Optimal bang-bang control of a mechanical double oscillator using averaging methods

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Abstract: This paper considers a semi-active Tuned Liquid Column Damper (TLCD) designed to damp a Floating Wind Turbine (FWT). This device contains a choke that can be controlled to damp the natural oscillations created by ocean waves on the FWT. The choke, which controls the flow inside the TLCD, defines, by construction, an input variable having bounded values (the restriction). These limitations spur an interest in optimal control strategies for this oscillating system subject to a sinusoidal wave. These strategies are the subject of the paper. First, we numerically solve an optimal control problem of interests for the application. When computed over long time horizons, the resulting control signal appears to be of periodic nature. To confirm this fact, we employ averaging methods on a simplified model. This study confirms that the optimal control is bang-bang, is periodic, and its frequency is the double of the incoming wave frequency.

1. INTRODUCTION

Wind power is the second fastest growing renewable source of electricity (National Renewable Energy Laboratory, 2012) in terms of installed power. The construction of offshore wind farms is booming all over the world. In Europe, offshore wind energy is expected to grow to 23.5 GW by 2020, thereby tripling the installed capacity in 2015 (Ernst & Young, 2015). The causes of this recent trend are the strength and regularity of the wind far from the shore, which should facilitate the mass production of electricity. Two types of technology may be employed to exploit offshore wind energy: fixed-bottom wind turbines (with the foundations fixed into the seabed) and floating wind turbines (FWTs). Fixed bottom offshore wind turbines are too expensive for waters deeper than 60 m (Musial et al., 2006), which prevents their use in the most interesting fields. Thus, FWTs are a more attractive alternative. In particular, FWTs have little dependence on the seabed conditions for installation and they can be moved to a harbour to perform maintenance. However, the main drawback of FWTs is their sensitivity to the surrounding water waves which subject the wind turbine to increased mechanical loads (Jonkman, 2007), thereby reducing the lifespan of the mechanical parts of the wind turbine.

Attached moving masses such as tuned mass dampers (TMDs) can be employed to improve the response of massive structures to external disturbances. One of the most economical and efficient variants of the TMD is the tuned liquid column damper (TLCD), which is also known as an anti-roll tank or U-tank. The TLCD was originally proposed by Frahm (Frahm, 1911; Moaleji and Greig, 2007) to limit ship roll: it is a U-shaped tube on a plane orthogonal to the ship roll axis, which is generally filled with water. The liquid inside the TLCD oscillates due to the movement of the structure and the liquid energy is dissipated via a restriction located in the horizontal section. A TLCD is usually employed to damp the natural frequency of the structure.

In (Coudurier et al., 2015), we considered damping a FWT with a semi-active TLCD (with a variable restriction) and derived the dynamics of the coupled system using a Lagrangian approach. A *reduced model* of the system was proposed along with a clipped LQR law. The clipping (saturation) of the feedback law was a simple, yet relatively efficient, solution as was shown in simulations that demonstrate the potential of this control law for reducing the pitch motion of the structure. The obtained performance was not optimal, but, as expected, was much better than the passive TLCD (without any actuation).

In this study, we seek performance improvement and consider the optimal control of the restriction of the TLCD. Mathematically, we aim to damp a FWT subject to a sinusoidal wave. Solving such optimization problem can serve to quantitatively estimate the best possible performance (and, in turn, the performance loss of suboptimal strategies such as the clipped LQR mentioned earlier). In addition, if its computational burden is not too heavy, it could serve as online control algorithm, following the large trend of model predictive control (Lee, 2011). These questions are left for future studies. The question at stake in this article is the observed periodic nature of the optimal solution.

The paper is organised as follows. First, we numerically solve the optimal control problem of the *reduced model*, as proposed in (Coudurier et al., 2015). Next, using a similar but simpler system (the *Toy Problem*), we analytically

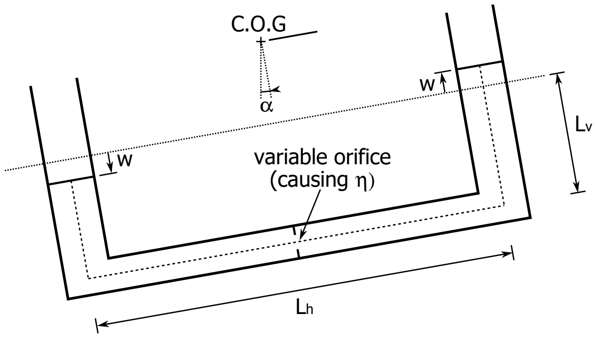

Fig. 1. Scheme of the TLCD in motion.

study the optimal control under a sinusoidal disturbance. For this, we employ averaging methods (Arnold, 2013; Guckenheimer and Holmes, 1983) on the stationarity conditions obtained from the application of Pontryagin Minimum Principle (Pontryagin, 1987). This analysis confirms the numerical observations of periodicity.

2. OPTIMAL CONTROL OF A FWT

To quantify the best performance that can be achieved by a semi-active TLCD on a FWT, we consider an optimal control formulation.

2.1 Formulation

Following the model presented in (Coudurier et al., 2015), the following optimal control problem is considered.

Problem 1. (Optimal control of the TLCD-FWT)

$$\min_{\eta(t),X_0} J = \int_0^T \alpha(t)^2 \, \mathrm{d}t$$

subject to

$$\dot{X} = \mathrm{A}X - \frac{1}{2}\rho A_h(\nu\dot{w})|\nu\dot{w}|\mathrm{B}\eta + \mathrm{E}F \qquad \forall t \in [0,T]$$

$$X_0 = 0_{4\times 1}$$

$$X_L \le X(t) \le X_U \qquad \forall t \in [0,T]$$

$$\eta_L \le \eta(t) \le \eta_U \qquad \forall t \in [0,T]$$

where T is the horizon of the problem (T is large w.r.t. the period of the sinusoidal wave denoted T_s). We have $X = (\alpha\ w\ \dot{\alpha}\ \dot{w})^\top$ where α is the pitch angle of the platform and w is the liquid displacement. The dynamics are detailed in (Coudurier et al., 2015, § 4.1). The control variable is η. It is positive at all times, so that $\eta_L = 0$. Also an upper bound $\eta_U = 1000$ is introduced to avoid numerical issues (a larger value for this parameter does not create any practical change in the solutions as the gain of the control variable is null for large values, asymptotically). To satisfy the model hypothesis stating that the vertical columns of the TLCD are never empty, we must have $X_U(2) = -X_L(2) = L_v$ with L_v the length of the vertical tubes of the TLCD as in Fig. 1, this is the only state constraint considered in this study . In practical numerical experimentations, this constraint is not active. We note that classically $X_0 \triangleq X(t=0)$. In Problem 1, F is the force created by the wave on the barge, which is a known time-varying signal (a sinusoid in this study).

2.2 Numerical resolution of the problem

We use the MATLAB toolbox ICLOCS (Imperial College of London Optimal Control Software) (Falugi et al., 2010) to solve optimal control problems with a direct approach (the dynamics of the system and the cost are discretized, and then the resulting finite-dimensional optimisation problem is solved with an interior point algorithm). The control signal is restricted to being a discrete-time signal sampled at 5 Hz (this sampling time being very small compared with the time constants of our system).

In Fig. 2, we report a typical solution $\eta(t)$ and $\alpha(t)$. One can see that the control is bang-bang (it commutes back and forth from η_L to η_U, which are the extremal admissible values) and it has two bangs per wave period. In the next section, we investigate this fact using analytic tools.

3. MATHEMATICAL ANALYSIS OF AN EQUIVALENT PROBLEM

The following investigations are conducted on a simplified version of Problem 1. The dynamics are structurally unchanged, but a change of input is introduced so that the dynamics become control-affine (with state-dependant gain).

Formally, the system at stake is a double mechanical oscillator with similar physical characteristics to those in Problem 1, but with significantly simpler dynamics. The ratio of the two masses is small. This system is called a *Toy Problem* and an illustration of this system is given in Fig. 3. The dynamics of this system are

$$M\ddot{x} + C\dot{x} + Kx = B_1(x)u(t) + E_1F(t)$$

with

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, \quad B_1(x) = \begin{pmatrix} 0 \\ c\dot{x}_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where x_1 (respectively x_2) is the displacement of the first (respectively second) mass.

We set

$$m_1 = 100\,\mathrm{kg} \qquad k_1 = 100\,\mathrm{N/m} \qquad c = 0.5\,\mathrm{N\,s/m}$$
$$m_2 = 2\,\mathrm{kg} \qquad k_2 = 3\,\mathrm{N/m}$$

The dynamics rewrite in a state space representation as

$$\dot{X} = AX + B(X)u(t) + EF(t)$$

with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}, \quad A = \begin{pmatrix} 0_{2\times 2} & \mathbb{I}_2 \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$$

$$E = \begin{pmatrix} 0_{2\times 1} \\ M^{-1}E_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0_{2\times 1} \\ M^{-1}B_1(X) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{c}{m_2}X_4 \end{pmatrix}$$

or, in details,

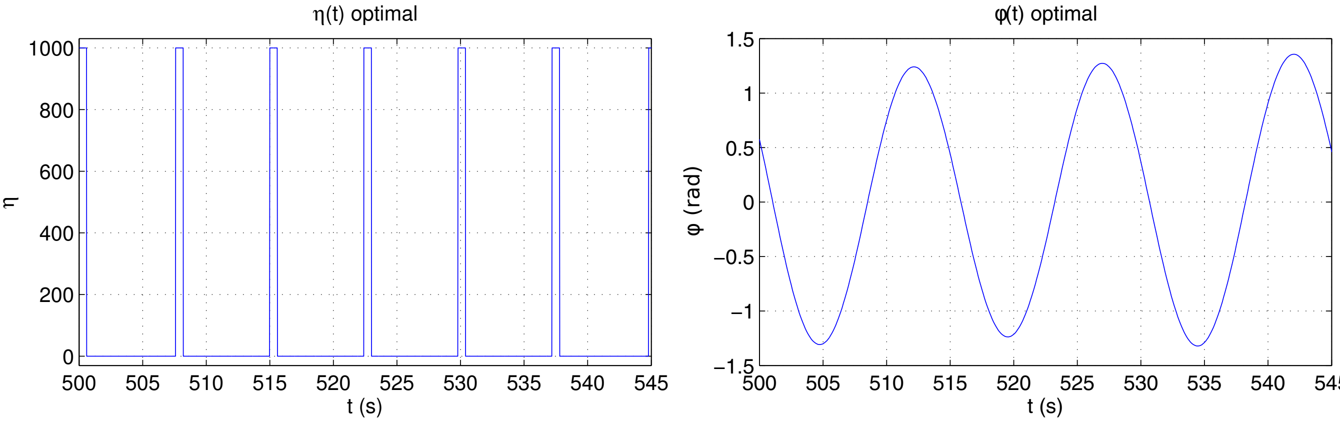

Fig. 2. Numerical solution for T_s=15 s

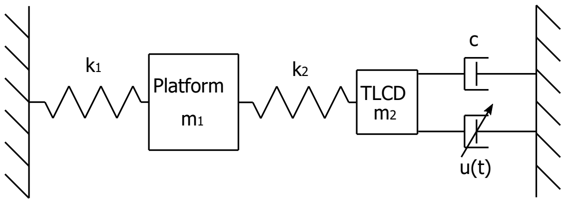

Fig. 3. Double mechanical oscillator *"Toy Problem"*

$$\dot{X}_1 = X_3 \tag{1}$$
$$\dot{X}_2 = X_4 \tag{2}$$
$$\dot{X}_3 = -\frac{(k_1+k_2)}{m_1}X_1 + \frac{k_2}{m_1}X_2 + \frac{1}{m_1}F(t) \tag{3}$$
$$\dot{X}_4 = \frac{k_2}{m_2}X_1 - \frac{k_2}{m_2}X_2 - \frac{c\,(1-u(t))}{m_2}X_4 \tag{4}$$

3.1 Optimal control problem

The following optimal control problem is considered

Problem 2. (Optimal control of the *Toy Problem*)

$$\min_{u(t),X_0} J(X,u) = \int_0^T \frac{1}{2} X(t)^\top Q\, X(t)\ \mathrm{d}t$$

subject to

$$\dot{X} = AX + B(X)u(t) + EF(t) \qquad \forall t \in [0,T]$$
$$X_0 = X(T)$$
$$-1 \le u(t) \le 1 \qquad \forall t \in [0,T]$$

where F is a sinusoid with a period of $T_s = 2\pi$ and T is the horizon of the problem. We set $Q_{11} = 1$ as the only non-zero coefficient of the cost matrix Q. Classically, the Hamiltonian of our problem is written as

$$H = -\frac{1}{2}X^\top Q\, X + \lambda^\top (AX + B(X)u(t) + EF(t)),$$

which depends in a linear manner on u; therefore, according to Pontryagin Minimum Principle (Pontryagin, 1987) u is bang-bang and it is defined as

$$u(\lambda, X) = \operatorname{sgn}\left(\lambda^\top B(X)\right) = \operatorname{sgn}(\lambda_4 B_4(X)) = \operatorname{sgn}(\lambda_4 X_4). \tag{5}$$

Classically, the other stationarity conditions give the adjoint dynamics

$$\dot{\lambda} = -\frac{\partial H}{\partial X} = -A^\top \lambda + QX - \lambda^\top \frac{\mathrm{d}B}{\mathrm{d}X} u \tag{6}$$

where $\lambda \in \mathbb{R}^4$ and

$$\frac{\mathrm{d}B}{\mathrm{d}X} = \begin{pmatrix} 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & \dfrac{c}{m_2} \end{pmatrix}.$$

We change the adjoint variables using $\lambda = P\mu$ such that $\dot{\mu}_1 = \mu_3$ and $\dot{\mu}_2 = \mu_4$. We select

$$P = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & \dfrac{k_2}{m_2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \dfrac{m_2}{m_1} & 0 & 0 & 1 \end{pmatrix}$$

With these new variables, we rewrite (5) and (6) as follows

$$u(\mu, X) = \operatorname{sgn}\left(\mu^\top P^\top B(X)\right) = \operatorname{sgn}\left(\left(\frac{\mu_1}{m_1} + \frac{\mu_4}{m_2}\right) X_4\right) \tag{7}$$

$$\dot{\mu} = P^{-1}\left(-A^\top P\mu + QX - \mu^\top P^\top \frac{\mathrm{d}B}{\mathrm{d}X} u\right) \tag{8}$$

We note that the system (8) is unstable as the real parts of the eigenvalues of A are negative. The equation (8) expands as

$$\dot{\mu}_1 = \mu_3 \tag{9}$$
$$\dot{\mu}_2 = \mu_4 \tag{10}$$
$$\dot{\mu}_3 = \frac{k_2}{m_2}\mu_4 - \frac{k_1}{m_1}\mu_1 - x_1 Q_{11} \tag{11}$$
$$\dot{\mu}_4 = \frac{c\,(1-u(\mu,X))}{m_1}\mu_4 + \frac{c\,(1-u(\mu,X))}{m_2}\mu_1 - \frac{k_2}{m_2}\mu_2 - \frac{m_2}{m_1}\mu_3 \tag{12}$$

We note that writing the adjoint variables under this form, i.e. finding the P matrix, is a step that could cause difficulties when attempting to treat the original FWT/TLCD system with averaging methods.

3.2 Sinusoidal form

We use averaging methods to solve Problem (2) (Arnold, 2013; Guckenheimer and Holmes, 1983). As the wave excitation is sinusoidal, the state of the system and the optimal control are periodic and can be written under the form $a(t)\cos(t + \phi(t))$ with $a(t)$ and $\phi(t)$ periodic. Without any loss of generality, we consider $T = 2\pi$. Therefore, we have

$$F = \varepsilon \sin(t)$$

and we search for solutions of the stationarity conditions under the form

$$\begin{aligned} X_1 &= a_1 \cos(t+\phi_1) & \mu_1 &= a_3 \cos(t+\phi_3) && (13)\\ X_2 &= a_2 \cos(t+\phi_2) & \mu_2 &= a_4 \cos(t+\phi_4) && (14)\\ X_3 &= -a_1 \sin(t+\phi_1) & \mu_3 &= -a_3 \sin(t+\phi_3) && (15)\\ X_4 &= -a_2 \sin(t+\phi_2) & \mu_4 &= -a_4 \sin(t+\phi_4) && (16) \end{aligned}$$

This periodic form implies that X and u stabilize the unstable system (8). Substituting (13–16) into (1–2) and (9–10) for $i = 1,..4$ yields

$$\dot{a}_i \cos(t+\phi_i) - a_i \dot{\phi}_i \sin(t+\phi_i) = 0 \tag{17}$$

Equations (3–4) and (11–12) for $i = 1,..4$ can be rewritten as

$$-\dot{a}_i \sin(t+\phi_i) - a_i \dot{\phi}_i \cos(t+\phi_i) = a_i \cos(t+\phi_i) + f_i(a,\phi,t) \tag{18}$$

where $f_i(a,\phi,t)$ are defined in (23–26). By solving the system having 8 equations and 8 unknown variables (17–18), we obtain

$$\begin{aligned} \dot{a}_i(a,\phi,t) &= -\left(a_i \cos(t+\phi_i) + f_i(a,\phi,t)\right)\sin(t+\phi_i)\\ \dot{\phi}_i(a,\phi,t) &= -\frac{1}{a_i}\left(a_i \cos(t+\phi_i) + f_i(a,\phi,t)\right)\cos(t+\phi_i). \end{aligned}$$

3.3 Averaging

As a (resp. ϕ) is periodic it can be written as the sum of its average value $\bar{a}$ (resp. $\bar{\phi}$) and an oscillating term εv. We are looking to find $\bar{a}$ and $\bar{\phi}$.

$$a_i = \bar{a}_i + \varepsilon v_i\left(\bar{a},\bar{\phi},t\right) + O\left(\varepsilon^2\right) \tag{19}$$

$$\phi_i = \bar{\phi}_i + \varepsilon w_i\left(\bar{a},\bar{\phi},t\right) + O\left(\varepsilon^2\right) \tag{20}$$

with

$$\dot{\bar{a}}_i = \frac{1}{T}\int_0^T \dot{a}_i\left(\bar{a},\bar{\phi},t\right)\mathrm{d}t + O\left(\varepsilon^2\right) \tag{21}$$

$$\dot{\bar{\phi}}_i = \frac{1}{T}\int_0^T \dot{\phi}_i\left(\bar{a},\bar{\phi},t\right)\mathrm{d}t + O\left(\varepsilon^2\right), \tag{22}$$

where $\bar{a}_i$ represents the mean part of a_i and $\varepsilon v_i\left(\bar{a},\bar{\phi},t\right)$ is the oscillating part. We have

$$\begin{aligned} u\left(\bar{a},\bar{\phi},t\right) &= -\operatorname{sgn}(\Pi)\\ \Pi &= \bar{a}_2 \sin(t+\bar{\phi}_2)\left(m_2\bar{a}_3\cos(t+\bar{\phi}_3) - m_1\bar{a}_4\sin(t+\bar{\phi}_4)\right)\\ f_1\left(\bar{a},\bar{\phi},t\right) &= -\frac{k_1+k_2}{m_1}\bar{a}_1\cos(t+\bar{\phi}_1)\\ &+ \frac{k_2}{m_1}\bar{a}_2\cos(t+\bar{\phi}_2) + \frac{\varepsilon}{m_1}\sin(t) && (23)\\ f_2\left(\bar{a},\bar{\phi},t\right) &= -\frac{k_2}{m_2}\bar{a}_2\cos(t+\bar{\phi}_2)\\ &+ \frac{k_2}{m_2}\bar{a}_1\cos(t+\bar{\phi}_1) + \frac{c\bar{a}_2}{m_2}\left(1-u\left(\bar{a},\bar{\phi},t\right)\right)\sin(t+\bar{\phi}_2) && (24)\\ f_3\left(\bar{a},\bar{\phi},t\right) &= -\frac{k_1}{m_1}\bar{a}_3\cos(t+\bar{\phi}_3)\\ &- \frac{k_2}{m_2}\bar{a}_4\sin(t+\bar{\phi}_4) - \bar{a}_1 Q_{11}\cos(t+\bar{\phi}_1) && (25)\\ f_4\left(\bar{a},\bar{\phi},t\right) &= -\bar{a}_4\frac{k_2}{m_2}\cos(t+\bar{\phi}_4)\\ &- \bar{a}_4\frac{c}{m_2}\left(1-u\left(\bar{a},\bar{\phi},t\right)\right)\sin(t+\bar{\phi}_4)\\ &+ \frac{\bar{a}_3}{m_1}c\left(1-u\left(\bar{a},\bar{\phi},t\right)\right)\cos(t+\bar{\phi}_3) + \bar{a}_3\frac{m_2}{m_1}\sin(t+\bar{\phi}_3) && (26) \end{aligned}$$

The term $\lambda^\top B(X)$, in (5) is a product of two sinusoids of period T, so it has a period of $T/2$, and thus u is bang-bang with a period of $T/2$. For the same reason, $\dot{a}_i\left(\bar{a},\bar{\phi},t\right)$ and $\dot{\phi}_i\left(\bar{a},\bar{\phi},t\right)$ also have a period of $T/2$. Therefore, we can rewrite (21) and(22) as follows.

$$\dot{\bar{a}}_i = \frac{2}{T}\int_0^{T/2} \dot{a}_i\left(\bar{a},\bar{\phi},t\right)\mathrm{d}t + O\left(\varepsilon^2\right) \tag{27}$$

$$\dot{\bar{\phi}}_i = \frac{2}{T}\int_0^{T/2} \dot{\phi}_i\left(\bar{a},\bar{\phi},t\right)\mathrm{d}t + O\left(\varepsilon^2\right) \tag{28}$$

According to (5), u switches when λ_4 or X_4 is zero. On the interval $\left[0, \frac{T}{2}\right]$ we have

$$\begin{aligned} &\lambda_4(t) = 0 \iff\\ &t = \arctan\left(\frac{m_1\bar{a}_4\sin(\bar{\phi}_3-\bar{\phi}_4) + m_2\bar{a}_3}{m_1\bar{a}_4\cos(\bar{\phi}_3-\bar{\phi}_4)}\right) - \bar{\phi}_3 + k_\lambda\pi \triangleq r_{\lambda_4}\\ &X_4(t) = 0 \iff t = -\bar{\phi}_2 + k_B\pi \triangleq r_{B_4} \end{aligned}$$

where $k_\lambda, k_B \in \mathbb{Z}$. We define $r_m \triangleq \min(r_{\lambda_4}, r_{B_4})$ and $r_M \triangleq \max(r_{\lambda_4}, r_{B_4})$.

Then, one can rewrite (27)-(28) as follows

$$\dot{\bar{a}}_1\left(\bar{a},\bar{\phi}\right) = -\frac{k_2\bar{a}_2}{2m_1}\sin\left(\bar{\phi}_1-\bar{\phi}_2\right) - \frac{\varepsilon}{2m_1}\cos(\bar{\phi}_1) + O\left(\varepsilon^2\right) \tag{29}$$

$$\begin{aligned} \dot{\bar{a}}_2\left(\bar{a},\bar{\phi}\right) &= u\left(\bar{a},\bar{\phi},0\right)\frac{c\bar{a}_2}{m_2 T}\times\\ &\left(\pi - 2\left(r_M - r_m\right) - \sin(2r_m + 2\bar{\phi}_2) + \sin(2r_M + 2\bar{\phi}_2)\right)\\ &+ \frac{\bar{a}_1 k_2}{2m_2}\sin(\bar{\phi}_1-\bar{\phi}_2) - \frac{\bar{a}_2 c}{2m_2} + O\left(\varepsilon^2\right) && (30)\\ \dot{\bar{a}}_3\left(\bar{a},\bar{\phi}\right) &= \frac{k_2}{2m_2}\bar{a}_4\cos(\bar{\phi}_3-\bar{\phi}_4)\\ &+ \frac{\bar{a}_1 Q_{11}}{2}\sin(\bar{\phi}_3-\bar{\phi}_1) + O\left(\varepsilon^2\right) && (31)\\ \dot{\bar{a}}_4\left(\bar{a},\bar{\phi}\right) &= \frac{\bar{a}_3}{2m_1}\left(c\sin(\bar{\phi}_3-\bar{\phi}_4) - m_2\cos(\bar{\phi}_3-\bar{\phi}_4)\right) + \frac{\bar{a}_4 c}{2m_2}\\ &+ u\left(\bar{a},\bar{\phi},0\right)\frac{c}{T}\frac{\bar{a}_3}{m_1}\times\\ &\left(-\pi - 2\left(r_M - r_m\right) - \cos(2r_m + \bar{\phi}_3 + \bar{\phi}_4) + \cos(2r_M + \bar{\phi}_3 + \bar{\phi}_4)\right)\\ &+ u\left(\bar{a},\bar{\phi},0\right)\frac{c}{T}\frac{\bar{a}_4}{m_2}\times\\ &\left(-\pi - 2\left(r_M - r_m\right) - \sin(2r_m + 2\bar{\phi}_4) + \sin(2r_M + 2\bar{\phi}_4)\right)\\ &+ O\left(\varepsilon^2\right) && (32) \end{aligned}$$

$\bar{a}_1$	$10.88\,\mu$m	$\bar{\phi}_1$	2.159
$\bar{a}_2$	$4.351\,\mu$m	$\bar{\phi}_2$	1.691
$\bar{a}_3$	$118.5\,10^{-6}$	$\bar{\phi}_3$	1.592
$\bar{a}_4$	$7.361\,10^{-6}$	$\bar{\phi}_4$	0.581

Table 1. Amplitudes and phases of the ICLOCS solution

$\bar{a}_1^{\text{avg}}$	$11.19\,\mu$m	$\bar{\phi}_1^{\text{avg}}$	2.137
$\bar{a}_2^{\text{avg}}$	$4.319\,\mu$m	$\bar{\phi}_2^{\text{avg}}$	1.710
$\bar{a}_3^{\text{avg}}$	$125.2\,10^{-6}$	$\bar{\phi}_3^{\text{avg}}$	$\pi/2$ ($\simeq 1.571$)
$\bar{a}_4^{\text{avg}}$	$7.459\,10^{-6}$	$\bar{\phi}_4^{\text{avg}}$	0.567

Table 2. Amplitudes and phases predicted by the averaging technique

$$\dot{\bar{\phi}}_1\left(\bar{a},\bar{\phi}\right) = \frac{1}{2}\left(\frac{k_1+k_2}{m_1}-1\right) - \frac{\bar{a}_2 k_2}{2\bar{a}_1 m_1}\cos(\bar{\phi}_1-\bar{\phi}_2) + \frac{\varepsilon}{2\bar{a}_1 m_1}\sin\left(\bar{\phi}_1\right) + O\left(\varepsilon^2\right) \tag{33}$$

$$\dot{\bar{\phi}}_2\left(\bar{a},\bar{\phi}\right) = u\left(\bar{a},\bar{\phi},0\right)\frac{c}{m_2 T}\times \left(-\cos(2r_m+2\bar{\phi}_2)+\cos(2r_M+2\bar{\phi}_2)\right) + \frac{1}{2}\left(\frac{k_2}{m_2}-1\right) - \frac{\bar{a}_1 k_2}{2\bar{a}_2 m_2}\cos(\bar{\phi}_1-\bar{\phi}_2) + O\left(\varepsilon^2\right) \tag{34}$$

$$\dot{\bar{\phi}}_3\left(\bar{a},\bar{\phi}\right) = \frac{1}{2}\left(\frac{k_1}{m_1}-1\right) + \frac{\bar{a}_1 Q_{11}}{2\bar{a}_3}\cos(\bar{\phi}_1-\bar{\phi}_3) - \frac{\bar{a}_4 k_2}{2\bar{a}_3 m_2}\sin(\bar{\phi}_3-\bar{\phi}_4) + O\left(\varepsilon^2\right) \tag{35}$$

$$\begin{aligned}\dot{\bar{\phi}}_4\left(\bar{a},\bar{\phi}\right) &= \frac{1}{2}\left(\frac{k_2}{m_2}-1\right) \\ &- \frac{\bar{a}_3}{2\bar{a}_4 m_1}\left(m_2\sin(\bar{\phi}_3-\bar{\phi}_4)+c\cos(\bar{\phi}_3-\bar{\phi}_4)\right) \\ &+ u\big(\bar{a},\bar{\phi},0\big)\frac{c}{T}\left(\cos(2r_m+2\bar{\phi}_4)-\cos(2r_M+2\bar{\phi}_4)+\frac{\bar{a}_3}{\bar{a}_4 m_1}\right) \\ &+ u\big(\bar{a},\bar{\phi},0\big)\frac{c}{T}\frac{\bar{a}_3}{\bar{a}_4 m_1}\times \\ &\big(-\cos(\bar{\phi}_3-\bar{\phi}_4)\left(-\pi-2\left(r_M-r_m\right)\right) \\ &\quad+\sin(2r_m+\bar{\phi}_3+\bar{\phi}_4)-\sin(2r_M+\bar{\phi}_3+\bar{\phi}_4)\big) + O\left(\varepsilon^2\right)\end{aligned} \tag{36}$$

Equations (29–36) are differential equations with equilibrium points, which define the quantities we are looking for. Below we detail the obtention of the numerical parameters of this averaged model.

3.4 Numerical results

For this numerical study, we set the oscillating force to be a sinusoid of amplitude $\varepsilon = 10^{-4}$N, which is insignificant for the mass of our system ($m_1 = 100$ kg and $m_2 = 2$ kg).

We search for the equilibrium point closest to the values identified based on the data provided by ICLOCS. The amplitudes and phases identified according to the numerical solution to Problem 2 are reported in Table 1.

This solution is actually very close to the solution estimated as the equilibrium point of the equations (29–36) obtained using the averaging technique, reported in Table 2.

Therefore, a trajectory exists [1] written as (13–16) and (19–20) close to that provided by ICLOCS verifying Pontryagin Minimum Principle. The results of ICLOCS are in accordance with the analysis obtained with the averaging method. We also confirmed that the optimal command is bang-bang with a period of $T/2$.

4. CONCLUSIONS AND PERSPECTIVES

In this study, we investigated the optimal control of a semi-active TLCD for damping a FWT subject to a sinusoidal wave. First, we numerically solved the optimal control problem. The optimal command appeared to have periodic nature, so we employed a simpler model with similar physical characteristics to the original problem in order to perform an analytic study of the optimal control using averaging methods. We showed that the optimal command was a bang-bang command with a period of $T/2$. We also found that the optimal control obtained analytically was very close to the command obtained by the numerical routine, which confirms the validity of the averaging technique employed here.

It seems very reasonable that an optimal control of a set of coupled oscillators subjected to a sinusoid input generates a periodic solution. What is less easy to anticipate is the frequency doubling effect. Interestingly, the frequency of the solution does not depend on the natural frequency of the system. Only its phases and amplitudes do.

We believe that this result is relatively general, and that one could benefit from studying the general question of optimal control of sets of oscillators, under periodic disturbances. More generally, if the disturbance is not monochromatic but contains more than one, say two, frequencies, the questions will certainly be more complex. If the frequencies are harmonic (i.e. define an integer ratio) then the calculations presented here could be generalized. The two frequencies will be coupled, implicitly, through the constraint, most likely involving some arithmetics connecting the various extremums of the signal. If the two frequencies define a rational ratio (ratio of two prime integers), the situation will also be governed by some arithmetic equation, defining the periodic distribution of extremums. However, these last two cases, of theoretical interest, are not well-suited to cover practical problems where the waves have widespread spectrum. For such cases, the questions of replacing a genuine real-time optimal control solver by an analytically derived approximate solution as is done in this article remains vastly opened.

However, the case of single frequency wave is of interest for applications. Future studies could use the properties of the optimal control to develop computationally efficient model predictive control (Mayne et al., 2000) to control the restriction of the TLCD in association with short-term wave forecasting algorithms (Fusco and Ringwood, 2010) with some adaptation on the frequency which could be considered as a slowly-varying parameter.

[1] If we initialize the `fsolve` function with random values, the algorithm either does not converge or it converges to the values given in Table 2. Therefore, it is reasonable to assume that this equilibrium point is unique.

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