ll \omega_{eg}|\boldsymbol{u}|$. Set $H_0 = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\epsilon H_1 = \frac{u}{2}\boldsymbol{\sigma_x}$

1. Consider $|\psi\rangle = e^{-\frac{i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ and show that $i\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with
$$\boldsymbol{H}_{\text{int}} = \frac{u(t)}{2}e^{i\omega_{eg}t}\,\frac{\overbrace{\boldsymbol{\sigma_x} + i\boldsymbol{\sigma_y}}^{\boldsymbol{\sigma_+} = |e\rangle\langle g|}}{2} + \frac{u(t)}{2}e^{-i\omega_{eg}t}\,\frac{\overbrace{\boldsymbol{\sigma_x} - i\boldsymbol{\sigma_y}}^{\boldsymbol{\sigma_-} = |g\rangle\langle e|}}{2}.$$
2. Show that up to second order terms one has $i\frac{d}{dt}|\phi\rangle = \boldsymbol{H}^{1\text{st}}_{\text{rwa}}|\phi\rangle$ with $\boldsymbol{H}^{1\text{st}}_{\text{rwa}} = \frac{\boldsymbol{u}^*\boldsymbol{\sigma_+} + \boldsymbol{u}\boldsymbol{\sigma_-}}{2}$.
3. Take constant control $\boldsymbol{u} = \Omega_r e^{i\theta}$ for $t \in [0, T]$, $T > 0$. Show that $|\phi\rangle$ is solution of (Σ) : $i\frac{d}{dt}|\phi\rangle = \frac{\Omega_r\left(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y}\right)}{2}|\phi\rangle$.
4. Set $\Theta_r = \frac{\Omega_r}{2}T$. Show that the solution at T of the propagator $\boldsymbol{U}_t \in SU(2)$, $i\frac{d}{dt}\boldsymbol{U} = \frac{\Omega_r\left(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y}\right)}{2}\boldsymbol{U}$, $\boldsymbol{U}_0 = \boldsymbol{I}$ is given by
$$\boldsymbol{U}_T = \cos\Theta_r\boldsymbol{I} - i\sin\Theta_r\left(\cos\theta\boldsymbol{\sigma_x} + \sin\theta\boldsymbol{\sigma_y}\right),$$
5. Take a wave function $|\bar{\phi}\rangle$. Show that exist Ω_r and θ such that $U_T|g\rangle = e^{i\alpha}|\bar{\phi}\rangle$, where α is some global phase.
6. Prove that for any given two wave functions $|\phi_a\rangle$ and $|\phi_b\rangle$ exists a piece-wise constant control $[0, 2T] \ni t \mapsto \boldsymbol{u}(t) \in \mathbb{C}$ such that the solution of (Σ) with $|\phi\rangle_0 = |\phi_a\rangle$ satisfies $|\phi\rangle_T = e^{i\beta}|\phi_b\rangle$ for some global phase β.

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Lecture 6
Chengdu, July 9, 2019

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Schrödinger dynamics $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}(t)|\psi\rangle$, with

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 + \sum_{k=1}^{m} u_k(t)\boldsymbol{H}_k, \qquad u_k(t) = \sum_{j=1}^{r} \boldsymbol{u}_{k,j} e^{i\omega_j t} + \boldsymbol{u}_{k,j}^* e^{-i\omega_j t}.$$

The Hamiltonian in interaction frame

$$\boldsymbol{H}_{\text{int}}(t) = \sum_{k,j} \left(\boldsymbol{u}_{k,j} e^{i\omega_j t} + \boldsymbol{u}_{k,j}^* e^{-i\omega_j t}\right) e^{i\boldsymbol{H}_0 t} \boldsymbol{H}_k e^{-i\boldsymbol{H}_0 t}$$

We define the first order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{H}_{\text{int}}(t)dt,$$

and the second order Hamiltonian

$$\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} - \overline{i\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t \left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\right)}$$

Choose the amplitudes $\boldsymbol{u}_{k,j}$ and the frequencies ω_j such that the propagators of $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}}$ or $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}}$ admit simple explicit forms that are used to find $t \mapsto u(t)$ steering $|\psi\rangle$ from one location to another one.

Outline

Outline

1. Averaging of spin/spring systems
   - The spin/spring model
   - Resonant interaction (Jaynes-Cummings system)
   - Dispersive interaction

2. Exercise: control of the Jaynes-Cummings system

The Schrödinger system

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})\right)|\psi\rangle$$

corresponds to two coupled scalar PDE's:

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega_{eg}}{2}\psi_e + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_g$$
$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_{eg}}{2}\psi_g + \frac{\omega_c}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g - i\frac{\Omega}{\sqrt{2}}\frac{\partial}{\partial x}\psi_e$$

since $\boldsymbol{a} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $|\psi\rangle$ corresponds to $(\psi_e(x,t), \psi_g(x,t))$ where $\psi_e(.,t), \psi_g(.,t) \in L^2(\mathbb{R}, \mathbb{C})$ and $\|\psi_e\|^2 + \|\psi_g\|^2 = 1$.

Resonant case: passage to the interaction frame

In $\frac{\boldsymbol{H}}{\hbar} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})$, $\omega_{eg} = \omega_c = \omega$ with $|\Omega| \ll \omega$. Then $\boldsymbol{H} = \boldsymbol{H}_0 + \epsilon\boldsymbol{H}_1$ where ϵ is a small parameter and

$$\frac{\boldsymbol{H}_0}{\hbar} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)$$

$$\epsilon\frac{\boldsymbol{H}_1}{\hbar} = i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}).$$

$\boldsymbol{H}_{\text{int}}$ is obtained by setting $|\psi\rangle = e^{-i\omega t\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)}e^{\frac{-i\omega t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ in $i\hbar\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$ to get $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with

$$\frac{\boldsymbol{H}_{\text{int}}}{\hbar} = i\frac{\Omega}{2}\left(e^{-i\omega t}\boldsymbol{\sigma_-} + e^{i\omega t}\boldsymbol{\sigma_+}\right)\left(e^{i\omega t}\boldsymbol{a}^\dagger - e^{-i\omega t}\boldsymbol{a}\right)$$

where we used

$$e^{\frac{i\theta}{2}\boldsymbol{\sigma_z}}\ \boldsymbol{\sigma_x}e^{-\frac{i\theta}{2}\boldsymbol{\sigma_z}} = e^{-i\theta}\boldsymbol{\sigma_-} + e^{i\theta}\boldsymbol{\sigma_+}, \quad e^{i\theta\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)}\ \boldsymbol{a}\ e^{-i\theta\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)} = e^{-i\theta}\boldsymbol{a}$$

Resonant spin/spring Hamiltonian and associated PDE

The secular terms in $\boldsymbol{H}_{\text{int}}$ are given by (RWA, first order approximation) $\boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}}/\hbar = i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)$. Since quantum state $|\phi\rangle = e^{+i\omega t\left(\boldsymbol{a}^\dagger\boldsymbol{a}+\frac{\mathbf{I}}{2}\right)} e^{\frac{+i\omega t}{2}\boldsymbol{\sigma_z}}|\psi\rangle$ obeys approximatively to $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}^{1^{\text{st}}}_{\text{rwa}}|\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by

$$i\frac{d}{dt}|\psi\rangle = \left(\frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)\right)|\psi\rangle$$

The Jaynes-Cummings Hamiltonian ($\omega_{\text{eg}} = \omega_c = \omega$) reads:

$$\boldsymbol{H}_{JC}/\hbar = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + i\frac{\Omega}{2}\left(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}\right)$$

The corresponding PDE is :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega}{2}\psi_e + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_e - i\frac{\Omega}{2\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)\psi_g$$

$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega}{2}\psi_g + \frac{\omega}{2}(x^2 - \frac{\partial^2}{\partial x^2})\psi_g + i\frac{\Omega}{2\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)\psi_e$$

Dispersive case: passage to the interaction frame

$$\frac{\boldsymbol{H}}{\hbar} = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a})$$

with $|\Omega| \ll |\omega_{\text{eg}} - \omega_c| \ll \omega_{\text{eg}}, \omega_c$.
Then $\boldsymbol{H} = \boldsymbol{H}_0 + \epsilon\boldsymbol{H}_1$ where ϵ is a small parameter and

$$\frac{\boldsymbol{H}_0}{\hbar} = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right), \quad \epsilon\frac{\boldsymbol{H}_1}{\hbar} = i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}).$$

$\boldsymbol{H}_{\text{int}}$ is obtained by setting $|\psi\rangle = e^{-i\omega_c t\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)} e^{\frac{-i\omega_{\text{eg}}t}{2}\boldsymbol{\sigma_z}}|\phi\rangle$ in $i\hbar\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$ to get $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\text{int}}|\phi\rangle$ with

$$\frac{\boldsymbol{H}_{\text{int}}}{\hbar} = i\frac{\Omega}{2}\left(e^{-i\omega_{\text{eg}}t}\boldsymbol{\sigma_-} + e^{i\omega_{\text{eg}}t}\boldsymbol{\sigma_+}\right)\left(e^{i\omega_c t}\boldsymbol{a}^\dagger - e^{-i\omega_c t}\boldsymbol{a}\right)$$

$$= i\frac{\Omega}{2}\left(e^{i(\omega_c-\omega_{\text{eg}})t}\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - e^{-i(\omega_c-\omega_{\text{eg}})t}\boldsymbol{\sigma_+}\boldsymbol{a} + e^{i(\omega_c+\omega_{\text{eg}})t}\boldsymbol{\sigma_+}\boldsymbol{a}^\dagger - e^{-i(\omega_c+\omega_{\text{eg}})t}\boldsymbol{\sigma_-}\boldsymbol{a}\right)$$

Thus $\boldsymbol{H}_{\text{rwa}}^{1^{\text{st}}} = \overline{\boldsymbol{H}_{\text{int}}} = 0$: no secular term. We have to compute $\boldsymbol{H}_{\text{rwa}}^{2^{\text{nd}}} = \overline{\boldsymbol{H}_{\text{int}}} - i\overline{\left(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}\right)\left(\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}})\right)}$ where $\int_t(\boldsymbol{H}_{\text{int}} - \overline{\boldsymbol{H}_{\text{int}}}/\hbar$ corresponds to

$$\frac{\Omega}{2}\left(\frac{e^{i(\omega_c-\omega_{\text{eg}})t}}{\omega_c-\omega_{\text{eg}}}\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger + \frac{e^{-i(\omega_c-\omega_{\text{eg}})t}}{\omega_c-\omega_{\text{eg}}}\boldsymbol{\sigma_+}\boldsymbol{a} + \frac{e^{i(\omega_c+\omega_{\text{eg}})t}}{\omega_c+\omega_{\text{eg}}}\boldsymbol{\sigma_+}\boldsymbol{a}^\dagger + \frac{e^{-i(\omega_c+\omega_{\text{eg}})t}}{\omega_c+\omega_{\text{eg}}}\boldsymbol{\sigma_-}\boldsymbol{a}\right)$$

Dispersive spin/spring Hamiltonian and associated PDE

The secular terms in $\boldsymbol{H}_{\rm rwa}^{\rm 2nd}$ are

$\frac{\Omega^2}{4(\omega_c-\omega_{\rm eg})}\left(\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+\boldsymbol{a}^\dagger\boldsymbol{a} - \boldsymbol{\sigma}_+\boldsymbol{\sigma}_-\boldsymbol{a}\boldsymbol{a}^\dagger\right) + \frac{\Omega^2}{4(\omega_c+\omega_{\rm eg})}\left(-\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{\sigma}_+\boldsymbol{\sigma}_-\boldsymbol{a}^\dagger\boldsymbol{a}\right)$

Since $|\Omega| \ll |\omega_{\rm eg}-\omega_c| \ll \omega_{\rm eg}, \omega_c$, we have $\frac{\Omega^2}{4(\omega_c+\omega_{\rm eg})} \ll \frac{\Omega^2}{4(\omega_c-\omega_{\rm eg})}$

$$\boldsymbol{H}_{\rm rwa}^{\rm 2nd}/\hbar \approx -\frac{\Omega^2}{4(\omega_c-\omega_{\rm eg})}\left(\boldsymbol{\sigma_z}\left(\boldsymbol{N}+\tfrac{\boldsymbol{I}}{2}\right)+\tfrac{\boldsymbol{I}}{2}\right).$$

Since quantum state $|\phi\rangle = e^{+i\omega_c t\left(\boldsymbol{N}+\frac{\boldsymbol{I}}{2}\right)} e^{\frac{+i\omega_{\rm eg}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle$ obeys approximatively to $i\hbar\frac{d}{dt}|\phi\rangle = \boldsymbol{H}_{\rm rwa}^{\rm 2nd}|\phi\rangle$, the original quantum state $|\psi\rangle$ is governed by $i\frac{d}{dt}|\psi\rangle = \left(\frac{\boldsymbol{H}_{disp}}{\hbar} - \frac{\Omega^2}{8(\omega_c-\omega_{\rm eg})}\right)|\psi\rangle$ with

$$\boldsymbol{H}_{disp}/\hbar = \frac{\omega_{\rm eg}}{2}\boldsymbol{\sigma_z} + \omega_c\left(\boldsymbol{N}+\tfrac{\boldsymbol{I}}{2}\right) - \frac{\chi}{2}\,\boldsymbol{\sigma_z}\left(\boldsymbol{N}+\tfrac{\boldsymbol{I}}{2}\right) \quad \text{and } \chi = \frac{\Omega^2}{2(\omega_c-\omega_{\rm eg})}$$

The corresponding PDE is :

$$i\frac{\partial\psi_e}{\partial t} = +\frac{\omega_{\rm eg}}{2}\psi_e + \frac{1}{2}(\omega_c - \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_e$$
$$i\frac{\partial\psi_g}{\partial t} = -\frac{\omega_{\rm eg}}{2}\psi_g + \frac{1}{2}(\omega_c + \frac{\chi}{2})(x^2 - \frac{\partial^2}{\partial x^2})\psi_g$$

Outline

Exercise: control of the Jaynes-Cummings system

Consider the spin-spring model with $\Omega \ll |\omega|$:

$$\frac{\boldsymbol{H}}{\hbar} = \frac{\omega}{2}\boldsymbol{\sigma_z} + \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + i\frac{\Omega}{2}\boldsymbol{\sigma_x}(\boldsymbol{a}^\dagger - \boldsymbol{a}) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)$$

with a real control input $u(t) \in \mathbb{R}$:

1. Show that with the resonant control $u(t) = \boldsymbol{u}e^{-i\omega t} + \boldsymbol{u}^* e^{+i\omega t}$ with complex amplitude $\boldsymbol{u}$ such that $|\boldsymbol{u}| \ll \omega$, the first order RWA approximation yields to the following dynamics in the interaction frame
$$i\frac{d}{dt}|\psi\rangle = \left(i\frac{\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + \boldsymbol{u}\boldsymbol{a}^\dagger + \boldsymbol{u}^*\boldsymbol{a}\right)|\psi\rangle$$
2. Set $\mathbf{v} \in \mathbb{C}$ solution of $\frac{d}{dt}\mathbf{v} = -i\boldsymbol{u}$ and consider the following change of frame $|\phi\rangle = D_{-\mathbf{v}}|\psi\rangle$ with the displacement operator $D_{-\mathbf{v}} = e^{-\mathbf{v}\boldsymbol{a}^\dagger + \mathbf{v}^*\boldsymbol{a}}$. Show that, up to a global phase change, we have, with $\tilde{\boldsymbol{u}} = i\frac{\Omega}{2}\mathbf{v}$,
$$i\frac{d}{dt}|\phi\rangle = \left(\frac{i\Omega}{2}(\boldsymbol{\sigma_-}\boldsymbol{a}^\dagger - \boldsymbol{\sigma_+}\boldsymbol{a}) + (\tilde{\boldsymbol{u}}\boldsymbol{\sigma_+} + \tilde{\boldsymbol{u}}^*\boldsymbol{\sigma_-})\right)|\phi\rangle$$
3. Take the orthonormal basis $\{|g,n\rangle, |e,n\rangle\}$ with $n \in \mathbb{N}$ being the photon number and where for instance $|g,n\rangle$ stands for the tensor product $|g\rangle \otimes |n\rangle$. Set $|\phi\rangle = \sum_n \phi_{g,n}|g,n\rangle + \phi_{e,n}|e,n\rangle$ with $\phi_{g,n}, \phi_{e,n} \in \mathbb{C}$ depending on t and $\sum_n |\phi_{g,n}|^2 + |\phi_{e,n}|^2 = 1$. Show that, for $n \geq 0$
$$i\frac{d}{dt}\phi_{g,n+1} = i\frac{\Omega}{2}\sqrt{n+1}\phi_{e,n} + \tilde{\boldsymbol{u}}^*\phi_{e,n+1}, \quad i\frac{d}{dt}\phi_{e,n} = -i\frac{\Omega}{2}\sqrt{n+1}\phi_{g,n+1} + \tilde{\boldsymbol{u}}\phi_{g,n}$$
and $i\frac{d}{dt}\phi_{g,0} = \tilde{\boldsymbol{u}}^*\phi_{e,0}$.
4. Assume that $|\phi\rangle_0 = |g,0\rangle$. Construct an open-loop control $[0,T] \ni t \mapsto \tilde{\boldsymbol{u}}(t)$ such that $|\phi\rangle_T \approx |g,1\rangle$ (hint: use an impulse for $t \in [0,\epsilon]$ followed by 0 on $[\epsilon, T]$ with $\epsilon \ll T$ and well chosen T).
5. Generalize the above open-loop control when the goal state $|\phi\rangle_T$ is $|g,n\rangle$ with any arbitrary photon number n.

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Lecture 7
Chengdu, July 10, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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Outline

1. Discrete-time dynamics of the LKB photon box
   - General structure based on three quantum features
   - Dispersive probe qubits
   - Resonant probe qubits
   - Density operator to cope with measurement imperfections

2. Exercise: Markov process including detection errors

Three quantum features emphasized by the LKB photon box [3]

1. Schrödinger ($\hbar = 1$): wave function $|\psi\rangle$ in Hilbert space $\mathcal{H}$,

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1.$$

   Unitary propagator $\boldsymbol{U}$ solution of $\frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}\boldsymbol{U}$ with $\boldsymbol{U}(0) = I$.

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle$ depending on $|\psi\rangle$, just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

---

[3] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \;\middle|\; (c_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity

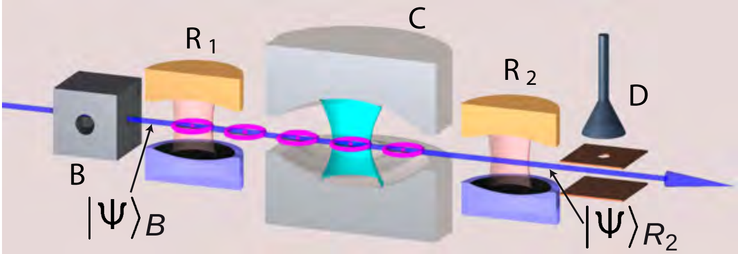
- Meter M is a qubit, a 2-level system: $\mathcal{H}_M = \mathcal{H}_a = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$;

- **State of the full system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M = \mathcal{H}_c \otimes \mathcal{H}_a$:**

$$|\Psi\rangle = \sum_{n=0}^{+\infty} c_{ng}|n\rangle \otimes |g\rangle + c_{ne}|n\rangle \otimes |e\rangle, \quad c_{ne}, c_{ng} \in \mathbb{C}.$$

  Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n\in\mathbb{N}}$.

Markov model (1)



- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \left\langle \psi | \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu | \psi \right\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\left\langle \psi | \boldsymbol{M}_y^\dagger \boldsymbol{M}_y | \psi \right\rangle}} |\psi\rangle \right) \otimes |y\rangle.$$

Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g|\psi_k\rangle}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle}} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle; \\ \frac{\boldsymbol{M}_e|\psi_k\rangle}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle}} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle. \end{cases}$$

Dispersive case

$$\boldsymbol{U}_{R_1} = \left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{|g\rangle - |e\rangle}{\sqrt{2}}\right)\langle e|$$

$$\boldsymbol{U}_{R_2} = \left(\frac{|g\rangle + e^{-i\phi_R}|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{e^{i\phi_R}|g\rangle - |e\rangle}{\sqrt{2}}\right)\langle e|$$

$$\boldsymbol{U}_C = e^{-i\frac{\phi_0}{2}\boldsymbol{N}}\,|g\rangle\langle g| + e^{i\frac{\phi_0}{2}\boldsymbol{N}}\,|e\rangle\langle e|$$

where ϕ_0 and ϕ_R are constant parameters.
The measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ are the following bounded operators:

$$\boldsymbol{M}_g = \cos\left(\tfrac{\phi_R + \phi_0 \boldsymbol{N}}{2}\right), \quad \boldsymbol{M}_e = \sin\left(\tfrac{\phi_R + \phi_0 \boldsymbol{N}}{2}\right)$$

up to irrelevant global phases.
Exercise: prove the above formulae for $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$.

$$\boldsymbol{U}_{R_1} = e^{-i\frac{\theta_1}{2}\boldsymbol{\sigma_y}} = \cos\left(\tfrac{\theta_1}{2}\right) + \sin\left(\tfrac{\theta_1}{2}\right)\left(|g\rangle\langle e| - |e\rangle\langle g|\right) \quad \text{and} \quad \boldsymbol{U}_{R_2} = \boldsymbol{I}$$

and

$$\boldsymbol{U}_C = |g\rangle\langle g| \cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) + |e\rangle\langle e| \cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right) + |g\rangle\langle e| \left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger - |e\rangle\langle g|\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

The measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ are the following bounded operators:

$$\boldsymbol{M}_g = \cos\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right) - \sin\left(\tfrac{\theta_1}{2}\right)\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)\boldsymbol{a}^\dagger$$

$$\boldsymbol{M}_e = -\sin\left(\tfrac{\theta_1}{2}\right)\cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}+1}\right) - \cos\left(\tfrac{\theta_1}{2}\right)\boldsymbol{a}\left(\frac{\sin\left(\frac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

Exercise: Show that $\boldsymbol{M}_g^\dagger\boldsymbol{M}_g + \boldsymbol{M}_e^\dagger\boldsymbol{M}_e = \boldsymbol{I}$.

Markov process with detection inefficiency

- With pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, we have

$$\boldsymbol{\rho}_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger$$

  when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)$.

- Detection efficiency: the probability to detect the atom is $\eta \in [0, 1]$. Three possible outcomes for y: $y = g$ if detection in g, $y = e$ if detection in e and $y = 0$ if no detection.

The only possible update is based on $\boldsymbol{\rho}$: expectation $\boldsymbol{\rho}_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\boldsymbol{\rho}$ and the outcome $y \in \{g, e, 0\}$.

$$\boldsymbol{\rho}_+ = \begin{cases} \frac{\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger}{\text{Tr}(\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g)} & \text{if } y = g, \text{ probability } \eta \, \text{Tr}\left(\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g\right) \\ \frac{\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger}{\text{Tr}(\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e)} & \text{if } y = e, \text{ probability } \eta \, \text{Tr}\left(\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e\right) \\ \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger & \text{if } y = 0, \text{ probability } 1 - \eta \end{cases}$$

For $\eta = 0$: $\boldsymbol{\rho}_+ = \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger = \mathbb{K}(\boldsymbol{\rho}) = \mathbb{E}\left(\boldsymbol{\rho}_+ \mid \boldsymbol{\rho}\right)$ defines a Kraus map.

Several operator spaces

- $\mathcal{H}$ separable Hilbert space. Pure states $|\psi\rangle$ are unitary vectors of $\mathcal{H}$ also called (probability amplitude) wave functions.
- $\mathcal{L}(\mathcal{H})$ is the space of linear operators from $\mathcal{H}$ to $\mathcal{H}$: it contains the spaces of
    - bounded operators (Banach space $\mathcal{B}(\mathcal{H})$ with sup-norm)
    - compact operators (space $\mathcal{K}^c(\mathcal{H})$)
    - Hilbert-Schmidt operators (Hilbert space $\mathcal{K}^2(\mathcal{H})$ with the Frobenius norm)
    - trace class operators (Banach space $\mathcal{K}^1(\mathcal{H})$ with the trace norm).
- the most general quantum state $\boldsymbol{\rho}$ is non negative Hermitian trace class operator of trace one. $\boldsymbol{\rho}$ live in a closed convex subset of $\mathcal{K}^1(\mathcal{H})$.
  If $\text{Tr}\left(\boldsymbol{\rho}^2\right) = 1$ then $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is pure state.

For $\mathcal{H}$ of finite dimension, these operator spaces coincide. For $\mathcal{H}$ of infinite dimension, they are all different:

$$\dim \mathcal{H} = \infty \quad \Rightarrow \quad \mathcal{K}^1(\mathcal{H}) \subsetneq \mathcal{K}^2(\mathcal{H}) \subsetneq \mathcal{K}^c(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H}) \subsetneq \mathcal{L}(\mathcal{H}).$$

- With pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, we have

$$\boldsymbol{\rho}_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu\boldsymbol{\rho}\boldsymbol{M}_\mu^\dagger\right)}\boldsymbol{M}_\mu\boldsymbol{\rho}\boldsymbol{M}_\mu^\dagger$$

when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu\boldsymbol{\rho}\boldsymbol{M}_\mu^\dagger\right)$.

- Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0,1]$ the probability of erroneous assignation to e when the atom collapses in g; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0,1]$ (given by the contrast of the Ramsey fringes).

Bayesian law: expectation $\boldsymbol{\rho}_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\boldsymbol{\rho}$ and the imperfect detection y.

$$\boldsymbol{\rho}_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right)} \text{if } y = g, \text{ prob. Tr}\left((1-\eta_g)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \eta_e\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right)} \text{if } y = e, \text{ prob. Tr}\left(\eta_g\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_e)\boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right). \end{cases}$$

$\boldsymbol{\rho}_+$ does not remain pure: the quantum state $\boldsymbol{\rho}_+$ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.

We get

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger\right)}, & \text{with prob. } \mathrm{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger\right); \\ \frac{\eta_g M_g\rho M_g^\dagger + (1-\eta_e)M_e\rho M_e^\dagger}{\mathrm{Tr}\left(\eta_g M_g\rho M_g^\dagger + (1-\eta_e)M_e\rho M_e^\dagger\right)} & \text{with prob. } \mathrm{Tr}\left(\eta_g M_g\rho M_g^\dagger + (1-\eta_e)M_e\rho M_e^\dagger\right). \end{cases}$$

Key point:

$$\mathrm{Tr}\left((1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger\right) \text{ and } \mathrm{Tr}\left(\eta_g M_g\rho M_g^\dagger + (1-\eta_e)M_e\rho M_e^\dagger\right)$$

are the probabilities to detect $y = g$ and e, knowing ρ.

Reformulation with quantum maps : set

$$\mathbb{K}_g(\rho) = (1-\eta_g)M_g\rho M_g^\dagger + \eta_e M_e\rho M_e^\dagger, \quad \mathbb{K}_e(\rho) = \eta_g M_g\rho M_g^\dagger + (1-\eta_e)M_e\rho M_e^\dagger.$$

$$\rho_+ = \frac{\mathbb{K}_y(\rho)}{\mathrm{Tr}\left(\mathbb{K}_y(\rho)\right)} \quad \text{when we detect } y$$

The probability to detect y knowing ρ is $\mathrm{Tr}\left(\mathbb{K}_y(\rho)\right)$.
We have the following Kraus map:

$$\mathbb{E}\left(\rho_+ \mid \rho\right) = \mathbb{K}_g(\rho) + \mathbb{K}_e(\rho) = \mathbb{K}(\rho) = M_g\rho M_g^\dagger + M_e\rho M_e^\dagger.$$

Exercise: Markov process including detection errors

Consider a set of N bounded operators $\boldsymbol{M}_\mu$ on an Hilbert space $\mathcal{H}$ such that $\sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$. Take the ideal Markov process $\boldsymbol{\rho}_{k+1} = \frac{\boldsymbol{M}_\mu \boldsymbol{\rho}_k \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho}_k \boldsymbol{M}_\mu^\dagger\right)}$ and ideal measurement outcomes $\mu \in \{1, \ldots, N\}$ of probability $\mathrm{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho}_k \boldsymbol{M}_\mu^\dagger\right)$. Assume that the real measurement process provides N_d different values $y \in \{1, \ldots, N_d\}$ correlated to the ideal measurement μ via the following conditional classical probabilities $\mathbb{P}\left(y \mid \mu\right) = \eta_{y,\mu} \in [0, 1]$ where η is a left stochastic matrix ($\sum_y \eta_{y,\mu} = 1$ for each μ).
Denote by $\widehat{\boldsymbol{\rho}}_k$ the expectation value of $\boldsymbol{\rho}_k$ knowing $\boldsymbol{\rho}_0$ and the real measurement outcomes $y_0, \ldots, y_{k-1}$ at steps $0, \ldots, k-1$. Consider the un-normalized ideal quantum state

$$\boldsymbol{\xi}_{\mu_0,\ldots,\mu_k} = \boldsymbol{M}_{\mu_k} \ldots \boldsymbol{M}_{\mu_0} \boldsymbol{\rho}_0 \boldsymbol{M}_{\mu_0}^\dagger \ldots \boldsymbol{M}_{\mu_k}^\dagger$$

associated to the ideal outcomes $\mu_0, \ldots, \mu_k$.

1. Show that $\mathbb{P}\left(\mu_0, \ldots, \mu_k \mid \rho_0\right) = \mathrm{Tr}\left(\boldsymbol{\xi}_{\mu_0,\ldots,\mu_k}\right)$.
2. Using Bayes law, prove that

$$\mathbb{P}\left(y_0, \ldots, y_k \mid \rho_0\right) = \sum_{\mu_k=1}^{N} \cdots \sum_{\mu_0=1}^{N} \eta_{y_0,\mu_0} \ldots \eta_{y_k,\mu_k} \,\mathrm{Tr}\left(\boldsymbol{\xi}_{\mu_0,\ldots,\mu_k}\right).$$

3. Using Bayes law, prove also that

$$\mathbb{P}\left(\mu_0, \ldots, \mu_k \mid y_0, \ldots, y_k, \rho_0\right) = \frac{\eta_{y_0,\mu_0} \ldots \eta_{y_k,\mu_k} \,\mathrm{Tr}\left(\boldsymbol{\xi}_{\mu_0,\ldots,\mu_k}\right)}{\mathbb{P}\left(y_0, \ldots, y_k \mid \rho_0\right)}$$

4. Prove for $\ell = 1, \ldots, k-1$ that $\widehat{\boldsymbol{\rho}}_{\ell+1} = \frac{\sum_{\mu=1}^N \eta_{y_\ell,\mu} \boldsymbol{M}_\mu \widehat{\boldsymbol{\rho}}_\ell \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^N \eta_{y_\ell,\mu} \boldsymbol{M}_\mu \widehat{\boldsymbol{\rho}}_\ell \boldsymbol{M}_\mu^\dagger\right)}$ and that $\mathbb{P}\left(y_\ell \mid y_0, \ldots, y_{\ell-1}, \boldsymbol{\rho}_0\right) = \mathrm{Tr}\left(\sum_{\mu=1}^N \eta_{y_\ell,\mu} \boldsymbol{M}_\mu^\dagger \widehat{\boldsymbol{\rho}}_\ell \boldsymbol{M}_\mu\right)$ (hint: use the un-normalized estimate $\widehat{\boldsymbol{\xi}}_{y_0,\ldots,y_\ell}$ colinear to $\widehat{\boldsymbol{\rho}}_{\ell+1}$).

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Lecture 8
Chengdu, July 10, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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Outline

Outline

1. Quantum measurement and filtering
   - Projective measurement
   - Positive Operator Valued Measurement (POVM)
   - Stochastic process attached to POVM
   - Quantum Filtering
2. Convergence issues with Schrödinger and Heisenberg pictures
3. Exercise: cooling with resonant qubits in $|g\rangle$

Projective measurement

For the system defined on Hilbert space $\mathcal{H}$, take

- an observable $\boldsymbol{O}$ (Hermitian operator) defined on $\mathcal{H}$:

$$\boldsymbol{O} = \sum_{\nu} \lambda_\nu \boldsymbol{P}_\nu,$$

  where λ_ν's are the eigenvalues of $\boldsymbol{O}$ and $\boldsymbol{P}_\nu$ is the projection operator over the associated eigenspace.

- a quantum state given by the wave function $|\psi\rangle$ in $\mathcal{H}$.

Projective measurement of the physical observable $\boldsymbol{O} = \sum_\nu \lambda_\nu \boldsymbol{P}_\nu$ for the quantum state $|\psi\rangle$:

1. The probability of obtaining the value λ_ν is given by $\mathbb{P}_\nu = \langle\psi|\boldsymbol{P}_\nu|\psi\rangle$; note that $\sum_\nu \mathbb{P}_\nu = 1$ as $\sum_\nu \boldsymbol{P}_\nu = \boldsymbol{I}_{\mathcal{H}}$ ($\boldsymbol{I}_{\mathcal{H}}$ represents the identity operator of $\mathcal{H}$).
2. After the measurement, the conditional (a posteriori) state $|\psi_+\rangle$ of the system, given the outcome λ_ν, is

$$|\psi_+\rangle = \frac{\boldsymbol{P}_\nu \, |\psi\rangle}{\sqrt{\mathbb{P}_\nu}} \quad \text{(collapse of the wave packet).}$$

System S of interest (a quantized electromagnetic field) interacts with the meter M (a probe atom), and the experimenter measures projectively the meter M (the probe atom). Need for a **Composite system**: $\mathcal{H}_S \otimes \mathcal{H}_M$ where $\mathcal{H}_S$ and $\mathcal{H}_M$ are Hilbert spaces of S and M. Measurement process in three successive steps:

1. Initially the quantum state is separable

$$\mathcal{H}_S \otimes \mathcal{H}_M \ni |\Psi\rangle = |\psi_S\rangle \otimes |\psi_M\rangle$$

with a well defined and known state $|\psi_M\rangle$ for M.

2. Then a Schrödinger evolution during a small time (unitary operator $\boldsymbol{U}_{S,M}$) of the composite system from $|\psi_S\rangle \otimes |\psi_M\rangle$ and producing $\boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\psi_M\rangle\big)$, entangled in general.

3. Finally a projective measurement of the meter M: $\boldsymbol{O}_M = \boldsymbol{I}_S \otimes \left(\sum_\nu \lambda_\nu \boldsymbol{P}_\nu\right)$ the measured observable for the meter. Projection operator $\boldsymbol{P}_\nu$ is a rank-1 projection in $\mathcal{H}_M$ over the eigenstate $|\xi_\nu\rangle \in \mathcal{H}_M$: $\boldsymbol{P}_\nu = |\xi_\nu\rangle\langle\xi_\nu|$.

Define the measurement operators $\boldsymbol{M}_\nu$ via

$$\forall |\psi_S\rangle \in \mathcal{H}_S, \quad \boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\psi_M\rangle\big) = \sum_\nu \big(\boldsymbol{M}_\nu |\psi_S\rangle\big) \otimes |\xi_\nu\rangle.$$

Then $\sum_\nu \boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu = \boldsymbol{I}_S$. The set $\{\boldsymbol{M}_\nu\}$ defines a Positive Operator Valued Measurement (POVM).
In $\mathcal{H}_S \otimes \mathcal{H}_M$, projective measurement of $\boldsymbol{O}_M = \boldsymbol{I}_S \otimes \big(\sum_\nu \lambda_\nu \boldsymbol{P}_\nu\big)$ with quantum state $\boldsymbol{U}_{S,M}\big(|\psi_S\rangle \otimes |\psi_M\rangle\big)$:

1. The probability of obtaining the value λ_ν is given by $\mathbb{P}_\nu = \langle \psi_S | \boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu | \psi_S \rangle$
2. After the measurement, the conditional (a posteriori) state of the system, given the outcome ν, is

$$|\psi_{S,+}\rangle = \frac{\boldsymbol{M}_\nu |\psi_S\rangle}{\sqrt{\mathbb{P}_\nu}}.$$

Stochastic processes attached to a POVM

- To the POVM $(\boldsymbol{M}_\nu)$ on $\mathcal{H}_S$ is attached a stochastic process of quantum state $|\psi\rangle$

$$|\psi_+\rangle = \frac{\boldsymbol{M}_\nu|\psi\rangle}{\sqrt{\mathbb{P}_\nu}} \text{ with probability } \mathbb{P}_\nu = \langle\psi|\boldsymbol{M}_\nu^\dagger \boldsymbol{M}_\nu|\psi\rangle$$

- For any observable $\boldsymbol{A}$ on $\mathcal{H}_S$, its conditional expectation value after the transition knowing the state $|\psi\rangle$

$$\mathbb{E}\left(\langle\psi_+|\boldsymbol{A}|\psi_+\rangle \;\middle|\; |\psi\rangle\right) = \langle\psi|(\sum_\nu \boldsymbol{M}_\nu^\dagger \boldsymbol{A}\boldsymbol{M}_\nu)|\psi\rangle = \mathrm{Tr}\left(A\,\boldsymbol{K}(|\psi\rangle\langle\psi|)\right)$$

with Kraus map $\boldsymbol{K}(\rho) = \sum_\nu \boldsymbol{M}_\nu \rho \boldsymbol{M}_\nu^\dagger$ with $\rho = |\psi\rangle\langle\psi|$ density operator corresponding to $|\psi\rangle$.

- Imperfection and errors described by left stochastic matrix $(\eta_{y,\nu})$ where $\eta_{y,\nu}$ is the probability of detector outcome y knowing that the ideal detection ν $(\sum_y \eta_{y,\nu} \equiv 1)$. Then Bayes law yields

$$\mathbb{E}\left(\rho_+ \mid \rho, y\right) = \frac{\boldsymbol{K}_y(\rho)}{\mathrm{Tr}\left(\boldsymbol{K}_y(\rho)\right)}$$

with completely positive linear maps $\boldsymbol{K}_y(\rho) = \sum_\nu \eta_{y,\nu}\boldsymbol{M}_\nu \rho \boldsymbol{M}_\nu^\dagger$ depending on y. Probability to detect y knowing ρ is $\mathrm{Tr}\left(\boldsymbol{K}_y(\rho)\right)$.

Stochastic Master Equation (SME) and quantum filtering

Discrete-time models are Markov processes

$$\rho_{k+1} = \frac{\boldsymbol{K}_{y_k}(\rho_k)}{\text{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)}, \text{ with proba. } \mathbb{P}_{y_k}(\rho_k) = \text{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)$$

where each $\boldsymbol{K}_y$ is a linear completely positive map depending on the measurement outcomes. $\boldsymbol{K} = \sum_y \boldsymbol{K}_y$ corresponds to a Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_y \boldsymbol{K}_y(\rho_k).$$

Quantum filtering (Belavkin quantum filters)

data: initial estimation $\hat{\rho}_0$ of the quantum state ρ at step $k = 0$, past measurement outcomes $\boldsymbol{y_l}$ for $l \in \{0, \ldots, k-1\}$;

goal: estimation $\hat{\rho}_k$ of ρ at step k via the recurrence (quantum filter)

$$\hat{\rho}_{l+1} = \frac{\boldsymbol{K}_{\boldsymbol{y_l}}(\hat{\rho}_l)}{\text{Tr}\left(\boldsymbol{K}_{\boldsymbol{y_l}}(\hat{\rho}_l)\right)}, \quad l = 0, \ldots, k-1.$$

stability If the initial estimate $\hat{\rho}_0$ of ρ differs from ρ_0, then $\hat{\rho}_k$, the quantum-filter state at step k tends to converge to ρ_k (the fidelity $F(\rho, \hat{\rho}) \triangleq \text{Tr}\left(\sqrt{\sqrt{\rho}\hat{\rho}\sqrt{\rho}}\right)$ between ρ and $\hat{\rho}$ is a sub-martingale [3]).

[3] PR: Fidelity is a Sub-Martingale for Discrete-Time Quantum Filters. IEEE Transactions on Automatic Control, 2011, 56, 2743-2747.

Outline

- Any open model of quantum system in discrete time is governed by a Markov chain of the form

$$\boldsymbol{\rho}_{k+1} = \frac{\mathbb{K}_{y_k}(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\mathbb{K}_{y_k}(\boldsymbol{\rho}_k)\right)},$$

  with the probability $\mathrm{Tr}\left(\mathbb{K}_{y_k}(\boldsymbol{\rho}_k)\right)$ to have the measurement outcome y_k knowing ρ_{k-1}.

- The structure of the super-operators $\mathbb{K}_y$ is as follows. Each $\mathbb{K}_y$ is a linear completely positive map (a quantum operation, a partial Kraus map[4]) and $\sum_y \mathbb{K}_y(\boldsymbol{\rho}) = \mathbb{K}(\boldsymbol{\rho})$ is a Kraus map, i.e. $\mathbb{K}(\boldsymbol{\rho}) = \sum_\mu \boldsymbol{K}_\mu \boldsymbol{\rho} \boldsymbol{K}_\mu^\dagger$ with $\sum_\mu \boldsymbol{K}_\mu^\dagger \boldsymbol{K}_\mu = \boldsymbol{I}$.

[4] Each $\mathbb{K}_y$ admits the expression

$$\mathbb{K}_y(\boldsymbol{\rho}) = \sum_\mu \boldsymbol{K}_{y,\mu} \boldsymbol{\rho} \boldsymbol{K}_{y,\mu}^\dagger$$

where $(\boldsymbol{K}_{y,\mu})$ are bounded operators on $\mathcal{H}$.

Schrödinger view point of ensemble average dynamics

- Without measurement record, the quantum state ρ_k obeys to the master equation
$$\boldsymbol{\rho}_{k+1} = \mathbb{K}(\boldsymbol{\rho}_k).$$
since $\mathbb{E}\left(\rho_{k+1} \mid \rho_k\right) = \mathbb{K}(\boldsymbol{\rho}_k)$ (ensemble average).

- $\mathbb{K}$ is always a contraction (not strict in general) for the following two such metrics. For any density operators $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ we have
$$\|\mathbb{K}(\boldsymbol{\rho}) - \mathbb{K}(\boldsymbol{\rho}')\|_1 \leq \|\boldsymbol{\rho} - \boldsymbol{\rho}'\|_1 \text{ and } F(\mathbb{K}(\boldsymbol{\rho}), \mathbb{K}(\boldsymbol{\rho}')) \geq F(\boldsymbol{\rho}, \boldsymbol{\rho}')$$
where the trace norm $\| \bullet \|_1$ and fidelity F are given by
$$\|\boldsymbol{\rho} - \boldsymbol{\rho}'\|_1 \triangleq \text{Tr}\left(|\boldsymbol{\rho} - \boldsymbol{\rho}'|\right) \text{ and } F(\boldsymbol{\rho}, \boldsymbol{\rho}') \triangleq \text{Tr}\left(\sqrt{\sqrt{\boldsymbol{\rho}}\boldsymbol{\rho}'\sqrt{\boldsymbol{\rho}}}\right).$$

Properties of the trace distance $D(\rho, \rho') = \mathrm{Tr}\left(|\rho - \rho'|\right)/2$.

1. Unitary invariance: for any unitary operator U ($U^\dagger U = I$), $D\big(U\rho U^\dagger, U\rho' U^\dagger\big) = D(\rho, \rho')$.
2. For any density operators ρ and ρ',

$$D(\rho, \rho') = \max_{\substack{P \text{such that} \\ 0 \le P = P^\dagger \le I}} \mathrm{Tr}\left(P(\rho - \rho')\right).$$

3. Triangular inequality: for any density operators ρ, ρ' and ρ''

$$D(\rho, \rho'') \le D(\rho, \rho') + D(\rho', \rho'').$$

Complement: Kraus maps are contractions for several "distances"[5]

For any Kraus map $\rho \mapsto \boldsymbol{K}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$ ($\sum_\mu M_\mu^\dagger M_\mu = I$) $d(\boldsymbol{K}(\rho), \boldsymbol{K}(\sigma)) \leq d(\rho, \sigma)$ with

- trace distance: $d_{tr}(\rho, \sigma) = \frac{1}{2} \operatorname{Tr}\left(|\rho - \sigma|\right)$.
- Bures distance: $d_B(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ with fidelity $F(\rho, \sigma) = \operatorname{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$.
- Chernoff distance: $d_C(\rho, \sigma) = \sqrt{1 - Q(\rho, \sigma)}$ where $Q(\rho, \sigma) = \min_{0 \leq s \leq 1} \operatorname{Tr}\left(\rho^s \sigma^{1-s}\right)$.
- Relative entropy: $d_S(\rho, \sigma) = \sqrt{\operatorname{Tr}\left(\rho(\log \rho - \log \sigma)\right)}$.
- χ^2-divergence: $d_{\chi^2}(\rho, \sigma) = \sqrt{\operatorname{Tr}\left((\rho - \sigma)\sigma^{-\frac{1}{2}}(\rho - \sigma)\sigma^{-\frac{1}{2}}\right)}$.
- Hilbert's projective metric: if $\operatorname{supp}(\rho) = \operatorname{supp}(\sigma)$ $d_h(\rho, \sigma) = \log\left(\left\|\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}\right\|_\infty \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty\right)$ otherwise $d_h(\rho, \sigma) = +\infty$.

[5] A good summary in M.J. Kastoryano PhD thesis: Quantum Markov Chain Mixing and Dissipative Engineering. University of Copenhagen, December 2011.

The Schrödinger approach $d_h(\rho,\sigma) = \log\left(\left\|\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}\right\|_\infty \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty\right)$

$$\boldsymbol{K}(\rho) = \sum M_\mu \rho M_\mu^\dagger, \quad \sum M_\mu^\dagger M_\mu = I$$

Contraction ratio: $\tanh\left(\frac{\Delta(\boldsymbol{K})}{4}\right)$ with $\Delta(\boldsymbol{K}) = \max_{\rho,\sigma>0} d_h(\boldsymbol{K}(\rho), \boldsymbol{K}(\sigma))$

The Heisenberg approach (dual of Schrödinger approach):

$$\boldsymbol{K}^*(A) = \sum M_\mu^\dagger A M_\mu, \quad \boldsymbol{K}^*(I) = I.$$

"Contraction of the spectrum":

$$\lambda_{min}(A) \leq \lambda_{min}(\boldsymbol{K}^*(A)) \leq \lambda_{max}(\boldsymbol{K}^*(A)) \leq \lambda_{max}(A).$$

[6] R. Sepulchre et al.: Consensus in non-commutative spaces. CDC 2010.

[7] D. Reeb et al.: Hilbert's projective metric in quantum information theory. J. Math. Phys. 52, 082201 (2011).

Heisenberg view point of ensemble average dynamics

- The "Heisenberg description" is given by iterates $\boldsymbol{A}_{k+1} = \mathbb{K}^*(\boldsymbol{A}_k)$ from an initial bounded Hermitian operator $\boldsymbol{A}_0$ of the the dual map $\mathbb{K}^*$ characterized as follows: $\text{Tr}\left(\boldsymbol{A}\mathbb{K}(\boldsymbol{\rho})\right) = \text{Tr}\left(\mathbb{K}^*(\boldsymbol{A})\boldsymbol{\rho}\right)$ for any bounded operator $\boldsymbol{A}$ on $\mathcal{H}$. Thus

$$\mathbb{K}^*(\boldsymbol{A}) = \sum_{\mu} \boldsymbol{K}_{\mu}^{\dagger}\boldsymbol{A}\boldsymbol{K}_{\mu} \quad \text{when } \mathbb{K}(\boldsymbol{\rho}) = \sum_{\mu} \boldsymbol{K}_{\mu}\boldsymbol{\rho}\boldsymbol{K}_{\mu}^{\dagger}.$$

  $\mathbb{K}^*$ is an unital map, i.e., $\mathbb{K}^*(\boldsymbol{I}) = \boldsymbol{I}$, and the image via $\mathbb{K}^*$ of any bounded operator is a bounded operator.

- When $\mathcal{H}$ is of finite dimension, we have, for any Hermitian operator $\boldsymbol{A}$:

$$\lambda_{min}(\boldsymbol{A}) \leq \lambda_{min}(\mathbb{K}^*(\boldsymbol{A})) \leq \lambda_{max}(\mathbb{K}^*(\boldsymbol{A})) \leq \lambda_{max}(\boldsymbol{A})$$

  where λ_{min} and λ_{max} correspond to the smallest and largest eigenvalues[8].

- If $\overline{\boldsymbol{A}} = \mathbb{K}^*(\overline{\boldsymbol{A}})$, then $\text{Tr}\left(\boldsymbol{\rho}_k\overline{\boldsymbol{A}}\right) = \text{Tr}\left(\boldsymbol{\rho}_0\overline{\boldsymbol{A}}\right)$ is a constant of motion of $\boldsymbol{\rho}$.

[8] R. Sepulchre et al.: Consensus in non-commutative spaces. Decision and Control (CDC), 2010 49th IEEE Conference on,2010, 6596-6601.

Convergence in Schrödinger and Heisenberg pictures

Take a Kraus map $\mathbb{K}$ and its adjoint unital map $\mathbb{K}^*$. When $\mathcal{H}$ is of finite dimension, the following two statements are equivalent :

- Global convergence towards the fixed point $\overline{\boldsymbol{\rho}} = \mathbb{K}(\overline{\boldsymbol{\rho}})$ of $\boldsymbol{\rho}_{k+1} = \mathbb{K}(\boldsymbol{\rho}_k)$: for any initial density operator $\boldsymbol{\rho}_0$, $\lim_{k \mapsto +\infty} \boldsymbol{\rho}_k = \overline{\boldsymbol{\rho}}$ for the trace norm $\| \bullet \|_1$.
- Global convergence of $\boldsymbol{A}_{k+1} = \mathbb{K}^*(\boldsymbol{A}_k)$: there exists a unique density operator $\overline{\boldsymbol{\rho}}$ such that, for any initial bounded operator $\boldsymbol{A}_0$, $\lim_{k \mapsto +\infty} A_k = \operatorname{Tr}(A_0 \overline{\boldsymbol{\rho}}) \boldsymbol{I}$ for the sup norm on the bounded operators on $\mathcal{H}$.

Exercise: cooling with resonant qubits in $|g\rangle$.

Consider the quantum channel $\rho_{k+1} = \mathbb{K}(\rho_k) \triangleq \boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger$ with Kraus operators given by

$$\boldsymbol{M}_g = \cos\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right), \quad \boldsymbol{M}_e = \boldsymbol{a}\left(\frac{\sin\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\right)$$

where $\boldsymbol{a}$ is the annihilation operator, $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$ and $\Theta > 0$ is a parameter. Take the Fock basis $(|n\rangle)_{n\in\mathbb{N}}$. The density operator ρ is said to be supported in the subspace $\{|n\rangle\}_{n=0}^{n^{\max}}$ when, for all $n > n^{\max}$, $\rho|n\rangle = 0$.

1. Verify that $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.
2. Show that
$$\operatorname{Tr}\left(\boldsymbol{N}\rho_{k+1}\right) = \operatorname{Tr}\left(\boldsymbol{N}\rho_k\right) - \operatorname{Tr}\left(\sin^2\left(\tfrac{\Theta}{2}\sqrt{\boldsymbol{N}}\right)\rho_k\right).$$
3. Assume that for any integer $0 < n \leq n^{\max}$, $\Theta\sqrt{n}/\pi$ is not an integer. Then prove that ρ_k tends to the vacuum state $|0\rangle\langle 0|$ whatever its initial condition with support in $\{|n\rangle\}_{n=0}^{n^{\max}}$.
4. When $\Theta\sqrt{n}/\pi$ is an integer for some $0 < \bar{n} \leq n^{\max}$, describe the possible Ω-limit sets for ρ_k for any initial condition ρ_0 with support in $\{|n\rangle\}_{n=0}^{n^{\max}}$.

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Lecture 9
Chengdu, July 10, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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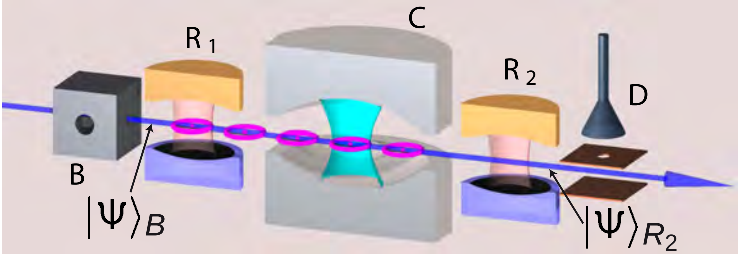
Outline

Outline



Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g|\psi_k\rangle}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle}} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\right\rangle; \\ \frac{\boldsymbol{M}_e|\psi_k\rangle}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle}} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\right\rangle, \end{cases}$$

with

$$\boldsymbol{M}_g = \cos\left(\tfrac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right), \quad \boldsymbol{M}_e = \sin\left(\tfrac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right).$$

QND measurement of photons

Markov process: density operator $\rho_k = |\psi_k\rangle\langle\psi_k|$ as state.

$$\rho_{k+1} = \begin{cases} \frac{\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right), \end{cases}$$

with

$$\boldsymbol{M}_g = \cos\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right), \quad \boldsymbol{M}_e = \sin\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right).$$

Quantum Monte Carlo simulations:

Matlab script: `IdealModelPhotonBox.m`

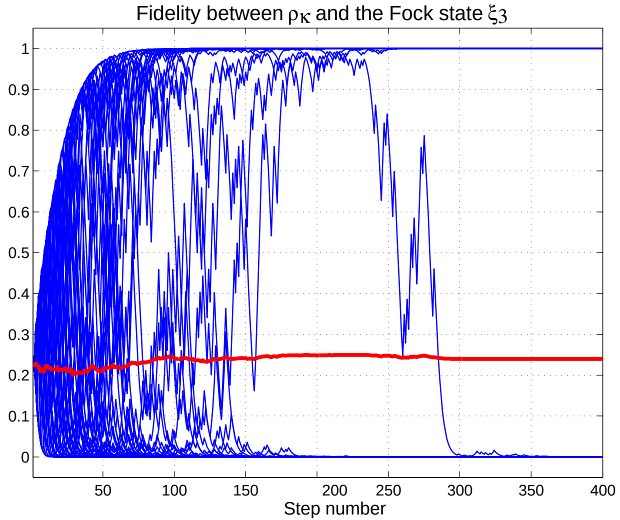
Experimental data

Quantum Non-Demolition (QND) measurement

The measurement operators $\boldsymbol{M}_{g,e}$ commute with the photon-number observable $\boldsymbol{N}$: photon-number states $|n\rangle\langle n|$ are fixed points of the measurement process. We say that the measurement is QND for the observable $\boldsymbol{N}$.

Asymptotic behavior: numerical simulations

100 Monte-Carlo simulations of $\mathrm{Tr}\left(\rho_k |3\rangle\langle 3|\right)$ versus k



Convergence of a random process

Consider (X_k) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a metric space $\mathcal{X}$. The random process X_k is said to,

1. converge in probability towards the random variable X if for all $\epsilon > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left(|X_k - X| > \epsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\omega \in \Omega \mid |X_k(\omega) - X(\omega)| > \epsilon\right) = 0;$$

   [Deterministic analogue with measurable real-valued functions $X(\omega)$ and $X_k(\omega)$ of $\omega \in \Omega \equiv \mathbb{R}$ and $p(\omega) \geq 0$ a probability density versus the Lebesgue measure $d\omega$ ($\int_{\mathbb{R}} p(\omega)d\omega = 1$): $\lim_{k\mapsto+\infty} \int_{\mathbb{R}} I_\epsilon(|X_k(\omega) - X(\omega)|)p(\omega)d\omega = 0$ with $I_\epsilon(x) = 1$ (resp. 0) for $|x| > \epsilon$ (resp. $|x| \leq \epsilon$).]

2. converge almost surely towards the random variable X if

$$\mathbb{P}\left(\lim_{k\to\infty} X_k = X\right) = \mathbb{P}\left(\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = X(\omega)\right) = 1;$$

   [$\forall\omega \in \mathbb{R}/W$ with $W \subset \mathbb{R}$ of zero measure ($\int_W p(\omega)d\omega = 0$), we have $\lim_{k\mapsto+\infty} X_k(\omega) = X(\omega)$.]

3. converge in mean towards the random variable X if $\lim_{k\to\infty} \mathbb{E}\left(|X_k - X|\right) = 0$.
   [$\lim_{k\mapsto+\infty} \int_{\mathbb{R}} |X_k(\omega) - X(\omega)|p(\omega)d\omega = 0$]

Some definitions

Markov process

The sequence $(X_k)_{k=1}^{\infty}$ is called a Markov process, if for all k and ℓ satisfying $k > \ell$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$,

$$\mathbb{E}\left(f(X_k) \mid X_1, \ldots, X_\ell\right) = \mathbb{E}\left(f(X_k) \mid X_\ell\right).$$

Martingales

The sequence $(X_k)_{k=1}^{\infty}$ is called respectively a *supermartingale*, a *submartingale* or a martingale, if $\mathbb{E}\left(|X_k|\right) < \infty$ for $k = 1, 2, \cdots$, and

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) \leq X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \geq \ell$$

or

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) \geq X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \geq \ell,$$

or finally,

$$\mathbb{E}\left(X_k \mid X_1, \ldots, X_\ell\right) = X_\ell \qquad (\mathbb{P} \text{ almost surely}), \qquad k \geq \ell.$$

Martingales asymptotic behavior

H.J. Kushner invariance Theorem

Let $\{X_k\}$ be a Markov chain on the compact state space S. Suppose that there exists a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_{k+1}) \mid X_k = x\right) - V(x) = -\sigma(x)$, where $\sigma(x) \geq 0$ is a positive continuous function of x. Then the ω-limit set (in the sense of almost sure convergence) of X_k is included in the following set

$$I = \{X \mid \sigma(X) = 0\}.$$

Trivially, the same result holds true for the case where $\mathbb{E}\left(V(X_{k+1}) \mid X_k = x\right) - V(x) = \sigma(x)$ with $\sigma(x) \geq 0$ and $V(x)$ bounded from above ($V(X_k)$ is a submartingale),.

Stochastic version of Lasalle invariance principle for Lyapunov function of deterministic dynamics.

Asymptotic behavior

Theorem

Consider for $\boldsymbol{M}_g = \cos\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)$ and $\boldsymbol{M}_e = \sin\left(\frac{\phi_0 \boldsymbol{N}+\phi_R}{2}\right)$

$$\rho_{k+1} = \begin{cases} \frac{\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right)} & \text{with } y_k = g, \text{ probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho_k \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e, \text{ probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho_k \boldsymbol{M}_e^\dagger\right), \end{cases}$$

with an initial density matrix ρ_0 defined on the subspace $\mathrm{span}\{|n\rangle \mid n = 0, 1, \cdots, n^{\max}\}$. Also, assume the non-degeneracy assumption $\forall n \neq m \in \{0, 1, \cdots, n^{\max}\}$, $\cos^2(\varphi_m) \neq \cos^2(\varphi_n)$ where $\varphi_n = \frac{\phi_0 n+\phi_R}{2}$.

Then

- for any $n \in \{0, \ldots, n^{\max}\}$, $\mathrm{Tr}\left(\rho_k |n\rangle\langle n|\right) = \langle n|\rho_k|n\rangle$ is a martingale
- ρ_k converges with probability 1 to one of the $n^{\max}+1$ Fock state $|n\rangle\langle n|$ with $n \in \{0, \ldots, n^{\max}\}$.
- the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $\mathrm{Tr}\left(\rho_0 |n\rangle\langle n|\right) = \langle n|\rho_0|n\rangle$.

Proof based on QND super-martingales

- For any function f, $V_f(\rho) = \mathrm{Tr}\left(f(\boldsymbol{N})\rho\right)$ is a martingale: $\mathbb{E}\left(V_f(\rho_{k+1}) \mid \rho_k\right) = V_f(\rho_k)$.
- $V(\rho) = \sum_{n \neq m} \sqrt{\langle n|\rho|n\rangle \langle m|\rho|m\rangle}$ is a strict super-martingale:

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) = \sum_{n \neq m} \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right) \sqrt{\langle n|\rho_k|n\rangle \langle m|\rho_k|m\rangle} \leq rV(\rho_k)$$

  with $r = \max_{n \neq m} \left(|\cos\phi_n \cos\phi_m| + |\sin\phi_n \sin\phi_m|\right)$ and $r < 1$.
- $V(\rho) \geq 0$ and $V(\rho) = 0$ means that exists n such that $\rho = |n\rangle\langle n|$.

Interpretation: for large k, $V(\rho_k)$ is very close to 0, thus very close to $|n\rangle\langle n|$ ("pure state" = maximal information state) for an a priori random n.
Information extracted by measurement makes state "less uncertain" *a posteriori* but not more predictable *a priori*.

Exercise: QND measurement of photons

We consider QND measurement of photons: detection $y \in \{e, g\}$ and Kraus operators

$$\boldsymbol{M}_g = \cos(\tfrac{\phi_0}{2}\boldsymbol{N}), \quad \boldsymbol{M}_e = \sin(\tfrac{\phi_0}{2}\boldsymbol{N})$$

with ϕ_0 parameter.

1. Take $\boldsymbol{\rho}_{k+1} = \frac{\boldsymbol{M}_{y_k}\boldsymbol{\rho}_k\boldsymbol{M}_{y_k}^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_{y_k}\boldsymbol{\rho}_k\boldsymbol{M}_{y_k}^\dagger\right)}$ with $y_k \in \{g, e\}$ of probability $\mathrm{Tr}\left(\boldsymbol{M}_{y_k}\boldsymbol{\rho}_k\boldsymbol{M}_{y_k}^\dagger\right)$.
    1. Take $\phi_0 = \pi/4$ and assume that $\boldsymbol{\rho}_0|n\rangle = 0$ for $n > 4$. Prove the almost sure convergence towards one of the Fock state $|n\rangle$, for $n \leq 4$.
    2. More generally, under which condition on ϕ_0 do we have, for any $\boldsymbol{\rho}$ such that $\boldsymbol{\rho}_0|n\rangle = 0$ for $n > n^{\max}$, almost sure convergence towards one of the Fock state $|n\rangle$, for $n \leq n^{\max}$.
    3. Take $n^{\max} = 4$ photons and $\phi_0 = \pi/4$. Write a computer program (e.g. a Scilab or Matlab script) to simulate over 100 sampling steps the Markov process starting from $\rho_0 = \frac{1}{5}\sum_{n=0}^{n^{\max}} |n\rangle\langle n|$. Check via the statistics over 1000 realizations that the probability to converge to $|n\rangle\langle n|$ is close to $1/5$ for $n \in \{0, 1, 2, 3, 4\}$.
2. Re-consider the above three questions with the Markov process

$$\boldsymbol{\rho}_{k+1} = \begin{cases} \frac{(1-\eta)\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger + \eta\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta)\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger + \eta\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right)}, & \text{with } y_k = g \text{ of probability } \mathrm{Tr}\left((1-\eta)\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger + \eta\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger + (1-\eta)\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\eta\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger + (1-\eta)\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right)} & \text{with } y_k = e \text{ of probability } \mathrm{Tr}\left(\eta\boldsymbol{M}_g\boldsymbol{\rho}_k\boldsymbol{M}_g^\dagger + (1-\eta)\boldsymbol{M}_e\boldsymbol{\rho}_k\boldsymbol{M}_e^\dagger\right). \end{cases}$$

including a symmetric detection error rate $\eta = 1/10$.

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Lecture 10
Chengdu, July 10, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
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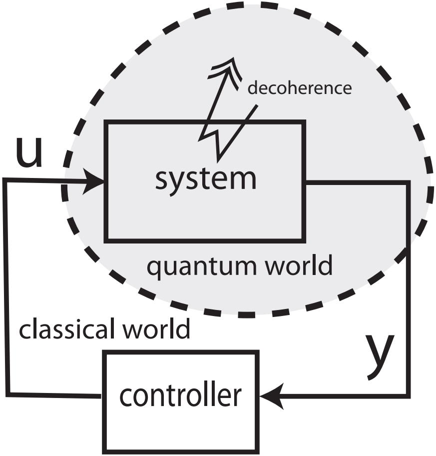
Outline

Outline



Measurement-based feedback: **controller is classical**; measurement back-action on the system $\mathcal{S}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.

Nonlinear hidden-state stochastic systems: convergence analysis, Lyapunov exponents, dynamic output feedback, delays, robustness, ...

Short sampling times limit feedback complexity

Question: how to stabilize deterministically a single photon-number state $|\bar{n}\rangle\langle\bar{n}|$?

Markov chain with classical control input u:

$$\rho_{k+1} = \begin{cases} \frac{M_{g,u_k}\rho_k M_{g,u_k}^\dagger}{\mathrm{Tr}\left(M_{g,u_k}\rho_k M_{g,u_k}^\dagger\right)} & \text{if } y_k = g, \text{ probability } \mathrm{Tr}\left(M_{g,u_k}\rho_k M_{g,u_k}^\dagger\right) \\ \frac{M_{e,u_k}\rho_k M_{e,u_k}^\dagger}{\mathrm{Tr}\left(M_{e,u_k}\rho_k M_{e,u_k}^\dagger\right)} & \text{if } y_k = e, \text{ probability } \mathrm{Tr}\left(M_{e,u_k}\rho_k M_{e,u_k}^\dagger\right) \end{cases}$$

where the Kraus operators depend on the control input u [3] $(\phi_0, \phi_R, \theta_0)$ constant parameters.

dispersive interaction for $u = 0$:

$$M_{g,0} = \cos\left(\frac{\phi_0 N + \phi_R}{2}\right) \text{ and } M_{e,0} = \sin\left(\frac{\phi_0 N + \phi_R}{2}\right),$$

resonant interaction with atom prepared in $|e\rangle$ for $u = 1$:

$$M_{g,1} = \frac{\sin\left(\frac{\theta_0}{2}\sqrt{N}\right)}{\sqrt{N}} a^\dagger \text{ and } M_{e,1} = \cos\left(\frac{\theta_0}{2}\sqrt{N+I}\right)$$

resonant interaction with atom prepared in $|g\rangle$ for $u =$ -1:

$$M_{g,\text{-}1} = \cos\left(\frac{\theta_0}{2}\sqrt{N}\right) \text{ and } M_{e,\text{-}1} = -a\frac{\sin\left(\frac{\theta_0}{2}\sqrt{N}\right)}{\sqrt{N}}$$

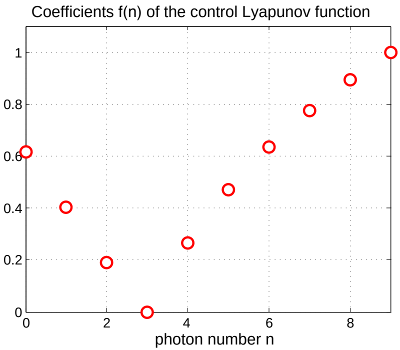
[3] Zhou, X.; Dotsenko, I.; Peaudecerf, B.; Rybarczyk, T.; Sayrin, C.; S. Gleyzes, J. R.; Brune, M.; Haroche, S. Field locked to Fock state by quantum feedback with single photon corrections. Physical Review Letter, 2012, 108, 243602.

Lyapunov function and quantum-state feedback

Idea: open-loop martingale

$$V(\rho) = \mathrm{Tr}\left(\rho f(\boldsymbol{N})\right)$$

with $f : [0, +\infty[\mapsto [0, +\infty[$ strictly decreasing on $[0, \bar{n}]$, strictly increasing on $[\bar{n}, +\infty[$ and $f(\bar{n}) = 0$ as candidate of closed-loop super-martingale with u_k function of ρ_k.



$$u_k = \Gamma(\boldsymbol{\rho}_k) := \underset{u\in\{-1,0,1\}}{\mathrm{argmin}} \left\{ \mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k, u_k = u \right) \right\}$$

$$= \underset{u\in\{-1,0,1\}}{\mathrm{argmin}} \left\{ \mathrm{Tr}\left(\left(\boldsymbol{M}_{g,u}\rho_k \boldsymbol{M}_{g,u}^\dagger + \boldsymbol{M}_{e,u}\rho_k \boldsymbol{M}_{e,u}^\dagger \right) f(\boldsymbol{N}) \right) \right\}$$

Closed-loop simulations `IdealFeedbackPhotonBox.m`: truncation to $n^{\max} = 7$ photons of the Hilbert space, $\bar{n} = 3$, $f(n) = (n - \bar{n})^2$, $\phi_0 = \pi/7$, $\phi_R = 0$, $\theta_0 = \frac{2\pi}{\sqrt{n^{\max}+1}}$.

Cavity decoherence: cavity decay, thermal photon(s)

Three possible outcomes:

- zero photon annihilation during ΔT: Kraus operator $\boldsymbol{M}_0 = \boldsymbol{I} - \frac{\Delta T}{2}\boldsymbol{L}_{-1}^\dagger \boldsymbol{L}_{-1} - \frac{\Delta T}{2}\boldsymbol{L}_1^\dagger \boldsymbol{L}_1$, probability $\approx \text{Tr}\left(\boldsymbol{M}_0 \boldsymbol{\rho} \boldsymbol{M}_0^\dagger\right)$ with back action $\boldsymbol{\rho}_{t+\Delta T} \approx \frac{\boldsymbol{M}_0 \boldsymbol{\rho}_t \boldsymbol{M}_0^\dagger}{\text{Tr}\left(\boldsymbol{M}_0 \boldsymbol{\rho} \boldsymbol{M}_0^\dagger\right)}$.
- one photon annihilation during ΔT: Kraus operator $\boldsymbol{M}_{-1} = \sqrt{\Delta T}\boldsymbol{L}_{-1}$, probability $\approx \text{Tr}\left(\boldsymbol{M}_{-1} \boldsymbol{\rho} \boldsymbol{M}_{-1}^\dagger\right)$ with back action $\boldsymbol{\rho}_{t+\Delta T} \approx \frac{\boldsymbol{M}_{-1} \boldsymbol{\rho}_t \boldsymbol{M}_{-1}^\dagger}{\text{Tr}\left(\boldsymbol{M}_{-1} \boldsymbol{\rho} \boldsymbol{M}_{-1}^\dagger\right)}$
- one photon creation during ΔT: Kraus operator $\boldsymbol{M}_1 = \sqrt{\Delta T}\boldsymbol{L}_1$, probability $\approx \text{Tr}\left(\boldsymbol{M}_1 \boldsymbol{\rho} \boldsymbol{M}_1^\dagger\right)$ with back action $\boldsymbol{\rho}_{t+\Delta T} \approx \frac{\boldsymbol{M}_1 \boldsymbol{\rho}_t \boldsymbol{M}_1^\dagger}{\text{Tr}\left(\boldsymbol{M}_1 \boldsymbol{\rho} \boldsymbol{M}_1^\dagger\right)}$

where

$$\boldsymbol{L}_{-1} = \sqrt{\tfrac{1+n_{th}}{T_{cav}}}\boldsymbol{a}, \quad \boldsymbol{L}_1 = \sqrt{\tfrac{n_{th}}{T_{cav}}}\boldsymbol{a}^\dagger$$

are the Lindbald operators associated to cavity decoherence : T_{cav} the photon life time, $\Delta T \ll T_{cav}$ the sampling period and n_{th} is the average of thermal photon(s) (vanishes with the environment temperature) ($\frac{\Delta T}{T_{cav}} \approx 5 \times 10^{-4}$, $n_{th} \approx 0.05$ for the LKB photon box).

LKB photon-box: controlled Markov process with errors and decoherence

Transition model with control u_k from $\boldsymbol{\rho}_k$ to $\boldsymbol{\rho}_{k+1}$ via $\boldsymbol{\rho}_{k+\frac{1}{2}}$: measurement back-action ($\eta \in [0,1]$ detection error probability and $\eta_{eff} \in [0,1]$ detection efficiency)

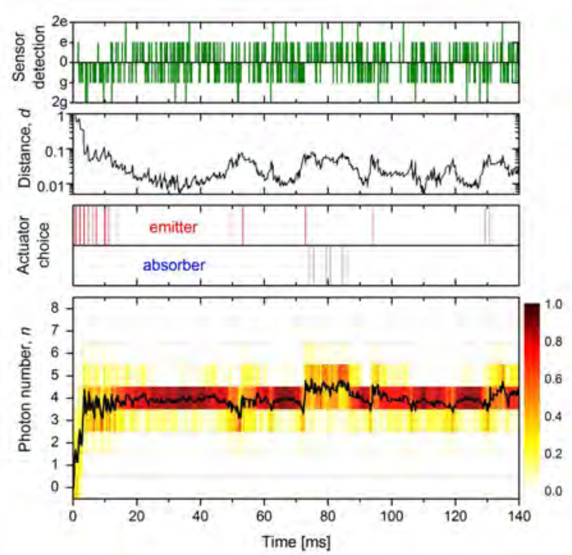
$$\boldsymbol{\rho}_{k+\frac{1}{2}} = \begin{cases} \frac{(1-\eta)\boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + \eta\boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger}{\mathrm{Tr}\left((1-\eta)\boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + \eta\boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger\right)}, & \text{prob. } \eta_{eff}\,\mathrm{Tr}\left((1-\eta)\boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + \eta\boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger\right); \\ \frac{\eta\boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + (1-\eta)\boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger}{\mathrm{Tr}\left(\eta\boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + (1-\eta)\boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger\right)} & \text{prob. } \eta_{eff}\,\mathrm{Tr}\left(\eta\boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + (1-\eta)\boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger\right) \\ \boldsymbol{M}_{g,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{g,u_k}^\dagger + \boldsymbol{M}_{e,u_k}\boldsymbol{\rho}_k\boldsymbol{M}_{e,u_k}^\dagger & \text{prob. } (1-\eta_{eff}) \end{cases}$$

is completed by cavity decoherence during the small sampling time ΔT:

$$\boldsymbol{\rho}_{k+1} = \boldsymbol{M}_{\text{-}1}\boldsymbol{\rho}_{k+\frac{1}{2}}\boldsymbol{M}_{\text{-}1}^\dagger + \boldsymbol{M}_0\boldsymbol{\rho}_{k+\frac{1}{2}}\boldsymbol{M}_0^\dagger + \boldsymbol{M}_1\boldsymbol{\rho}_{k+\frac{1}{2}}\boldsymbol{M}_1^\dagger.$$

Model used in simulation to test the robustness of the Lyapunov feedback $u_k = \Gamma(\boldsymbol{\rho}_k)$ with $\eta = 1/10$, $\eta_{eff} = 4/10$, $\frac{\Delta T}{T_{cav}} \approx 5 \times 10^{-4}$ and $n_{th} \approx 0.05$

Closed-loop experimental results



Zhou et al. Field locked to Fock state by quantum feedback with single photon corrections. Physical Review Letter, 2012, 108, 243602.

See the closed-loop quantum Monte Carlo simulations of the Matlab script: `RealisticFeedbackPhotonBox.m`.

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Lecture 11
Chengdu, July 11, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
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Outline

Discrete-time Stochastic Master Equations (SME)

Trace preserving Kraus map $\boldsymbol{K}_u$ depending on the classical control input u:

$$\boldsymbol{K}_u(\boldsymbol{\rho}) = \sum_{\mu} \boldsymbol{M}_{u,\mu} \boldsymbol{\rho} \boldsymbol{M}_{u,\mu}^{\dagger} \quad \text{with} \quad \sum_{\mu} \boldsymbol{M}_{u,\mu}^{\dagger} \boldsymbol{M}_{u,\mu} = \boldsymbol{I}.$$

Take a left stochastic matrix $\left[\eta_{y,\mu}\right]$ ($\eta_{y,\mu} \geq 0$ and $\sum_y \eta_{y,\mu} \equiv 1, \ \forall \mu$) and set $\boldsymbol{K}_{u,y}(\boldsymbol{\rho}) = \sum_{\mu} \eta_{y,\mu} \boldsymbol{M}_{u,\mu} \boldsymbol{\rho} \boldsymbol{M}_{u,\mu}^{\dagger}$. The associated Markov chain reads:

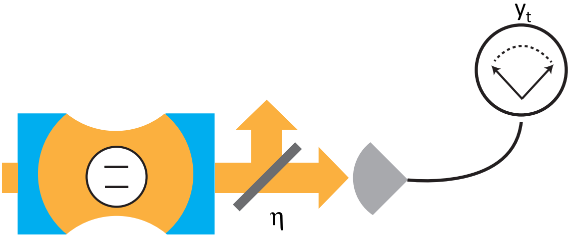
$$\boldsymbol{\rho}_{k+1} = \frac{\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)}{\operatorname{Tr}\left(\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)\right)} \quad \text{measurement } y_k \text{ with probability } \operatorname{Tr}\left(\boldsymbol{K}_{u_k,y_k}(\boldsymbol{\rho}_k)\right).$$

Classical input u, hidden state ρ, measured output y.
Ensemble average given by $\boldsymbol{K}_u$ since $\mathbb{E}\left(\boldsymbol{\rho}_{k+1} \,\middle|\, \boldsymbol{\rho}_k, u_k\right) = \boldsymbol{K}_{u_k}(\boldsymbol{\rho}_k)$.
Markov model useful for:

1. Monte-Carlo simulations of quantum trajectories (decoherence, measurement back-action).
2. quantum filtering to get the quantum state ρ_k from ρ_0 and $(y_0, \ldots, y_{k-1})$ (Belavkin quantum filter developed for diffusive models).
3. feedback design and Monte-Carlo closed-loop simulations.

Outline

Markov process under continuous measurement



Inverse setup of photon-box: photons read out a qubit.

Two major differences

- measurement output taking values from a continuum of possible outcomes
$$dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right)dt + dW_t.$$
- Time continuous dynamics.

Stochastic master equation: Markov process under continuous measurement

$$d\boldsymbol{\rho}_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) \right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t \right) \boldsymbol{\rho}_t \right) dW_{\nu,t},$$

where $W_{\nu,t}$ are independent Wiener processes, associated to measured signals

$$dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_\nu}\, \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t \right) dt.$$

Wiener process W_t:

- $W_0 = 0$;
- $t \to W_t$ is almost surely everywhere continuous;
- For $0 \leq s_1 < t_1 \leq s_2 < t_2$, $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables satisfying $W_t - W_s \sim N(0, t-s)$.

Average dynamics: Lindblad master equation

$$d\mathbb{E}\left(\boldsymbol{\rho}_t\right) =$$
$$\left(-\frac{i}{\hbar}[\boldsymbol{H}, \mathbb{E}\left(\boldsymbol{\rho}_t\right)] + \sum_\nu \boldsymbol{L}_\nu \mathbb{E}\left(\boldsymbol{\rho}_t\right) \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \mathbb{E}\left(\boldsymbol{\rho}_t\right) + \mathbb{E}\left(\boldsymbol{\rho}_t\right) \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) \right) dt.$$

Itō stochastic calculus

Given a SDE

$$dX_t = F(X_t, t)dt + \sum_\nu G_\nu(X_t, t)dW_{\nu,t},$$

we have the following chain rule:

Itō's rule

Defining $f_t = f(X_t)$ a C^2 function of X, we have

$$df_t = \left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right) dt \\ + \sum_\nu \frac{\partial f}{\partial X}\Big|_{X_t} G_\nu(X_t, t)dW_{\nu,t}.$$

Furthermore

$$\frac{d}{dt}\mathbb{E}(f_t) = \mathbb{E}\left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right).$$

Link to partial Kraus maps (1)

$$d\rho_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) \rho_t\right) dW_{\nu,t},$$

equivalent to

$$\rho_{t+dt} = \frac{\boldsymbol{M}_{dy_t} \rho_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t} \rho_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger dt\right)}$$

with

$$\boldsymbol{M}_{dy_t} = \boldsymbol{I} + (-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \boldsymbol{L}_\nu.$$

Moreover, defining $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$:

$$\mathbb{P}(s_t \in \prod_\nu [s_\nu, s_\nu + ds_\nu] \mid \rho_t) = \left(\mathrm{Tr}\left(\boldsymbol{M}_{s\sqrt{dt}} \rho_t \boldsymbol{M}_{s\sqrt{dt}}^\dagger\right) + \sum_\nu (1-\eta_\nu)\,\mathrm{Tr}\left(\boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger\right) dt\right) \prod_\nu \frac{e^{-\frac{|s_\nu|^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

Link to partial Kraus maps (2)

- $\mathbb{P}$ defines a probability density up to a correction of order dt^2:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbb{P}\left(s_t \in \prod_\nu [s_\nu, s_\nu + ds_\nu] \mid \boldsymbol{\rho}_t\right) \prod_\nu ds_\nu = 1 + O(dt^2).$$

- Mean value of measured signal

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} s_\nu \mathbb{P}\left(s_t \in \prod_\nu [s_\nu, s_\nu + ds_\nu] \mid \boldsymbol{\rho}_t\right) \prod_\nu ds_\nu = \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right)\sqrt{dt} + O(dt^{3/2}).$$

- Variance of measured signal

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} s_\nu^2 \mathbb{P}\left(s_t \in \prod_\nu [s_\nu, s_\nu + ds_\nu] \mid \boldsymbol{\rho}_t\right) \prod_\nu ds_\nu = 1 + O(dt).$$

Compatible with $dy_{\nu,t} = s_{\nu,t}\sqrt{dt} = dW_{\nu,t} + \sqrt{\eta_\nu}\,\mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right) dt$.

Link to partial Kraus maps (3)

$$d\boldsymbol{\rho}_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(\boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\boldsymbol{\rho}_t\right) \boldsymbol{\rho}_t\right) dW_{\nu,t},$$

equivalent to

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t} \boldsymbol{\rho}_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt}{\operatorname{Tr}\left(\boldsymbol{M}_{dy_t} \boldsymbol{\rho}_t \boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger dt\right)}$$

- Indicates that the solution remains in the space of semi-definite positive Hermitian matrices;
- Provides a time-discretized numerical scheme preserving non-negativity of $\boldsymbol{\rho}$.

Theorem

The above master equation admits a unique solution in $\{\boldsymbol{\rho} \in \mathbb{C}^{N \times N} \;:\; \boldsymbol{\rho} = \boldsymbol{\rho}^\dagger,\ \boldsymbol{\rho} \geq 0,\ \operatorname{Tr}(\boldsymbol{\rho}) = 1\}$.

The quantum state ρ_t is usually mixed and obeys to (measurement outcomes in blue)

$$d\rho_t = \left(-i[H, \rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu) \right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger) \rho_t \right) \rho_t \right) \boldsymbol{dW_{\nu,t}}$$
$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_\mu \rho_t V_\mu^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right)} - \rho_t \right) \left(\boldsymbol{dN_\mu(t)} - \left(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right) \right) dt \right)$$

where $\eta_\nu \in [0, 1]$, $\overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some μ, $\boldsymbol{N_\mu(t+dt) - N_\mu(t) = 1}$, we have $\rho_{t+dt} = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \mathrm{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right)}$.

When $\forall \mu$, $\boldsymbol{dN_\mu(t) = 0}$, we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1 - \eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1 - \overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\mathrm{Tr}\left(M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1 - \eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1 - \overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt \right)}$$

with $M_{\boldsymbol{dy_t}} = I + \left(-iH - \frac{1}{2} \sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2} \sum_\mu \left(\overline{\eta}_\mu \mathrm{Tr}\left(V_\mu \rho_t V_\mu^\dagger \right) I - V_\mu^\dagger V_\mu \right) \right) dt + \sum_\nu \sqrt{\eta_\nu} \boldsymbol{dy_{\nu,t}} L_\nu$ and where $\boldsymbol{dy_{\nu,t}} = \sqrt{\eta_\nu} \mathrm{Tr}\left((L_\nu + L_\nu^\dagger) \rho_t \right) dt + \boldsymbol{dW_{\nu,t}}$.

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Lecture 12
Chengdu, July 11, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
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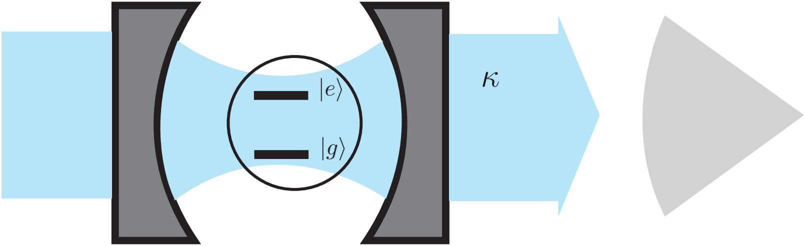
Outline

Outline

Dispersive measurement of a qubit



Inverse setup of photon-box: photons read out a qubit.

Approximate model

Cavity's dynamics are removed (singular perturbation techniques) to achieve a qubit SME:

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt$$
$$+ \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t)dW_t,$$
$$dy_t = dW_t + \sqrt{\eta\Gamma_m}\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)dt.$$

Quantum Non-Demolition measurement

$$
\begin{aligned}
d\rho_t &= -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt \\
&\quad + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t, \\
dy_t &= dW_t + \sqrt{\eta\Gamma_m}\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,dt.
\end{aligned}
$$

Uncontrolled case: $\boldsymbol{H}/\hbar = \omega_{\text{eg}}\boldsymbol{\sigma_z}/2$.

Interpretation as a Markov process with Kraus operators

$$
\boldsymbol{M}_{dy_t} = \boldsymbol{I} - \left(i\frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \frac{\Gamma_m}{8}\boldsymbol{I}\right)dt + \frac{\sqrt{\eta\Gamma_m}}{2}\boldsymbol{\sigma_z}dy_t,
$$

$$
\sqrt{(1-\eta)dt}\boldsymbol{L} = \frac{\sqrt{(1-\eta)\Gamma_m dt}}{2}\boldsymbol{\sigma_z}.
$$

QND measurement

Kraus operators $\boldsymbol{M}_{dy_t}$ and $\sqrt{(1-\eta)dt}\boldsymbol{L}$ commute with observable $\boldsymbol{\sigma_z}$: qubit states $|g\rangle\langle g|$ and $|e\rangle\langle e|$ are fixed points of the measurement process. The measurement is QND for the observable $\boldsymbol{\sigma_z}$.

QND measurement: asymptotic behavior

Theorem

Consider the SME

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t)dW_t,$$

with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}$ and $\eta > 0$.

- For any initial state ρ_0, the solution ρ_t converges almost surely as $t \to \infty$ to one of the states $|g\rangle\langle g|$ or $|e\rangle\langle e|$.
- The probability of convergence to $|g\rangle\langle g|$ (respectively $|e\rangle\langle e|$) is given by $p_g = \mathrm{Tr}\,(|g\rangle\langle g|\rho_0)$ (respectively $\mathrm{Tr}\,(|e\rangle\langle e|\rho_0)$).
- The convergence rate is given by $\eta\Gamma_M/2$.

Proof based on the Lyapunov function $V(\rho) = \sqrt{\mathrm{Tr}\,(\boldsymbol{\sigma_z}^2\rho) - \mathrm{Tr}^2\,(\boldsymbol{\sigma_z}\rho)}$ with

$$\frac{d}{dt}\mathbb{E}\,(V(\rho)) = -\frac{\eta\Gamma_M}{2}\mathbb{E}\,(V(\rho))$$

Matlab open-loop simulations: `RealisticModelQubit.m`

Quantum feedback

Question: how to stabilize deterministically a single qubit state $|g\rangle\langle g|$ or $|e\rangle\langle e|$?
Controlled SME:

$$d\rho_t = -\frac{i}{\hbar}[\boldsymbol{H}, \rho_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\rho_t\boldsymbol{\sigma_z} - \rho_t)dt + \frac{\sqrt{\eta\Gamma_m}}{2}(\boldsymbol{\sigma_z}\rho_t + \rho_r\boldsymbol{\sigma_z} - 2\operatorname{Tr}(\boldsymbol{\sigma_z}\rho_t)\rho_t)dW_t,$$

with

$$\boldsymbol{H} = \frac{u(\rho_t)}{2}\boldsymbol{\sigma_x} + \frac{v(\rho_t)}{2}\boldsymbol{\sigma_y},$$

$$u = g\,\text{sign}(\operatorname{Tr}(\rho\boldsymbol{\sigma_y}))(1 - \operatorname{Tr}(\rho\boldsymbol{\sigma_z})), \quad v = -g\,\text{sign}(\operatorname{Tr}(\rho\boldsymbol{\sigma_x}))(1 - \operatorname{Tr}(\rho\boldsymbol{\sigma_z}))$$

stabilizes with gain $g > 0$ large enough the target state $\rho_{\text{tag}} = |e\rangle\langle e|$ (based on the control Lyapunov function $1 - \operatorname{Tr}(\rho\boldsymbol{\sigma_z})$).

Matlab closed-loop simulations: `RealisticFeedbackQubit.m`

Exercise: continuous-time QND measurement[3]

Take a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$ with the Hermitian operator L of spectral decomposition $L = \sum_{k=1}^{d} \lambda_k \Pi_k$ where $\lambda_1, \ldots \lambda_d$ are the distinct ($d \leq n$), real eigenvalues of L with corresponding orthogonal projection operators $\Pi_1, \ldots, \Pi_d$ resolving the identity, i.e. $\sum_{k=1}^{d} \Pi_k = \boldsymbol{I}$. Assume that the density operator ρ obeys to

$$d\rho = (L\rho L - (L^2\rho + \rho L^2)/2)\,dt + \sqrt{\eta}(L\rho + \rho L - 2\,\mathrm{Tr}\,(L\rho)\,\rho)dW$$

with diffusive measurement $dy = 2\sqrt{\eta}\,\mathrm{Tr}\,(L\rho)\,dt + dW$ and $\eta > 0$.

1. For each k, set $p_k(\rho) = \mathrm{Tr}\,(\rho\Pi_k)$. Show that

$$dp_k = 2\sqrt{\eta}\left(\lambda_k - \sum_{k'=1}^{d} \lambda_{k'} p_{k'}\right) p_k dW$$

2. Deduce that $\xi_k = \sqrt{p_k}$ obeys to

$$d\xi_k = -\tfrac{1}{2}\eta(\lambda_k - \varpi(\xi))^2 \xi_k dt + \sqrt{\eta}(\lambda_k - \varpi(\xi))\xi_k dW,$$

   with $\varpi(\xi) = \sum_{k=1}^{d} \lambda_k \xi_k^2$

3. Prove that

$$d(\xi_k\xi_{k'}) == -\tfrac{1}{2}\eta(\lambda_k - \lambda_{k'})^2 \xi_{k'}\xi_k dt + \sqrt{\eta}(\lambda_k + \lambda_{k'} - 2\varpi(\xi))\xi_k\xi_{k'} dW.$$

4. Set $V(\rho) = \sum_{1 \leq k < k' \leq d} \sqrt{p_k(\rho)}\sqrt{p_{k'}(\rho)}$. Show that
   $\mathbb{E}(dV \mid \rho) = -\frac{\eta}{2} \sum_{k'=1}^{d} \sum_{k'<k} (\lambda_k - \lambda_{k'})^2\ \xi_k\xi_{k'}\ dt \leq -\frac{\eta}{2} \left(\min_{k',k\neq k'} (\lambda_k - \lambda_{k'})^2\right) V(\rho)dt.$

5. Conclude that $\mathbb{E}(V(\rho_t) \mid \rho_0) \leq V(\rho_0)e^{-rt}$ with $r > 0$ to be defined.

[3] G. Cardona, A. Sarlette,PR: Exponential stabilization of quantum systems under continuous non-demolition measurements. https://arxiv.org/abs/1906.07403

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Lecture 13
Chengdu, July 12, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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Outline

Outline

The Lindblad master differential equation (finite dimensional case)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_{\nu} \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) \triangleq \mathcal{L}(\rho)$$

where

- $\boldsymbol{H}$ is the Hamiltonian that could depend on t (Hermitian operator on the underlying Hilbert space $\mathcal{H}$)
- the $\boldsymbol{L}_\nu$'s are operators on $\mathcal{H}$ that are not necessarily Hermitian.

Qualitative properties ($\mathcal{H}$ of finite dimension):

1. Positivity and trace conservation: if ρ_0 is a density operator, then $\rho(t)$ remains a density operator for all $t > 0$.
2. For any $t \geq 0$, the propagator $e^{t\mathcal{L}}$ is a Kraus map: exists a collection of operators $(M_{\mu,t})$ such that $\sum_\mu M_{\mu,t}^\dagger M_{\mu,t} = I$ with $e^{t\mathcal{L}}(\rho) = \sum_\mu M_{\mu,t} \rho M_{\mu,t}^\dagger$ (Kraus theorem characterizing completely positive linear maps).
3. Contraction for many distances such as the nuclear distance: take two trajectories ρ and ρ'; for any $0 \leq t_1 \leq t_2$,

$$\text{Tr}\left(|\rho(t_2) - \rho'(t_2)|\right) \leq \text{Tr}\left(|\rho(t_1) - \rho'(t_1)|\right)$$

where for any Hermitian operator A, $|A| = \sqrt{A^2}$ and $\text{Tr}\left(|A|\right)$ corresponds to the sum of the absolute values of its eigenvalues.

Link between Lindblad differential equation and Kraus map

$$\rho_{k+1} = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \tfrac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

Take $dt > 0$ small. Set

$$\boldsymbol{M}_{dt,0} = \boldsymbol{I} - dt\left(\tfrac{i}{\hbar}\boldsymbol{H} + \tfrac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu\right), \quad \boldsymbol{M}_{dt,\nu} = \sqrt{dt}\,\boldsymbol{L}_\nu.$$

Since $\rho(t+dt) = \rho(t) + dt\left(\frac{d}{dt}\rho(t)\right) + O(dt^2)$, we have

$$\rho(t+dt) = \boldsymbol{M}_{dt,0}\rho(t)\boldsymbol{M}_{dt,0}^\dagger + \sum_\nu \boldsymbol{M}_{dt,\nu}\rho(t)\boldsymbol{M}_{dt,\nu}^\dagger + O(dt^2).$$

Since $\boldsymbol{M}_{dt,0}^\dagger \boldsymbol{M}_{dt,0} + \sum_\nu \boldsymbol{M}_{dt,\nu}^\dagger \boldsymbol{M}_{dt,\nu} = \boldsymbol{I} + 0(dt^2)$ the super-operator

$$\rho \mapsto \boldsymbol{M}_{dt,0}\rho \boldsymbol{M}_{dt,0}^\dagger + \sum_\nu \boldsymbol{M}_{dt,\nu}\rho \boldsymbol{M}_{dt,\nu}^\dagger$$

can be seen as an infinitesimal Kraus map.

Outline

Controlled Lindblad master equation

$$\frac{d}{dt}\rho = -i\left[\tfrac{\Delta}{2}\boldsymbol{\sigma_z}\ ,\ \rho\right] + \left[u\boldsymbol{\sigma_+} - u^*\boldsymbol{\sigma_-}\ ,\ \rho\right] + \tfrac{1}{T_1}\left(\boldsymbol{\sigma_-}\rho\boldsymbol{\sigma_+} - \tfrac{1}{2}(\boldsymbol{\sigma_+}\boldsymbol{\sigma_-}\rho + \rho\boldsymbol{\sigma_+}\boldsymbol{\sigma_-})\right) + \tfrac{1}{2T_\phi}(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho)$$

with

- Coherent drive of complex amplitude u at a pulsation $\omega_{eg} + \Delta$ detuned by Δ with respect to the qubit pulsation ω_{eg}.
- T_1 life-time of the excited state $|e\rangle$.
- T_ϕ dephasing time destroying the coherence $\langle e|\rho|g\rangle$.

Exercise: For $u = 0$ show that $\lim_{t\mapsto+\infty} \rho(t) = |g\rangle\langle g|$.

Outline

The driven and damped classical oscillator

Dynamics in the (x', p') phase plane with $\omega \gg \kappa, \sqrt{u_1^2 + u_2^2}$:

$$\frac{d}{dt}x' = \omega p', \quad \frac{d}{dt}p' = -\omega x' - \kappa p' - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t)$$

Define the frame rotating at ω by $(x', p') \mapsto (x, p)$ with

$$x' = \cos(\omega t)x + \sin(\omega t)p, \quad p' = -\sin(\omega t)x + \cos(\omega t)p.$$

Removing highly oscillating terms (rotating wave approximation), from

$$\frac{d}{dt}x = -\kappa \sin^2(\omega t)x + 2u_1 \sin^2(\omega t) + (\kappa p - 2u_2) \sin(\omega t)\cos(\omega t)$$
$$\frac{d}{dt}p = -\kappa \cos^2(\omega t)p + 2u_2 \cos^2(\omega t) + (\kappa x - 2u_1) \sin(\omega t)\cos(\omega t)$$

we get, with $\alpha = x + ip$ and $u = u_1 + iu_2$:

$$\frac{d}{dt}\alpha = -\tfrac{\kappa}{2}\alpha + u.$$

With $x' + ip' = \alpha' = e^{-i\omega t}\alpha$, we have $\frac{d}{dt}\alpha' = -(\frac{\kappa}{2} + i\omega)\alpha' + ue^{-i\omega t}$

Driven and damped quantum oscillator ($n_{th} = 0$)

- The Lindblad master equation:

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

- Consider $\rho = \boldsymbol{D}_{\overline{\alpha}}\xi\boldsymbol{D}_{-\overline{\alpha}}$ with $\overline{\alpha} = 2u/\kappa$ and $\boldsymbol{D}_{\overline{\alpha}} = e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}}$. We get

$$\frac{d}{dt}\xi = \kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\xi - \tfrac{1}{2}\xi\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

  since $\boldsymbol{D}_{-\overline{\alpha}}\boldsymbol{a}\boldsymbol{D}_{\overline{\alpha}} = \boldsymbol{a} + \overline{\alpha}$.

- Informal convergence proof with the strict Lyapunov function $V(\xi) = \mathrm{Tr}\,(\xi\boldsymbol{N})$:

$$\frac{d}{dt}V(\xi) = -\kappa V(\xi) \Rightarrow V(\xi(t)) = V(\xi_0)e^{-\kappa t}.$$

  Since $\xi(t)$ is Hermitian and non-negative, $\xi(t)$ tends to $|0\rangle\langle 0|$ when $t \mapsto +\infty$.

Theorem

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

Assume that the initial state ρ_0 is a density operator with finite energy $\mathrm{Tr}\left(\rho_0 \boldsymbol{N}\right) < +\infty$. *Then exists a unique solution to the Cauchy problem in the Banach space $\mathcal{K}^1(\mathcal{H})$, the set of trace class operators on $\mathcal{H}$. It is defined for all $t > 0$ with $\rho(t)$ a density operator (Hermitian, non-negative and trace-class) that remains in the domain of the Lindblad super-operator*

$$\rho \mapsto [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

This means that $t \mapsto \rho(t)$ is differentiable in the Banach space $\mathcal{K}^1(\mathcal{H})$. Moreover $\rho(t)$ converges for the trace-norm towards $|\overline{\alpha}\rangle\langle\overline{\alpha}|$ when t tends to $+\infty$, where $|\overline{\alpha}\rangle$ is the coherent state of complex amplitude $\overline{\alpha} = \frac{2u}{\kappa}$.

Link with the classical oscillator

Lemma

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{\rho}] + \kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \boldsymbol{\rho}(0) = \boldsymbol{\rho}_0.$$

1. *for any initial density operator $\boldsymbol{\rho}_0$ with $\text{Tr}(\boldsymbol{\rho}_0\boldsymbol{N}) < +\infty$, we have $\frac{d}{dt}\alpha = -\frac{\kappa}{2}(\alpha - \overline{\alpha})$ where $\alpha = \text{Tr}(\boldsymbol{\rho}\boldsymbol{a})$ and $\overline{\alpha} = \frac{2u}{\kappa}$.*
2. *Assume that $\boldsymbol{\rho}_0 = |\beta_0\rangle\langle\beta_0|$ where β_0 is some complex amplitude. Then for all $t \geq 0$, $\boldsymbol{\rho}(t) = |\beta(t)\rangle\langle\beta(t)|$ remains a coherent state of amplitude $\beta(t)$ solution of the following equation: $\frac{d}{dt}\beta = -\frac{\kappa}{2}(\beta - \overline{\alpha})$ with $\beta(0) = \beta_0$.*

Statement 2 relies on:

$$\boldsymbol{a}|\beta\rangle = \beta|\beta\rangle, \quad |\beta\rangle = e^{-\frac{\beta\beta^*}{2}} e^{\beta\boldsymbol{a}^\dagger}|0\rangle \quad \frac{d}{dt}|\beta\rangle = \left(-\tfrac{1}{2}(\beta^*\dot{\beta} + \beta\dot{\beta}^*) + \dot{\beta}\boldsymbol{a}^\dagger\right)|\beta\rangle.$$

Outline

Driven and damped quantum oscillator with thermal photon(s)

Parameters $\omega \gg \kappa, |u|$ and $n_{\text{th}} > 0$:

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{\rho}] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

Key issue: $\lim_{t\mapsto+\infty}\boldsymbol{\rho}(t) = ?$.
With $\bar{\alpha} = 2u/k$, we have

$$\frac{d}{dt}\boldsymbol{\rho} = (1+n_{\text{th}})\kappa\left((\boldsymbol{a}-\bar{\alpha})\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha})^\dagger - \tfrac{1}{2}(\boldsymbol{a}-\bar{\alpha})^\dagger(\boldsymbol{a}-\bar{\alpha})\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha})^\dagger(\boldsymbol{a}-\bar{\alpha})\right) + n_{\text{th}}\kappa\left((\boldsymbol{a}-\bar{\alpha})^\dagger\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha}) - \tfrac{1}{2}(\boldsymbol{a}-\bar{\alpha})(\boldsymbol{a}-\bar{\alpha})^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}(\boldsymbol{a}-\bar{\alpha})(\boldsymbol{a}-\bar{\alpha})^\dagger\right).$$

Using the unitary change of frame $\boldsymbol{\xi} = \boldsymbol{D}_{-\bar{\alpha}}\boldsymbol{\rho}\boldsymbol{D}_{\bar{\alpha}}$ based on the displacement $\boldsymbol{D}_{\bar{\alpha}} = e^{\bar{\alpha}\boldsymbol{a}^\dagger - \bar{\alpha}^\dagger\boldsymbol{a}}$, we get the following dynamics on $\boldsymbol{\xi}$

$$\frac{d}{dt}\boldsymbol{\xi} = (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\xi}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\xi}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{a}\boldsymbol{a}^\dagger\right)$$

since $\boldsymbol{a} + \bar{\alpha} = \boldsymbol{D}_{-\bar{\alpha}}\boldsymbol{a}\boldsymbol{D}_{\bar{\alpha}}$.

Asymptotic convergence towards the thermal equilibrium

The thermal mixed state $\xi_{\text{th}} = \frac{1}{1+n_{\text{th}}} \left(\frac{n_{\text{th}}}{1+n_{\text{th}}}\right)^{\boldsymbol{N}}$ is an equilibrium of

$$\frac{d}{dt}\xi = \kappa(1+n_{\text{th}})\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\xi - \tfrac{1}{2}\xi\boldsymbol{a}^\dagger\boldsymbol{a}\right) + \kappa n_{\text{th}}\left(\boldsymbol{a}^\dagger\xi\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\xi - \tfrac{1}{2}\xi\boldsymbol{a}\boldsymbol{a}^\dagger\right)$$

with $\text{Tr}\left(\boldsymbol{N}\xi_{\text{th}}\right) = n_{\text{th}}$. Following [3], set ζ the solution of the Sylvester equation: $\xi_{\text{th}}\zeta + \zeta\xi_{\text{th}} = \xi - \xi_{\text{th}}$. Then $V(\xi) = \text{Tr}\left(\xi_{\text{th}}\zeta^2\right)$ is a strict Lyapunov function. It is based on the following computations that can be made rigorous with an adapted Banach space for ξ:

$$\frac{d}{dt}V(\xi) = -\kappa(1+n_{\text{th}})\,\text{Tr}\left([\zeta,\boldsymbol{a}]\xi_{\text{th}}[\zeta,\boldsymbol{a}]^\dagger\right) - \kappa n_{\text{th}}\,\text{Tr}\left([\zeta,\boldsymbol{a}^\dagger]\xi_{\text{th}}[\zeta,\boldsymbol{a}^\dagger]^\dagger\right) \leq 0.$$

When $\frac{d}{dt}V = 0$, ζ commutes with $\boldsymbol{a}$, $\boldsymbol{a}^\dagger$ and $\boldsymbol{N}$. It is thus a constant function of $\boldsymbol{N}$. Since $\xi_{\text{th}}\zeta + \zeta\xi_{\text{th}} = \xi - \xi_{\text{th}}$, we get $\xi = \xi_{\text{th}}$.

[3] PR and A. Sarlette: Contraction and stability analysis of steady-states for open quantum systems described by Lindblad differential equations. Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on, 10-13 Dec. 2013, 6568-6573.

Outline

Parameters $\omega \gg \kappa, |u|$ and $n_{\text{th}} \geq 0$:

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{\rho}] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

Key issue: $\lim_{t\mapsto+\infty}\boldsymbol{\rho}(t) = \,?$.
The passage to another representation via the Wigner function:

- Since $\boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}}\boldsymbol{D}_{-\alpha}$ bounded and Hermitian operator (the dual of $\mathcal{K}^1(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$),

  $$W^{\{\boldsymbol{\rho}\}}(x,p) = \tfrac{2}{\pi}\operatorname{Tr}\left(\boldsymbol{\rho}\boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}}\boldsymbol{D}_{-\alpha}\right) \quad \text{with} \quad \alpha = x+ip \in \mathbb{C},$$

  defines a real and bounded function $|W^{\{\boldsymbol{\rho}\}}(x,p)| \leq \frac{2}{\pi}$.
- For a coherent state $\boldsymbol{\rho} = |\beta\rangle\langle\beta|$ with $\beta \in \mathbb{C}$:

  $$W^{\{|\beta\rangle\langle\beta|\}}(x,p) = \tfrac{2}{\pi}e^{-2|\beta-(x+ip)|^2}.$$

The partial differential equation satisfied by the Wigner function (1)

With $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} e^{-\alpha\alpha^*/2} = e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger} e^{\alpha\alpha^*/2}$ we have:

$$\tfrac{\pi}{2} W^{\{\boldsymbol{\rho}\}}(\alpha, \alpha^*) = \text{Tr}\left(\boldsymbol{\rho} e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} e^{i\pi \boldsymbol{N}} e^{\alpha^* \boldsymbol{a}} e^{-\alpha \boldsymbol{a}^\dagger}\right)$$

where α and α^* are seen as independent variables:

$$\frac{\partial}{\partial \alpha} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial \alpha^*} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right)$$

We have $\frac{\pi}{2}\frac{\partial}{\partial\alpha} W^{\{\boldsymbol{\rho}\}}(\alpha, \alpha^*) = \text{Tr}\left((\boldsymbol{\rho}\boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger \boldsymbol{\rho})\boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}\right)$ Since $\boldsymbol{a}^\dagger \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha} = \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}(2\alpha^* - \boldsymbol{a}^\dagger)$, we get

$$\frac{\partial}{\partial \alpha} W^{\{\boldsymbol{\rho}\}}(\alpha, \alpha^*) = 2\alpha^* W^{\{\boldsymbol{\rho}\}}(\alpha, \alpha^*) - 2 W^{\{\boldsymbol{a}^\dagger \boldsymbol{\rho}\}}(\alpha, \alpha^*).$$

Thus $W^{\{\boldsymbol{a}^\dagger \boldsymbol{\rho}\}}(\alpha, \alpha^*) = \alpha^* W^{\{\boldsymbol{\rho}\}}(\alpha, \alpha^*) - \frac{1}{2}\frac{\partial}{\partial \alpha} W^{\{\boldsymbol{\rho}\}}(\alpha, \alpha^*)$, i.e.

$$W^{\{\boldsymbol{a}^\dagger \boldsymbol{\rho}\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\boldsymbol{\rho}\}}.$$

The partial differential equation satisfied by the Wigner function (2)

Similar computations yield to the following correspondence rules:

$$W^{\{\rho a\}} = \left(\alpha - \tfrac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W^{\{\rho\}}, \quad W^{\{a\rho\}} = \left(\alpha + \tfrac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W^{\{\rho\}}$$

$$W^{\{\rho a^\dagger\}} = \left(\alpha^* + \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}, \quad W^{\{a^\dagger\rho\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}.$$

Thus

$$\frac{d}{dt}\rho = [u a^\dagger - u^* a, \rho] + (1+n_{\text{th}})\kappa\left(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a\right) + n_{\text{th}}\kappa\left(a^\dagger\rho a - \tfrac{1}{2}a a^\dagger\rho - \tfrac{1}{2}\rho a a^\dagger\right).$$

becomes

$$\frac{\partial}{\partial t}W^{\{\rho\}} = \frac{\kappa}{2}\left(\frac{\partial}{\partial \alpha}(\alpha - \overline{\alpha}) + \frac{\partial}{\partial \alpha^*}(\alpha^* - \overline{\alpha}^*) + (1+2n_{\text{th}})\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) W^{\{\rho\}}$$

Solutions of the quantum Fokker-Planck equation

Since the Green function of

$$\frac{\partial}{\partial t}W^{\{\rho\}} = \frac{\kappa}{2}\Big(\frac{\partial}{\partial x}\Big((x-\overline{x})W^{\{\rho\}}\Big) + \frac{\partial}{\partial p}\Big((p-\overline{p})W^{\{\rho\}}\Big) + \tfrac{1+2n_{\text{th}}}{4}\left(\frac{\partial^2 W^{\{\rho\}}}{\partial x^2} + \frac{\partial^2 W^{\{\rho\}}}{\partial p^2}\right)\Big)$$

is the following time-varying Gaussian function

$$G(x,p,t,x_0,p_0) = \frac{\exp\left(-\frac{\left(x-\overline{x}-(x_0-\overline{x})e^{-\frac{\kappa t}{2}}\right)^2+\left(p-\overline{p}-(p_0-\overline{p})e^{-\frac{\kappa t}{2}}\right)^2}{(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}\right)}{\pi(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}$$

we can compute $W_t^{\{\rho\}}$ from $W_0^{\{\rho\}}$ for all $t > 0$:

$$W_t^{\{\rho\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\rho\}}(x',p')G(x,p,t,x',p')\,dx'dp'.$$

Combining

- $W_t^{\{\rho\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\rho\}}(x',p') G(x,p,t,x',p')\, dx' dp'$.
- G uniformly bounded and $\lim_{t\mapsto+\infty} G(x,p,t,x',p') = \frac{1}{\pi(n_{\text{th}}+\frac{1}{2})} \exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right)$
- $W_0^{\{\rho\}}$ in L^1 with $\iint_{\mathbb{R}^2} W_0^{\{\rho\}} = 1$
- dominate convergence theorem

shows that all the solutions converge to a unique steady-state Gaussian density function, centered in $(\overline{x},\overline{p})$ with variance $\frac{1}{2}+n_{\text{th}}$:

$$\forall (x,p)\in\mathbb{R}^2, \quad \lim_{t\mapsto+\infty} W_t^{\{\rho\}}(x,p) = \frac{1}{\pi(n_{\text{th}}+\frac{1}{2})} \exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right).$$

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Lecture 14
Chengdu, July 12, 2019

[1] An important part of these slides gathered at the following web page have been elaborated with Mazyar Mirrahimi:
http://cas.ensmp.fr/~rouchon/ChengduJuly2019/index.html

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Outline

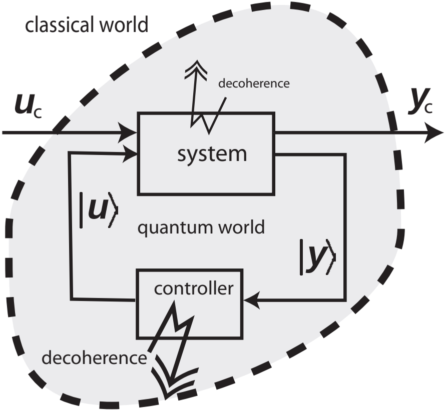
1. Coherent feedback stabilisation

2. Slow measurement-based feedback

Outline

Coherent (autonomous) feedback (dissipation engineering)

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [3]



Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, ...)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[3] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Coherent feedback underlying the cat-qubit (1) [4]

System: high quality oscillator with annihilation operator $\boldsymbol{a}$:

$$\frac{d}{dt}\rho = -i\omega_a[\boldsymbol{a}^\dagger \boldsymbol{a}, \rho] + \kappa_a \left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right).$$

Controller: low quality oscillator $\kappa_a \ll \kappa_b$ with annihilation operator $\boldsymbol{b}$ with resonant drive

$$\frac{d}{dt}\rho = -i\omega_b[\boldsymbol{b}^\dagger \boldsymbol{b}, \rho] + [-ue^{i\omega_b t}\boldsymbol{b}^\dagger + u^* e^{-i\omega_b t}\boldsymbol{b}, \rho] + \kappa_b \left(\boldsymbol{b}\rho\boldsymbol{b}^\dagger - \tfrac{1}{2}(\boldsymbol{b}^\dagger\boldsymbol{b}\rho + \rho\boldsymbol{b}^\dagger\boldsymbol{b})\right).$$

Coupling Hamiltonian term $g[\boldsymbol{a}^2\boldsymbol{b}^\dagger - (\boldsymbol{a}^\dagger)^2\boldsymbol{b}, \rho]$ yields to the closed-loop Lindblad equation

$$\begin{aligned}\frac{d}{dt}\rho = &-i[\omega_a\boldsymbol{a}^\dagger\boldsymbol{a} + \omega_b\boldsymbol{b}^\dagger\boldsymbol{b}] + [-ue^{-i\omega_b t}\boldsymbol{b}^\dagger + u^* e^{+i\omega_b t}\boldsymbol{b}, \rho] + g[\boldsymbol{a}^2\boldsymbol{b}^\dagger - (\boldsymbol{a}^\dagger)^2\boldsymbol{b}, \rho]\\ &+ \kappa_a \left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right) + \kappa_b \left(\boldsymbol{b}\rho\boldsymbol{b}^\dagger - \tfrac{1}{2}(\boldsymbol{b}^\dagger\boldsymbol{b}\rho + \rho\boldsymbol{b}^\dagger\boldsymbol{b})\right)\end{aligned}$$

[4] M. Mirrahimi, Z. Leghtas, . . . , M.H. Devoret: Dynamically protected cat-qubits: a new paradigm for universal quantum computation.New Journal of Physics,2014, 16:045014.

Coherent feedback underlying the cat-qubit (2)

• For $\omega_b = 2\omega_a$ one gets in the the frame rotating at ω_a for mode a and ω_b for mode b (unitary transformation: $\rho_{old} = e^{-i\omega_a t \boldsymbol{a}^\dagger \boldsymbol{a} - i\omega_b t \boldsymbol{b}^\dagger \boldsymbol{b}} \rho_{new} e^{i\omega_a t \boldsymbol{a}^\dagger \boldsymbol{a} + i\omega_b t \boldsymbol{b}^\dagger \boldsymbol{b}}$):

$$\frac{d}{dt}\rho = g\left[(\boldsymbol{a}^2 - \tfrac{u}{g})\boldsymbol{b}^\dagger - ((\boldsymbol{a}^\dagger)^2 - \tfrac{u^*}{g})\boldsymbol{b}, \rho\right] + \kappa_a\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right) + \kappa_b\left(\boldsymbol{b}\rho\boldsymbol{b}^\dagger - \tfrac{1}{2}(\boldsymbol{b}^\dagger\boldsymbol{b}\rho + \rho\boldsymbol{b}^\dagger\boldsymbol{b})\right).$$

• If we neglect κ_a in front of κ_b, any $\bar{\rho}$ of the form $\bar{\rho} = \bar{\rho}_a \otimes |0_b\rangle\langle 0_b|$ with $\bar{\rho}_a$ density operator on mode a with support in span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ where $\alpha = \sqrt{\frac{u}{g}} \in \mathbb{C}$, is a steady-state of the above Lindbald equation with $\kappa_a = 0$.

• If additionally, $g \ll \kappa_b$, the strongly damped mode b can be eliminated via singular perturbation techniques (quasi-static or adiabatic approximation) to get the following slow Lindblad equation on mode a only:

$$\frac{d}{dt}\rho = \frac{4g^2}{\kappa_b}\left(L\rho L^\dagger - \tfrac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) + \kappa_a\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right)$$

with Lindblad operator $L = \boldsymbol{a}^2 - \alpha^2$.

Coherent feedback underlying the cat-qubit (3)

- If $g \gg \sqrt{\kappa_a \kappa_b}$ then we can still neglect κ_a. Any solution of

$$\frac{d}{dt}\rho = \frac{4g^2}{\kappa_b}\left(L\rho L^\dagger - \tfrac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right)$$

converges to a steady state $\bar{\rho}_a$ with support in span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ (use the Lyapunov function $V(\rho) = \text{Tr}\left(L\rho L^\dagger\right)$ [5]).

- For $\frac{d}{dt}\rho = \frac{4g^2}{\kappa_b}\left(L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) + \kappa_a\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \frac{1}{2}(\boldsymbol{a}^\dagger\boldsymbol{a}\rho + \rho\boldsymbol{a}^\dagger\boldsymbol{a})\right)$ with $g \gg \sqrt{\kappa_a\kappa_b}$, a reduction to the sub-space span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ is possible to describe the very slow evolution due to κ_a. With the orthonormal basis,

$$|c_\alpha^+\rangle = \frac{|\alpha\rangle + |\text{-}\alpha\rangle}{\sqrt{2(1+e^{-2|\alpha|^2})}} \text{ (even cat) and } |c_\alpha^-\rangle = \frac{|\alpha\rangle - |\text{-}\alpha\rangle}{\sqrt{2(1-e^{-2|\alpha|^2})}} \text{ (odd cat)},$$

define the swap operator $X_c = |c_\alpha^+\rangle\langle c_\alpha^-| + |c_\alpha^-\rangle\langle c_\alpha^+|$. Since $\boldsymbol{a}|c_\alpha^+\rangle = \alpha|c_\alpha^-\rangle$ and $\boldsymbol{a}|c_\alpha^-\rangle = \alpha|c_\alpha^+\rangle$, the reduced dynamics on $\mathcal{H}_c \triangleq \text{span}\{|c_\alpha^+\rangle, |c_\alpha^-\rangle\}$ reads

$$\frac{d}{dt}\rho_c = \kappa_a|\alpha|^2(X_c\rho_c X_c - \rho_c)$$

where ρ_c a density operator on $\mathcal{H}_c$.

[5] R. Azouit, A. Sarlette, and PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. ESAIM: COCV, 2016, 22(4):1353 –1369.

Outline

Final slow measurement-based feedback stabilisation

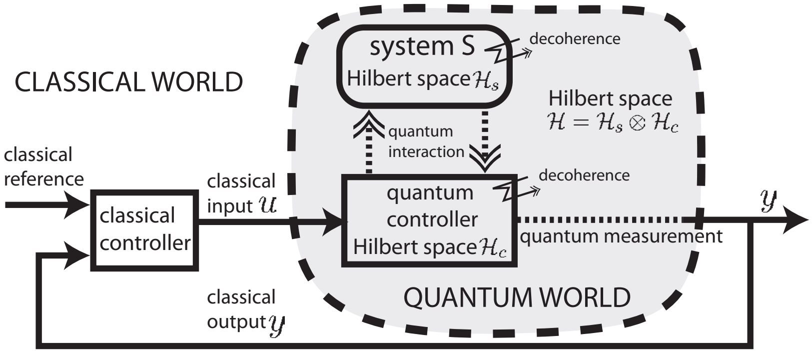
Assume that one can continuously and weakly measure the parity $e^{i\pi \boldsymbol{a}^\dagger \boldsymbol{a}}$ of mode a with a rate $\gamma_a \gg \kappa_a |\alpha|^2$. Then we have the following stochastic master equation ($Z_c = |c_\alpha^+\rangle\langle c_\alpha^+| - |c_\alpha^-\rangle\langle c_\alpha^-|$)

$$d\rho_c = \kappa_a |\alpha|^2 (X_c \rho_c X_c - \rho_c) dt + \gamma_a (Z_c \rho_c Z_c - \rho_c) dt + \sqrt{\eta_c \gamma_a} (Z_c \rho_c + \rho_c Z_c - 2 \operatorname{Tr}(Z_c \rho_c) \rho_c) dW$$

with continuous-time measurement output y_c of efficiency $\eta_c > 0$ and given by $dy_c = 2\sqrt{\eta_c \gamma_a} \operatorname{Tr}(Z_c \rho_c) dt + dW$.

One can stabilize either $|c_\alpha^+\rangle\langle c_\alpha^+|$ or $|c_\alpha^-\rangle\langle c_\alpha^-|$ if we have at our disposal a classical input signal u_c attached to an Hamiltonian H_c on $\mathcal{H}_c$ independent of Z_c.

Exercise: design a measurement-based feedback stabilizing $|c_\alpha^+\rangle\langle c_\alpha^+|$ with $H_c = X_c$ and based on the Lyapunov function $V_c(\rho_c) = \sqrt{\langle c_\alpha^+ | \rho_c | c_\alpha^+ \rangle}$ for $\kappa_a = 0$. Analyse the impact of $\kappa_a > 0$ with closed-loop Monte-Carlo simulations.



To stabilize the quantum information localized in system S:

- fast decoherence addressed by a **quantum controller** (coherent feedback);
- slow decoherence and perturbation tackled by a *classical controller* (measurement-based feedback).