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Adaptive control scheme for uncertain time-delay systems☆

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ABSTRACT

The paper exposes some of the potential of a recently introduced backstepping transformation for linear uncertain time-delay systems to address the classic problems of equilibrium regulation under partial measurements, disturbance rejection, parameter or delay adaptation. For each of these problems, an implementable control strategy is proposed. It is analyzed through a convergence analysis of infinite dimensional dynamics, based on a transport partial differential equation representation of the estimated input delay and the mentioned backstepping transformation. The considered controllers contain this representation of the system, under the form of a distributed parameter system, and use it to determine relevant adaptation strategies. An illustration example is proposed.

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1. Introduction

This paper addresses the general problem of equilibrium regulation of (potentially) unstable linear systems with an uncertain input time-delay. For such systems, predictor approaches (see e.g. in Artstein, 1982, Kwon & Pearson, 2002, Manitius & Olbrot, 2002 and Smith, 1959) are often considered. As established in numerous recent surveys and research papers (e.g. Huang & Lin, 1995, Richard, 2003 or Zhong, 2006), the lack of robustness of this technique with respect to the uncertainty on the delay is still a major concern in automatic control theory, especially in view of implementation, in which it often appears as a performance bottleneck (Mondie & Michiels, 2003).

Lately (see Krstic, 2008; Krstic & Bresch-Pietri, 2009; Krstic & Smyshlyaev, 2008 or also Krstic, 2009), a new class of predictor-based techniques has been proposed to address this uncertainty for single input time-delay systems. This methodology is based on a modeling of the actuator delay as a transport partial differential equation (PDE), assuming that the full actuator state (i.e. the past values of the input) is known over an interval of length equal to the delay, as in Bresch-Pietri and Krstic (2009) for example.

In this paper, we follow this methodology and develop an implementable form of the resulting controller in the spirit of Bresch-Pietri and Krstic (2010). In details, we use a backstepping boundary control method on a transport PDE introduced to model the delay. This transformation allows to use systematic Lyapunov design tools for robust stabilization and adaptation. Several classic cases are treated. While the resulting controllers could seem intuitive in their formulation, the results stated here and the corresponding proofs are, up to our knowledge, totally new.

A series of classic issues in linear automatic control is considered: model uncertainties, disturbance rejection and partial state measurement. We address these difficulties separately and propose a dedicated implementable solution for each of them, along with analysis of convergence that stress the role of the various adaptation and feedback components. The goal of this paper is to present a unified framework of these various techniques, for the sake of comparisons of their merits and limitations in the light of their mathematical analysis. In view of application, the interested reader and the practitioner can simply make its own selection to address a vast class of possible problems.

The paper is organized as follows. In Section 2, we formulate the general problem under consideration, along with basic assumptions used throughout the article. The considered control problem is briefly discussed. Then, in Section 3, we introduce the uncertain distributed parameter system representation of the delay system which was originally proposed in Bresch-Pietri and Krstic (2010) and emphasize its relation with a prediction feedback controller. In Section 4, we focus on delay on-line adaptation scheme before, in Section 5, studying the structure of the proposed Lyapunov proof of convergence to emphasize common points and differences between the considered cases. Then, at the light of this analysis, in Section 6, we employ an output feedback design. In Section 7, we introduce parameter adaptation. In Section 8,

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we propose a disturbance rejection strategy. In Section 9, the versatility of the proposed approach is underlined by a simulation example covering various cases.

Notations

In the following, m, n and p are strictly positive integers, $|\cdot|$ refers to the usual Euclidean norm whereas the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are defined as

$$\|f(t)\| = \sqrt{\int_0^1 f(x,t)^2 dx}, f : (x,t) \in [0;1] \times \mathbb{R}_+ \to \mathbb{R}$$

$$\|f\|_\infty = \sup_{\hat{\theta} \in \Pi} |f(\hat{\theta})|, f : \Pi \to \mathbb{R}^l \quad (l \in \mathbb{N}^*).$$

The matrix norm is defined, for $M \in \mathcal{M}_l(\mathbb{R})$ $(l \in \mathbb{N}^*)$, as $|M| = \sup_{|x|\leq 1} |Mx|$.

For sake of analysis, several positive constants $M_1, \ldots, M_{11}$, two functions f and g and a functional V_0 are introduced (and redefined) in each section.

2. Problem statement

We consider the following potentially open-loop unstable linear delay system

$$\dot{X}(t) = A(\theta)X(t) + B(\theta)[U(t-D) + d] \tag{1}$$

$$Y(t) = CX(t), \tag{2}$$

where $Y \in \mathbb{R}^m$, $X \in \mathbb{R}^n$ and U is a scalar input. $D > 0$ is an unknown (potentially long) constant delay, d is a constant input disturbance and we assume that the system matrix $A(\theta)$ and the input vector $B(\theta)$ are linearly parametrized under the form

$$A(\theta) = A_0 + \sum_{i=1}^{p} A_i \theta_i \quad \text{and} \quad B(\theta) = B_0 + \sum_{i=1}^{p} B_i \theta_i, \tag{3}$$

where θ is a constant parameter belonging to a convex closed set $\Pi = \{\theta \in \Pi | \mathbb{R}^p(\theta) \leq 0\} \subset \mathbb{R}^p$, where $\mathscr{P} : \mathbb{R}^p \to \mathbb{R}$ is a smooth convex function.

The control objective is to have system (1)–(2) track a given constant set point Y^r despite uncertainties on the delay D and several other difficulties the system may encounter: (i) uncertainty on θ, (ii) unmeasured state X of the plant, (iii) unknown input disturbance d. Each of these situations represents a different challenge and calls for the introduction of some sort of adaptation (in practice an estimate). This paper successively addresses the three presented problems separately. Each of them is theoretically studied, through a formal proof of convergence. For applications, it is certainly necessary to combine some of the proposed techniques. Convergence of every possible combination is not in the scope of this paper. We refer the interested reader to some previous works (Bresch-Pietri, Chauvin, & Petit, 2010, 2011a,b) where several combinations have been studied, both theoretically and practically on industrial applications. In particular, one theoretical question of interest is the combination of unknown parameter with constant disturbance rejection.

Following (Bresch-Pietri & Krstic, 2009), several assumptions are formulated. It is considered that they hold throughout the paper. The first two yield well-posedness of the problem, while the last one serves in the Lyapunov analysis.

Assumption 1. The set Π is known and bounded. An upper bound $\bar{D}$ and a lower bound $\underline{D} > 0$ of the delay D are known.

Assumption 2. For a given set point Y^r, there exists known functions $X^r(\theta)$ and $U^r(\theta)$ continuously differentiable in the parameter $\theta \in \Pi$ which satisfy, for all $\theta \in \Pi$,

$$0 = A(\theta)X^r(\theta) + B(\theta)U^r(\theta) \tag{4}$$

$$Y^r = CX^r(\theta). \tag{5}$$

Assumption 3. The pair $(A(\theta), B(\theta))$ is controllable for every $\theta \in \Pi$ and there exists a triple of vector/matrix functions $(K(\theta), P(\theta), Q(\theta))$ such that, for all $\theta \in \Pi$,

(i) $P(\theta)$ and $Q(\theta)$ are positive definite and symmetric
(ii) the following Lyapunov equation is satisfied
$$P(\theta)(A+BK)(\theta) + (A+BK)(\theta)^T P(\theta) = -Q(\theta)$$
(iii) $(K, P) \in C^1(\Pi)^2$ and $Q \in C^0(\Pi)$.

Assumption 4. The following quantities are well-defined

$$\underline{\lambda} = \inf_{\theta \in \Pi} \min\{\lambda_{\min}(P(\theta)), \lambda_{\min}(Q(\theta))\}$$

$$\bar{\lambda} = \sup_{\theta \in \Pi} \lambda_{\max}(P(\theta)).$$

Only one of these assumptions is restrictive: Assumption 3 requires the equivalent delay-free form of the system (1)–(2) to be controllable. This is a reasonable assumption to guarantee the possibility of regulation about the constant reference Y^r. As a final remark, we wish to stress that neither the considered reference U^r, nor the state reference X^r depend on time or delay, because the reference Y^r is constant. This point is important in the control design.

3. Distributed input estimate

We start our analysis by introducing the distributed input $u(x,t) = U(t + D(x-1))$, $x \in [0,1]$. The plant (1)–(2) can be represented under the form

$$\dot{X}(t) = A(\theta)X(t) + B(\theta)[u(0,t) + d] \tag{6}$$

$$Du_t(x,t) = u_x(x,t) \tag{7}$$

$$u(1,t) = U(t) \tag{8}$$

$$Y = CX(t) \tag{9}$$

where the delay is accounted for by the transport equation (a first-order hyperbolic PDE) whose speed of propagation is $1/D$. Unfortunately, because this speed is uncertain, even if the applied input $U(t)$ is fully known, one cannot deduce the value of $u(x,t)$ for each $x \in [0,1]$ from it. Consequently, if this distributed input is not measured (which is seldom the case in applications), one needs to introduce an estimate $\hat{u}(x,t) = U(t + \hat{D}(t)(x-1))$ of the distributed input, using an estimate $\hat{D}$ of the delay,

$$\hat{D}(t)\hat{u}_t(x,t) = \hat{u}_x(x,t) + \dot{\hat{D}}(t)(x-1)\hat{u}_x(x,t) \tag{10}$$

$$\hat{u}(1,t) = U(t). \tag{11}$$

These elements above have been presented in the paper Bresch-Pietri and Krstic (2010) and are the base of the following proposed methodology. They are represented as gray blocks in Fig. 1.

As will appear in Section 5.4.2, the Lyapunov-based synthesis of the delay update law introduced in Bresch-Pietri and Krstic (2009) and Krstic and Bresch-Pietri (2009) cannot be done in this context, i.e. without the knowledge of the distributed input u. Instead, to improve the delay estimation, one can use an optimization-based update law (e.g. gradient methods) or exploit delay stochastic properties, in the sense of the methods presented in O'Dwyer (2000). These techniques are shown here to be compliant with the proposed adaptive scheme, but at the expense of extra assumptions bearing on the delay estimate initialization (required

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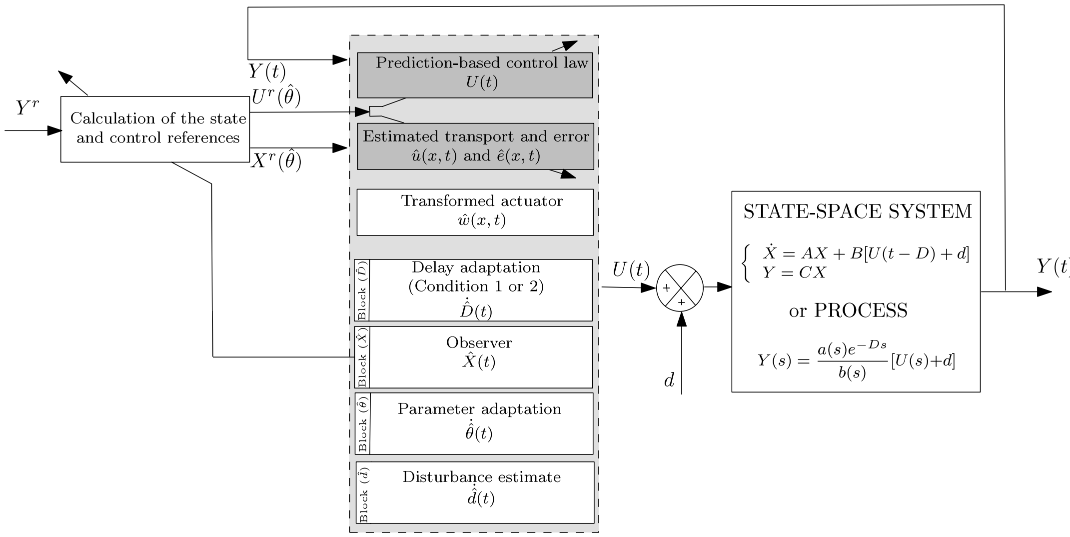

Fig. 1. The proposed adaptive control scheme. The closed-loop algorithm still uses a prediction-based control law jointly with distributed parameters, i.e. the estimated waiting line (in gray, Section 3 also presented in Bresch-Pietri and Krstic (2010)). According to the context, it can use a combination of the remaining blocks (in white), namely a delay estimate update law (Section 4) or a parameter estimate one (Section 7) or a system state observer (Section 6) or a disturbance estimate (Section 8). This may also requires the computation of the transformed state of the actuator.

to be close enough to the unknown delay value). In this context, one might as well prefer to choose a constant delay estimate. This is why (besides pedagogical reasons), in the following, we present first and separately a control strategy using a varying delay estimate and choose it as constant after that.

When D, d and θ are perfectly known and X is measured, the following controller (see Artstein, 1982 and Smith, 1959) achieves asymptotic stabilization of system (1) toward 0

$$U(t) = KX_P(t+D) - d = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-s)}BU(s)ds\right] - d. \tag{12}$$

This controller can be viewed as a delay-version of the delay-free controller $U(t) = KX(t) - d$, where $X_P(t+D)$ should be understood as a D-units of time ahead prediction of the system state, starting from $X(t)$ as initial condition, and driven by the control history over the D-units of time window. The presence of the term d aims at counteracting the input bias in (1).

This prediction control has been interpreted in Krstic (2008) as the result of a backstepping transformation of the transport equation[2] (7) and systematic Lyapunov tools have been used on it, to analyze the asymptotic stability of input time-delay systems (e.g. in Bresch-Pietri & Krstic, 2009 and Bresch-Pietri et al., 2010). These are the elements we propose to develop further. In the following, applying the certainty equivalence principle, we employ different versions of the controller (12), depending on the case under consideration, jointly with the corresponding backstepping transformation of the distributed input.

Finally, we introduce several error variables used below

$$\tilde X(t) = X(t) - X^r(\hat\theta) \tag{13}$$

$$e(x,t) = u(x,t) - u^r(\hat\theta) \tag{14}$$

$$\hat e(x,t) = \hat u(x,t) - u^r(\hat\theta) \tag{15}$$

$$\tilde e(x,t) = u(x,t) - \hat u(x,t). \tag{16}$$

In details, (13) represents the tracking errors, (14) and (15) are the distributed input tracking errors, and, (16) is the distributed input estimation error. These quantities characterize (possibly only partially, depending on the considered case) both the system state and the control performances.

We now pursue with a first case that we treat in details.

4. Control strategy with time-delay on-line update law

In this section, we focus on the delay estimation and its integration in the proposed prediction-based control. With this aim in view, we assume here that both θ and d are known ($d = 0$ for sake of simplicity, as it can be easily compensated for) and that X is measured. For sake of conciseness, we denote $A = A(\theta)$, $B = B(\theta)$ and so on for every quantity depending on $\hat\theta = \theta$.

The main result of this section (Theorem 1) is based on Conditions 1 and 2 presented below, which allow to consider a relatively general delay update law.

4.1. Control design

Following (12) and the certainty equivalence principle, we employ the control law

$$U(t) = U^r - KX^r + K\hat D(t)\int_0^1 e^{A\hat D(t)(1-x)}B\hat u(x,t)dx + Ke^{A\hat D(t)}X(t) \tag{17}$$

where the update law of the delay estimate is characterized by the following "growth condition".

Condition 1. *There exists positive constants $\gamma_D > 0$ and $M > 0$ such that*

$$\dot{\hat D}(t) = \gamma_D \mathrm{Proj}_{[\underline{D},\bar D]}\{\tau_D(t)\}$$

$$|\tau_D(t)| \le M\left(|\tilde X(t)|^2 + \|e(t)\|^2 + \left\|\hat e(t)\right\|^2 + \left\|\hat e_x(t)\right\|^2\right)$$

where $\mathrm{Proj}_{[\underline{D},\bar D]}$ is the standard projection operator on the interval $[\underline{D},\bar D]$.

[2] This transformation converts the plant (6)–(9) into the target system $\dot X(t) = (A+BK)X(t) + Bw(0,t)$, $Y(t) = CX(t)Dw_t(x,t) = w_x(x,t)$ with the boundary condition $w(1,t) = 0$.

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Condition 2. *There exists positive constants $\gamma_D > 0$ and $M > 0$ such that*

$$\dot{\hat{D}}(t) = \gamma_D \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau_D(t)\}$$
$$\forall t \geq 0,\ \tau_D(t)\tilde{D}(t) \geq 0 \quad \text{and} \quad |\tau_D(t)| \leq M$$

where $\mathrm{Proj}_{[\underline{D},\bar{D}]}$ *is the standard projection operator on the interval* $[\underline{D}, \bar{D}]$.

The following result can be found in a less general form in the seminal paper Bresch-Pietri and Krstic (2010). Despite the fact that this paper only addresses the stabilization problem of the plant, the main difference is the form of the employed delay update law, as Bresch-Pietri and Krstic (2010) only proposes a particular one,[3] whereas both Conditions 1 and 2 allow to consider a large amount of it. This more general result serves as prototype for the rest of the article.

Theorem 1. *Consider the closed-loop system consisting of (6)–(9), the control law (17), with the estimate actuator state (10)–(11) with a delay update law satisfying either Condition 1 or Condition 2. Let us define*

$$\Gamma_1(t) = |\tilde{X}(t)|^2 + \|e(t)\|^2 + \left\|\hat{e}(t)\right\|^2 + \left\|\hat{e}_x(t)\right\|^2 + \tilde{D}(t)^2.$$

Then, there exists $\gamma^* > 0$, $R > 0$ and $\rho > 0$ such that, if $0 < \gamma_D < \gamma^*$ and if the initial state satisfies $\Gamma_1(0) < \rho$ then

$$\forall t \geq 0, \quad \Gamma_1(t) \leq R\Gamma_1(0) \tag{18}$$

$$Y(t) \underset{t\to\infty}{\rightarrow} Y^r, \qquad X(t) \underset{t\to\infty}{\rightarrow} X^r \quad \text{and} \quad U(t) \underset{t\to\infty}{\rightarrow} U^r. \tag{19}$$

Condition 1 allows to update the delay estimate while guaranteeing the stability property (18) of the controller. This condition cannot be checked directly, as some of the signals involved in the upper bound are in fact unavailable. For strict implementability, a constructive choice could be to satisfy the more restrictive assumption $\tau_D(t) \leq M\left(|\tilde{X}(t)| + \left\|\hat{e}(t)\right\|^2 + \left\|\hat{e}_x(t)\right\|^2\right)$. On the other hand, Condition 2 allows to consider more sharp update laws provided that they point in the direction of estimation improvement, which is consistent with numerous delay identification techniques (see e.g. in O'Dwyer, 2000).

In a nutshell, employing a time-delay on-line update-law satisfying Condition 2 would hopefully provide an identification of the unknown delay and then allow a larger leeway for control (i.e. advanced feedforward strategies). Condition 1 would then, in this context, ensure that small computational errors in the delay update law do not jeopardize the stability of the controller.

For both cases, Theorem 1 requires this delay update law to be slow enough ($\gamma_D < \gamma^*$) not to interfere maliciously with the controller.

Theorem 1 also introduces a functional Γ_1, which evaluates the system error. Equivalently, the second condition ($\Gamma_1(0)$ being sufficiently small) requires that, initially, each of the state variables are sufficiently close to their corresponding trajectories (namely, X^r, U^r and the unknown delay D). In particular, one can notice the presence of the spatial derivative of the estimate queue $\hat{e}_x$ in the statement. Indeed, this quantity is involved in the state variables dynamics presented below, as a result of the estimation of the distributed input.

[3] This particular delay update law originates from the case where the (infinite) state of the transport PDE is known, applying the certainty equivalence principle. In the case of regulation, this update law can be expressed as

$$\tau_D(t) = -\int_0^1 (1+x)\hat{w}(x,t)Ke^{A\hat{D}(t)x}dx\left[A\tilde{X}(t) + B\hat{e}(0,t)\right]$$

which can be shown to satisfy Condition 1.

4.2. Lyapunov analysis

In the following, we use the backstepping transformation of the actuator state, satisfying a Volterra integral equation of the second kind,

$$\hat{w}(x,t) = \hat{e}(x,t) - K\hat{D}(t)\int_0^x e^{A\hat{D}(t)(x-y)}B\hat{e}(y,t)dy - Ke^{A\hat{D}(t)x}\tilde{X}(t) \tag{20}$$

jointly with the inverse transformation

$$\hat{e}(x,t) = \hat{w}(x,t) + K\hat{D}(t)\int_0^x e^{(A+BK)\hat{D}(t)(x-y)}B\hat{w}(y,t)dy + Ke^{(A+BK)\hat{D}(t)x}\tilde{X}(t) \tag{21}$$

designed to fulfill the boundary condition $\hat{w}(1,t) = 0$, from the chosen control law (17). This property is suitable for Lyapunov analysis and motivates the definition of the following functional candidate

$$V_1(t) = \tilde{X}(t)^T P\tilde{X}(t) + b_1 D\int_0^1 (1+x)\tilde{e}(x,t)^2dx + b_2\hat{D}(t)\int_0^1 (1+x)\hat{w}(x,t)^2dx + b_2\hat{D}(t)\int_0^1 (1+x)\hat{w}_x(x,t)^2dx + \tilde{D}(t)^2 \tag{22}$$

where P is defined in Assumption 3 and $b_{1,1}$ and $b_{1,2}$ are positive coefficients. The boundary conditions of the set $(\tilde{e}, \hat{w}, \hat{w}_x)$ are handy through integrations by parts, involving the factor $(1+x)$ under the integrals, to create bounding negative terms.

First, consider the dynamics of the variables involved in (22), which can be written, using (20) and (21), as

$$\dot{\tilde{X}}(t) = (A+BK)\tilde{X}(t) + B\tilde{e}(0,t) + B\hat{w}(0,t) \tag{23}$$

$$D\tilde{e}_t(x,t) = \tilde{e}_x(x,t) - \tilde{D}(t)f(x,t) - \dot{\hat{D}}(t)D(x-1)f^1(x,t) \tag{24}$$

$$\tilde{e}(1,t) = 0 \tag{25}$$

$$\hat{D}(t)\hat{w}_t(x,t) = \hat{w}_x(x,t) - \hat{D}(t)\dot{\hat{D}}(t)g(x,t) - \hat{D}(t)Ke^{A\hat{D}(t)x}B\tilde{e}(0,t) \tag{26}$$

$$\hat{w}(1,t) = 0 \tag{27}$$

$$\hat{D}(t)\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t) - \hat{D}(t)\dot{\hat{D}}(t)g_x(x,t) - \hat{D}(t)^2KAe^{A\hat{D}(t)x}B\tilde{e}(0,t) \tag{28}$$

$$\hat{w}_x(1,t) = \hat{D}(t)\dot{\hat{D}}(t)g(1,t) + \hat{D}(t)Ke^{A\hat{D}(t)}B\tilde{e}(0,t) \tag{29}$$

with the functions f and g expressed, thanks to (20) and (21), with the set of variables $(\tilde{e}, \hat{w}, \hat{w}_x)$ as given in Appendix A. Taking a time-derivative of V_1 and using suitable integrations by parts, one obtains with (23)–(29)

$$\dot{V}_1(t) = -\tilde{X}(t)^T Q\tilde{X}(t) + 2\tilde{X}(t)^T PB\left[\tilde{e}(0,t) + \hat{w}(0,t)\right] + b_1\Bigg(-\left\|\tilde{e}(t)\right\|^2 - \tilde{e}(0,t)^2 - 2\tilde{D}(t) \times \int_0^1 (1+x)f(x,t)\tilde{e}(x,t)dx + 2\dot{\hat{D}}(t)D\int_0^1 (1-x^2)f(x,t)\tilde{e}(x,t)dx\Bigg)$$

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$$+ b_2\Bigg(- \left\|\hat{w}(t)\right\|^2 - \hat{w}(0,t)^2 - 2\hat{D}(t)\dot{\hat{D}}(t)\int_0^1 (1+x)g(x,t)\hat{w}(x,t)dx - 2\hat{D}(t)\int_0^1 (1+x)Ke^{A\hat{D}(t)x}B\tilde{e}(0,t)\hat{w}(x,t)dx\Bigg)$$
$$+ b_2\Bigg(2\hat{w}_x(1,t)^2 - \hat{w}_x(0,t)^2 - \left\|\hat{w}_x(t)\right\|^2 - 2\hat{D}(t)\dot{\hat{D}}(t)\int_0^1 (1+x)g_x(x,t)\hat{w}_x(x,t)dx - 2\hat{D}(t)^2\int_0^1 (1+x)KAe^{A\hat{D}(t)x}B\tilde{e}(0,t)\hat{w}_x(x,t)dx\Bigg)$$
$$+ b_2\dot{\hat{D}}(t)\int_0^1 (1+x)[\hat{w}(x,t)^2 + \hat{w}_x(x,t)^2]dx - 2\tilde{D}(t)\dot{\hat{D}}(t).$$

The magnitude of the resulting non-negative terms can be bounded. Indeed, using Young and Cauchy–Schwarz inequalities, one can obtain the following inequalities, where $M_1, \ldots, M_6$ are positive constants independent on initial conditions,

$$2\tilde{X}(t)^T PB\left[\tilde{e}(0,t) + \hat{w}(0,t)\right] \le \frac{\lambda_{\min}(Q)}{2}|\tilde{X}(t)|^2 + \frac{4\left\|PB\right\|^2}{\lambda_{\min}(Q)}(\tilde{e}(0,t)^2 + \hat{w}(0,t)^2)$$
$$2\int_0^1 (1+x)|f(x,t)\tilde{e}(x,t)|dx \le M_1\left(|\tilde{X}(t)|^2 + \left\|\tilde{e}(t)\right\|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2\right)$$
$$2\int_0^1 (1-x^2)|f(x,t)\tilde{e}(x,t)|dx \le M_1\left(|\tilde{X}(t)|^2 + \left\|\tilde{e}(t)\right\|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2\right)$$
$$2\hat{D}(t)\int_0^1 (1+x)|g(x,t)\hat{w}(x,t)|dx \le M_2\left(|\tilde{X}(t)|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2\right)$$
$$2\hat{D}(t)\int_0^1 (1+x)\left|Ke^{A\hat{D}(t)x}B\tilde{e}(0,t)\hat{w}(x,t)\right|dx \le M_3\tilde{e}(0,t)^2 + \left\|\hat{w}(t)\right\|^2/2$$
$$2\hat{w}_x(1,t)^2 \le M_4\left(\dot{\hat{D}}(t)^2\left(|\tilde{X}(t)|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2\right) + \tilde{e}(0,t)^2\right)$$
$$2\hat{D}(t)\int_0^1 (1+x)|g_x(x,t)\hat{w}_x(x,t)|dx \le M_5\left(|\tilde{X}(t)|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2 + \hat{w}_x(0,t)^2\right)$$
$$2\hat{D}(t)^2\int_0^1 (1+x)\left|KAe^{A\hat{D}(t)x}B\tilde{e}(0,t)\hat{w}_x(x,t)\right|dx \le M_6\tilde{e}(0,t)^2 + \left\|\hat{w}_x(t)\right\|^2/2.$$

Consequently, if one defines $V_0(t) = |\tilde{X}(t)|^2 + \left\|\tilde{e}(t)\right\|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2$, the previous equality yields

$$\dot{V}_1(t) \le -\frac{\lambda_{\min}(Q)}{2}|\tilde{X}(t)|^2 - \left(b_2 - \frac{4\left\|PB\right\|^2}{\lambda_{\min}(Q)}\right)\hat{w}(0,t)^2 - \left(b_1 - \frac{4\left\|PB\right\|^2}{\lambda_{\min}(Q)} - b_2M_3 - b_2M_4 - b_2M_6\right)$$
$$\times \tilde{e}(0,t)^2 - b_1\left\|\tilde{e}(t)\right\|^2 - \frac{b_2}{2}\left\|\hat{w}(t)\right\|^2 - \frac{b_2}{2}\left\|\hat{w}_x(t)\right\|^2 + \left(b_1|\tilde{D}(t)|M_1 + b_1\bar{D}|\dot{\hat{D}}(t)|M_1 + b_2|\dot{\hat{D}}(t)|M_2 + b_2M_4\dot{\hat{D}}(t)^2 + b_2M_5|\dot{\hat{D}}(t)| + 2b_2|\dot{\hat{D}}(t)|\right)V_0(t)$$
$$- 2\tilde{D}(t)\dot{\hat{D}}(t) - b_2\left(1 - |\dot{\hat{D}}(t)|M_5\right)\hat{w}_x(0,t)^2. \tag{30}$$

To make the terms in $\tilde{e}(0,t)^2$ and $\hat{w}(0,t)^2$ vanish, one can choose the constant coefficients b_1 and b_2 such that $b_2 = \frac{8\|PB\|^2}{\lambda_{\min}(Q)}$ and $b_1 > b_2\left(\frac{1}{2} + M_3 + M_4 + M_6\right)$ The techniques to treat the remaining non-negative terms are slightly depending on which condition is satisfied (Condition 1 or Condition 2). We now distinguish the two cases.

4.2.1. *Condition 1*

First, considering (20)–(21) and applying Young inequality, one can establish the following inequalities

$$\left\|\hat{e}(t)\right\|^2 \le r_1|\tilde{X}(t)|^2 + r_2\left\|\hat{w}(t)\right\|^2 \tag{31}$$
$$\left\|\hat{e}_x(t)\right\|^2 \le r_3|\tilde{X}(t)|^2 + r_4\left\|\hat{w}(t)\right\|^2 + r_5\left\|\hat{w}_x(t)\right\|^2 \tag{32}$$
$$\left\|\hat{w}(t)\right\|^2 \le s_1|\tilde{X}(t)|^2 + s_2\left\|\hat{e}(t)\right\|^2 \tag{33}$$
$$\left\|\hat{w}_x(t)\right\|^2 \le s_3|\tilde{X}(t)|^2 + s_4\left\|\hat{e}(t)\right\|^2 + s_5\left\|\hat{e}_x(t)\right\|^2 \tag{34}$$

where $r_1, r_2, r_3, r_4, r_5, s_1, s_2, s_3, s_4$ and s_5 are positive constants. Using (31) and (32), Condition 1 can be reformulated as (with $M_1 > 0$)

$$|\dot{\hat{D}}(t)| \le \gamma_D M V_0(t)$$

which yields, with $\eta = \min\{\lambda_{\min}(Q)/2, b_1, b_2/2\}$,

$$\dot{V}_1(t) \le -\left(\eta - b_1|\tilde{D}(t)|M_1 - 2|\tilde{D}(t)|\gamma_D M_1\right)V_0(t) + \gamma_D M\left(b_2M_2 + b_1\bar{D}M_1 + b_2M_5 + 2b_2\right)V_0(t)^2 + b_2M_4\gamma_D^2M^2V_0(t)^3 - b_2\left(1 - \gamma_D MV_0(t)M_5\right)\hat{w}_x(0,t)^2.$$

Further, we employ the following bound, where $\varepsilon_1 > 0$,

$$|\tilde{D}(t)| \le \frac{\varepsilon_1}{2} + \frac{1}{2\varepsilon}\left(V_1(t) - \eta_1 V_0(t)\right) \tag{35}$$

and obtain

$$\dot{V}_1(t) \le -\left(\eta - (b_1M_1 + 2\gamma_D M)\left(\frac{\varepsilon_1}{2} + \frac{V_1(t)}{2\varepsilon_1}\right)\right)V_0(t) - \left((b_1M_1 + 2\gamma_D M_1)\frac{\eta_1}{2\varepsilon_1} - \gamma_D M(b_2M_2 + b_1\bar{D}M_1 + b_2M_5 + 2b_2) - b_2M_4\gamma_D^2M^2V_0(t)\right)V_0(t)^2 - b_2\left(1 - \gamma_D MV_0(t)M_5\right)\hat{w}_x(0,t)^2.$$

By choosing the parameter ε_1 such that

$$\varepsilon_1 < \min\left\{\frac{2\eta}{b_1M_1 + 2\gamma_D M} \times \frac{\eta_1(b_1M + 2\gamma_D M)}{2\gamma_D M(b_2M_2 + b_1\bar{D}M_1 + b_2M_5 + 2b_2)}\right\}$$

and restricting the initial condition as

$$V_1(0) < \min\left\{\varepsilon_1\left(\frac{2\eta}{b_1M_1+2\gamma_D M}-\varepsilon_1\right),\right. \times \frac{\eta_1}{b_2M_4\gamma_D^2M^2}\left((b_1M_1+2\gamma_D M)\frac{\eta_1}{2\varepsilon_1}-\gamma_D M(b_2M_2 + b_1\bar{D}M_1+b_2M_5+2b_2)\right), \left.\frac{\eta_1}{\gamma_D MM_5}\right\}$$

we conclude that there exists non-negative functions μ_1 and μ_2 such that

$$\dot{V}_1(t) \leq -\mu_1(t)V_0(t)-\mu_2(t)V_0(t)^2 \quad (36)$$

and finally

$$\forall t\geq 0, \quad V_1(t)\leq V_1(0). \quad (37)$$

4.2.2. Condition 2

Inequality (30) gives, together with (35)

$$\dot{V}_1(t) \leq -\left(\eta - b_1M_1\left(\frac{\varepsilon_1}{2}+\frac{V_1(t)}{2\varepsilon_1}\right)-\gamma_D M \times (b_1\bar{D}M_1+b_2(M_2+\gamma_D MM_4+M_5+2))\right)V_0(t) - b_2(1-\gamma_D MM_5)\,\hat{w}_x(0,t)^2.$$

Consequently, by choosing the delay update gain γ_D and the parameter ε_1 such that

$$\gamma_D < \min\left\{\frac{\eta}{M(b_1\bar{D}M_1+b_2(M_2+MM_4+M_5+2))}, \times\frac{1}{MM_5},1\right\}$$

$$\frac{\varepsilon_1}{2} < \frac{\eta-\gamma_D M(b_1\bar{D}M_1+b_2(M_2+\gamma_D MM_4+M_5+2))}{b_1M_1+2\gamma_D M}$$

and restricting the initial condition to satisfy

$$V_1(0) < 2\varepsilon_1\left(\frac{\eta-\gamma_D Mb_1\bar{D}M_1}{b_1M_1}-\frac{\varepsilon_1}{2} + \frac{\gamma_D Mb_2(M_2+\gamma_D MM_4+M_5+2)}{b_1M_1}\right)$$

one can finally obtain

$$\dot{V}_1(t)\leq -\mu(t)V_0(t) \quad (38)$$

where μ is a non-negative function and consequently

$$\forall t\geq 0, \quad V_1(t)\leq V_1(0). \quad (39)$$

4.2.3. Equivalence

Above, stability results for the Lyapunov function V_1 have been provided, in (37) and (39) respectively. In view of proving Theorem 1, we now show the equivalence of the two functionals V_1 and Γ_1, i.e. the existence of $(a,b)\in {\mathbb{R}_+^*}^2$ such that, for $t\geq 0$, $aV_1(t)\leq \Gamma_1(t)\leq bV_1(t)$.

Using (31)–(34), one directly obtains this property as follows

$$\Gamma_1(t) \leq |\tilde{X}(t)|^2+2\left\|\tilde{e}(t)\right\|^2+3\left\|\hat{e}(t)\right\|^2+\left\|\hat{e}_x(t)\right\|^2+\tilde{D}(t)^2 \leq \frac{\max\{1+3r_1+r_3, 3r_2+r_4, r_5, 2\}}{\min\{\lambda_{\min}(P), b_{1,1}D, b_{1,2}\underline{D}, 1\}}V_1(t)$$

$$V_1(t) \leq \max\left\{\lambda_{\max}(P)+2s_1b_1\bar{D}+2s_3b_2\bar{D}, \times 4b_1D+2s_2b_1\bar{D}+2s_4b_2\bar{D}, 2s_5b_2\bar{D}, 1\right\}\Gamma_1(t).$$

This gives the desired stability property (18) with $R=b/a$.

4.2.4. Convergence

One can now conclude the proof of Theorem 1, by applying Barbalat's lemma on the variables $|\tilde{X}(t)|^2$ and $\tilde{U}(t)^2$. Integrating (36) or (38) from 0 to $+\infty$, one directly concludes that both quantities are integrable. Further, from (23), one has

$$\frac{d|\tilde{X}(t)|^2}{dt} = 2\tilde{X}(t)^T((A+BK)\tilde{X}(t)+B\tilde{e}(0,t)+B\hat{w}(0,t)).$$

From (37) or (39), it follows that $|\tilde{X}(t)|$, $\left\|\tilde{e}(t)\right\|$, $\left\|\hat{w}(t)\right\|$ and $\left\|\hat{w}_x(t)\right\|$ are uniformly bounded. Then, with (31), we obtain the uniform boundedness of $\left\|\hat{e}(t)\right\|$ and, consequently, of $\left\|\hat{u}(t)\right\|$. With (17), we conclude that $U(t)$ is uniformly bounded, and, therefore, that $\tilde{e}(0,t) = U(t-D)-U(t-\hat{D}(t))$ is bounded for $t\geq \bar{D}$. Further, from the definition (20), we get the uniform boundedness of $\hat{w}(0,t)$ for $t\geq\bar{D}$ and, finally, the one of $d(|\tilde{X}(t)|^2)/dt$ for $t\geq\bar{D}$. Finally, we conclude, with Barbalat's lemma, that $\tilde{X}(t)\to 0$ as $t\to\infty$.

Similarly, one can obtain, from (17),

$$\frac{d\tilde{U}(t)^2}{dt} = 2\tilde{U}(t)\left(Ke^{A\hat{D}(t)}\dot{\tilde{X}}(t)+\dot{\hat{D}}(t)G_0(t)+H_0(t)\right)$$

with

$$G_0(t) = K\left[e^{A\hat{D}(t)}A\tilde{X}(t)+\int_0^1 e^{A\hat{D}(y)(1-y)}B(y-1)\hat{e}_x(y,t)dy + \int_0^1 (I+A\hat{D}(t)(1-y))e^{A\hat{D}(t)(1-y)}B\hat{e}(y,t)dy\right]$$

$$H_0(t) = K\int_0^1 e^{A\hat{D}(t)(1-y)}B\hat{e}_x(y,t)dy.$$

Using (32), we deduce from above that $\left\|\hat{e}_x(t)\right\|$ is uniformly bounded. Therefore, it is straightforward to obtain the uniform boundedness of G_0 and H_0 and the one of $d\tilde{U}(t)^2/dt$, using Assumption 1 and the previous arguments. Then, with Barbalat's Lemma, we conclude the $\tilde{U}(t)\to 0$ as $t\to\infty$. This concludes the proof of Theorem 1.

Some important comments can be made concerning the previous proof. First, one can notice the key role played by the transformed state of the actuator into the appearance of negative bounding terms in the Lyapunov analysis. Besides, in this context, the main difficulties result in the treatment of the delay update law $\dot{\hat{D}}(t)$ and the delay estimate error $\tilde{D}(t)$. Actually, these two difficulties arise from the same fact: the dynamics of $\tilde{e}$ results in a bilinear term (namely a $\tilde{D}(t)\tilde{e}(.,t)$ term) which is well-known to be difficult to handle in a Lyapunov design (Ioannou & Fidan, 2006) (employed e.g. in Bresch-Pietri & Krstic, 2009 and Krstic & Bresch-Pietri, 2009 in which this term is linear as $\tilde{e}(.,t)$ is assumed to be measured). The first difficulty is addressed by the formulation of the Conditions 1 and 2. The second one implies both the definition of the intermediate functional V_0 and the restriction bearing upon the initial condition. Further, a direct consequence is the necessity to invoke Barbalat's lemma to transform the stability result (18) into the asymptotic convergence one (19).

5. Comments about the proof structure

Now that we have developed a proof of convergence for the first case under consideration in this paper, we wish to outline its structure, which will also guide the proof of the other cases. One can notice above the following main steps:

(1) definition of a backstepping transformation $w(.,t)$ of the PDE state, based on the certainty equivalence principle

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Table 1
Comparison of the presented results and the corresponding elements of proof.

Problem under consideration	Error variables in the Lyapunov analysis	Main technicality in the Lyapunov analysis	Solution	CV
Delay adaptation new block ($\hat{D}$)	$\tilde{X}$, $\tilde{e}$, $\hat{w}$, $\hat{w}_x$, $\tilde{D}$	Bounding of $\lvert\dot{\hat{D}}(t)\rvert$	Formulation of Conditions 1 and 2	lo. & as.
Observer new block ($\hat{X}$)	$\tilde{X}$, $\Delta\hat{X}$, $\tilde{e}$, $\hat{w}$, $\hat{w}_x$	Extra state variable	Study of an extra Lyapunov equation	gl. & exp.
Parameter adaptation new block ($\hat{\theta}$)	$\tilde{X}$, $\tilde{e}$, $\hat{w}$, $\hat{w}_x$, $\tilde{\theta}$	Creation of error variables (completion of non-vanishing terms)	Introduction of a parameter update law	lo. & as.
Disturbance estimate new block ($\hat{d}$)	$\tilde{X}$, $\tilde{e}_0$, $\hat{w}_0$, $\hat{w}_{0,x}$, $\tilde{d}$	Additive disturbance estimate in the control law	Incorporation of a double integral term in the functional	gl. & as.

(In Table 1, "CV" stands for "convergence", "lo" for "local", "gl" for "global", "as" for "asymptotic" and "exp" for "exponential".).

(2) definition of a Lyapunov equation, involving a suitable set of error variables and alternative spatial integral norms of some of them
(3) setting of the corresponding differential error equations
(4) time-derivative of the Lyapunov equation and integration by parts to create negative bounding terms
(5) bounding of the remaining positive error terms, using Young and Cauchy–Schwarz inequality.

These are the common steps that are used through the paper, for each considered case. The choice of the variables set in step (2) and the bounding realized in step (5) are the most elaborate parts. In particular, this last point is different in each of the contexts and sections, due to specific difficulties that are listed in Table 1.

In the following sections, some of the steps listed above are omitted or briefly introduced in the proofs, in order to reduce the calculations.

6. Output feedback strategy

In this section, we treat a second case where it is assumed that the system state X is not fully measured (as considered for strict observation problems in Krstic & Smyshlyaev, 2008). We design an output feedback controller. The considered problem is relatively simple compared to the much complex case of delayed stated variables (Bhat & Koivo, 1976; Watanabe & Ouchi, 1985). Our design directly originates from the delay-compensation observer form derived in Klamka (1982), Watanabe and Ito (1981), with an input in the plant.

To do so, in the following, we assume that θ and d are known ($d = 0$ for sake of simplicity). Further, the delay update law is again simply chosen as constant (i.e. $\dot{\hat{D}}(t) = 0$). For sake of conciseness, we denote $A = A(\theta), B = B(\theta)$ and so on for every quantity depending on $\hat{\theta}$ (naturally chosen as θ).

First, using the estimate waiting line introduced in (10)–(11), we define the following observer of the system state

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) - L(Y(t) - C\hat{X}(t)) \tag{40}$$

where the vector gain L is defined through the following additional assumption.

Assumption 5. The pair (A, C) is observable and $L \in \mathbb{R}^{n\times m}$ is a stabilizing gain.

6.1. Controller and convergence result

Using again the certainty equivalence principle, we employ the control law

$$U(t) = U^r - KX^r + K\left[e^{A\hat{D}}\hat{X}(t) + \hat{D}\int_0^1 e^{A\hat{D}(1-y)}B\hat{u}(y,t)dy\right] \tag{41}$$

and introduce several additional error variables,

$$\Delta X(t) = X(t) - X^r, \qquad \Delta\hat{X}(t) = \hat{X}(t) - X^r,$$
$$\tilde{X}(t) = X(t) - \hat{X}(t).$$

Theorem 2. *Consider the closed-loop system, consisting of (6)–(9), the control law (41) and the estimate plant (40). Define*

$$\Gamma_2(t) = |\Delta X(t)|^2 + |\Delta\hat{X}(t)|^2 + \|e(t)\|^2 + \left\|\hat{e}(t)\right\|^2 + \left\|\hat{e}_x(t)\right\|^2. \tag{42}$$

Then, there exists $\delta^* > 0$, $R > 0$ and $\rho > 0$ such that, for any initial conditions, provided that $|\tilde{D}| < \delta^*$,

$$\forall t \geq 0, \quad \Gamma_2(t) \leq R\Gamma_2(0)e^{-\rho t} \tag{43}$$

and, consequently, $Y(t) \to Y^r$ and $\tilde{X}(t) \to 0$ as $t \to \infty$.

Comparing this result to Theorem 1, several remarks can be formulated. First, one can notice that similar tools are introduced in the two statements. In details, the functional Γ_2 can be related to Γ_1 as it evaluates the system state, but, as could be expected, the state observation error has been included in the functional Γ_2. This is consistent with the fact that we focus on a system state observer design and keep the delay estimate as constant.

On the other hand, two main differences can be noticed: the convergence is now global (provided that the delay estimate is chosen close enough to the uncertain delay, which can be understood here as a robustness property to delay mismatch) and exponential. The reasons of these differences are detailed above.

6.2. Error variable dynamics and Lyapunov Analysis

Following along lines similar to the proof of the previous result summarized in Table 1, we define the following Lyapunov functional candidate

$$V_2(t) = \Delta\hat{X}(t)^T P_1 \Delta\hat{X}(t) + b_0\tilde{X}(t)P_2\tilde{X}(t) + b_1 D\int_0^1 (1+x)\tilde{e}(x,t)^2 dx + b_2\hat{D}\int_0^1 (1+x)[\hat{w}(x,t)^2 + \hat{w}_x(x,t)^2]dx \tag{44}$$

where $b_{3,0}, b_{3,1}$ and $b_{3,2}$ are positive coefficients, $(P_1, Q_1) = (P, Q)$ defined in Assumption 3 and the symmetric definite matrix P_2 satisfies the following Lyapunov equation, with Q_2 a given symmetric definite positive matrix,

$$P_2(A + LC) + (A + LC)^T P_2 = -Q_2.$$

The transformed state of the actuator is defined, as previously, through the following Volterra integral equation

$$\hat{w}(x,t) = \hat{e}(x,t) - \hat{D}\int_0^x Ke^{A\hat{D}(x-y)}B\hat{e}(y,t)dy - Ke^{A\hat{D}x}\Delta\hat{X}(t) \tag{45}$$

which satisfies the boundary property $\hat{w}(1,t) = 0$, taking into account the control law (41). First, we consider this transformation,

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jointly with its inverse to obtain the dynamics of the variables involved in (44)

$$\dot{\tilde{X}}(t) = (A + LC)\tilde{X}(t) + B\tilde{e}(0, t)$$
$$\frac{d\Delta\hat{X}}{dt} = (A + BK)\Delta\hat{X}(t) + B\hat{w}(0, t) - LC\tilde{X}(t)$$
$$D\tilde{e}(x, t) = \tilde{e}_x(x, t) - \tilde{D}f(x, t)$$
$$\tilde{e}(1, t) = 0$$
$$\hat{D}\hat{w}_t(x, t) = \hat{w}_x(x, t) + \hat{D}Ke^{A\hat{D}x}LC\tilde{X}(t)$$
$$\hat{w}(1, t) = 0$$
$$\hat{D}\hat{w}_{xt}(x, t) = \hat{w}_{xx}(x, t) + \hat{D}^2KAe^{A\hat{D}x}LC\tilde{X}(t)$$
$$\hat{w}_x(1, t) = -\hat{D}Ke^{A\hat{D}}LC\tilde{X}(t)$$

where the function f can be expressed, in terms of the $(\hat{w}, \hat{w}_x)$-variables, and is given in Appendix B. Taking a time-derivative of V_2 and using suitable integrations by parts, one obtains

$$\begin{aligned}\dot{V}_2(t) = {} & -\Delta\hat{X}(t)^TQ_1\Delta\hat{X}(t) + 2\Delta\hat{X}(t)^TP_1B\hat{w}(0, t)\\ & -2\Delta\hat{X}(t)^TP_1LC\tilde{X}(t) + b_0\Big(-\tilde{X}(t)^TQ_2\tilde{X}(t)\\ & +2\tilde{X}(t)P_2B\tilde{e}(0, t)\Big) + b_1\Big(-\left\|\tilde{e}(t)\right\|^2 - \tilde{e}(0, t)^2\\ & -2\tilde{D}\int_0^1(1 + x)\tilde{e}(x, t)f(x, t)dx\Big)\\ & +b_2\Big(-\left\|\hat{w}(t)\right\|^2 - \hat{w}(0, t)^2\\ & +2\hat{D}\int_0^1(1 + x)\hat{w}(x, t)Ke^{A\hat{D}x}LC\tilde{X}(t)dx\Big)\\ & +b_2\Big(2\hat{w}_x(1, t)^2 - \hat{w}_x(0, t)^2 - \left\|\hat{w}_x(t)\right\|^2\\ & +2\hat{D}^2\int_0^1(1 + x)\hat{w}_x(x, t)KAe^{A\hat{D}x}LC\tilde{X}(t)dx\Big).\end{aligned}$$

Choosing $b_2 \geq 4|P_1B|^2/\lambda_1$, $b_0 \geq 16\frac{|P_1LC|^2}{\lambda_1\lambda_2}$, one gets

$$\begin{aligned}\dot{V}_2(t) \leq {} & -\frac{\lambda_1}{4}|\Delta\hat{X}(t)|^2 - \frac{b_0\lambda_2}{4}|\tilde{X}(t)|^2 - b_1\left\|\tilde{e}(t)\right\|^2\\ & -\left(b_1 - \frac{2|P_2B|^2}{\lambda_2b_0}\right)\tilde{e}(0, t)^2 - b_{3,2}\left\|\hat{w}(t)\right\|^2\\ & -\frac{b_2}{2}\hat{w}(0, t)^2 + 2b_2\hat{w}_x(1, t)^2 - b_2\left\|\hat{w}_x\right\|^2\\ & -b_2\hat{w}_x(0, t)^2 + 2b_1|\tilde{D}|\int_0^1(1 + x)|\tilde{e}(x, t)\ \|\ f(x, t)|dx\\ & +2b_2\hat{D}\int_0^1(1 + x)|Ke^{A\hat{D}x}LC\tilde{X}(t)\hat{w}(x, t)|dx\\ & +2b_2\hat{D}(t)^2\int_0^1(1 + x)|KAe^{A\hat{D}(t)x}LC\tilde{X}(t)\hat{w}_x(x, t)|dx.\end{aligned}$$

Applying Young and Cauchy–Schwarz inequalities, one can show that there exist positive constants M_1, M_2, M_3 and M_4 independent on initial conditions such that

$$\begin{aligned}& 2\int_0^1(1 + x)|\tilde{e}(x, t)||f(x, t)|dx\\ & \quad\leq M_1\left(|\Delta\hat{X}(t)|^2 + \left\|\tilde{e}(t)\right\|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2\right)\\ & 2\hat{D}\int_0^1(1 + x)|Ke^{A\hat{D}x}LC\hat{w}(x, t)dx|\\ & \quad\leq M_2|\tilde{X}(t)|^2 + \left\|\hat{w}(t)\right\|^2/2\\ & 2\hat{w}_x(1, t)^2 \leq M_3|\tilde{X}(t)|^2\\ & 2\hat{D}^2\int_0^1(1 + x)|KAe^{A\hat{D}x}LC\hat{w}_x(x, t)|dx\\ & \quad\leq M_4|\tilde{X}(t)|^2 + \left\|\hat{w}_x(t)\right\|^2/2.\end{aligned}$$

One can use these inequalities to bound the resulting positive terms. By choosing $b_0 \geq \frac{8b_2}{\lambda_2}(M_2 + M_3 + M_4)$ and $b_1 \geq \frac{2|P_2B|^2}{\lambda_2b_0}$, we define the following quantities

$$\begin{aligned}V_0(t) &= |\Delta\hat{X}(t)|^2 + |\tilde{X}(t)|^2 + \left\|\tilde{e}(t)\right\|^2\\ &\quad + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2 \qquad (46)\\ \eta &= \min\{\lambda_1/4, b_0\lambda_2/8, b_1, b_2/2\}\end{aligned}$$

and obtain

$$\dot{V}_2(t) \leq -\left(\eta - b_1M_1|\tilde{D}|\right)V_0(t).$$

Consequently, if we assume $\tilde{D} < \frac{\eta}{2b_1M_1} = \delta^*$ we finally conclude that

$$\begin{aligned}\dot{V}_2(t) &\leq -\frac{\eta}{2}V_0(t)\\ &\leq -\frac{\eta V_2(t)}{2\max\left\{\lambda_{\max}(P_1), b_0\lambda_{\max}(P_2), 2b_1D, 2b_2\hat{D}\right\}}.\end{aligned}$$

This establishes the existence of $\rho > 0$ such that

$$\forall t \geq 0, \quad V_2(t) \leq V_2(0)e^{-\rho t}. \qquad (47)$$

6.2.1. Equivalence

In view of obtaining the exponential stability result stated in Theorem 2, we prove that the two functionals Γ_2 and V_2 are equivalent. First, considering (45) and its inverse and applying Young inequality, one can establish inequalities similar to (31)–(34), replacing $\tilde{X}$ with $\Delta\hat{X}$, and with new positive constant coefficients r'_1, r'_2, r'_3, r'_4, r'_5, s'_1, s'_2, s'_3, s'_4 and s'_5. Using these inequalities, one directly obtains

$$\begin{aligned}\Gamma_2(t) &\leq 2|\tilde{X}(t)|^2 + 3|\Delta\hat{X}(t)|^2\\ &\quad + 2\left\|\tilde{e}(t)\right\|^2 + 3\left\|\hat{e}(t)\right\|^2 + \left\|\hat{e}_x(t)\right\|^2\\ &\leq \frac{\max\{3 + 3r'_1 + r'_3, 3r'_2 + r'_4, r'_5\}}{\min\{\lambda_{\max}(P_1) + b_0\lambda_{\max}(P_2), 2b_1\bar{D}, 2b_2\bar{D}\}}V_2(t)\\ V_2(t) &\leq \max\{\lambda_{\max}(P_1) + 2b_2\bar{D}s'_1 + 2b_2\bar{D}s'_3, b_0\lambda_{\max}(P_2),\\ &\quad \times\ 2b_2\bar{D}, 2b_2\bar{D}(s'_2 + s'_4), 2b_2\bar{D}s'_5\}\ \Gamma_2(t).\end{aligned}$$

Hence, one can obtain the existence of $a > 0$ and $b > 0$ such that $\forall t \geq 0$, $aV_2(t) \leq \Gamma_2(t) \leq bV_2(t)$. Then, one easily gets, using (47),

$$\Gamma_2(t) \leq bV_2(t) \leq bV_2(0)e^{-\rho t} \leq \frac{b}{a}\Gamma_2(0)e^{-\rho t}$$

which gives the desired exponential convergence result, with $R = b/a$. This concludes the proof of Theorem 2, without need to invoke Barbalat's lemma as the Lyapunov analysis directly provides the asymptotic stability.

6.2.2. Main specificity and other comments

The main challenge in this section has been the introduction of a second error state variable, accounting for system state estimation error. This additional variable is treated in the proof thanks to a dedicated Lyapunov equation, highlighting the stability of its internal dynamics.

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Actually, this stability is directly related to the global and exponential convergence of the overall system. Namely, in the previous section and the following ones, the existence of estimation error variables, which were impossible to compensate, motivates the definition of a "truncated" functional V_0 to express a restrictive condition upon the initial condition. The resulting bound of the time-derivative of the Lyapunov functional is then a function of this truncated functional, which cannot be directly compared to the original Lyapunov one. Nevertheless, in the present case, there is no such remaining estimation error variables, as the error of system state estimation is naturally stable. Therefore, the intermediate functional V_0 defined in (46) is not truncated and is directly equivalent to the Lyapunov functional V_3.

More generally, the analysis is based on the definition of a transformed actuator (step (1) in Section 5), already defined in (20) and which is introduced here as (45). The desirable property $\hat{w}(1,t)=0$ holds, taking into account the control law (41). In the two previous sections, this backstepping transformation is only a generic Lyapunov tools to study stability, whereas, in the following, it is explicitly used in the control design.

7. Control strategy with parameter adaptation

This section addresses the case of plant parameters adaptation despite uncertainties on the delay. Several works (see Palmor, 1996 and more recently Evesque, Annaswamy, Niculescu, & Dowling, 2001, 2003, Niculescu & Annaswamy, 2003) have dealt with an adaptive framework for input-delay systems, but a few have simultaneously considered delay uncertainties. Lately, Zhou, Wen, and Wang (2009) has proposed an approach in a similar context and with comparable tools, but the developed feedback is not prediction-based and does not aim at compensating the delay effect.

Here, to present our control strategy in this framework, we consider again that d is measured, or more conveniently that $d=0$, for sake of simplicity, and that X is measured. Further, no particular effort is made to update the delay estimate, that is kept constant (i.e. $\dot{\hat{D}}(t) = 0$, which trivially satisfies either Condition 1 or Condition 2).

7.1. Controller and convergence result

Applying again the certainty equivalence principle to (12), we employ here the control law

$$U(t) = U^r(\hat{\theta}) - K(\hat{\theta})X^r(\hat{\theta}) + K(\hat{\theta})\Big[e^{A(\hat{\theta})\hat{D}}X(t) + \hat{D}\int_0^1 e^{A(\hat{\theta})\hat{D}(1-x)}B(\hat{\theta})\hat{u}(x,t)dx\Big] \quad (48)$$

where the parameter update law is chosen as

$$\dot{\hat{\theta}}(t) = \gamma_\theta \mathrm{Proj}_\Pi(\tau_\theta(t)) \quad (49)$$

$$\tau_{\theta,i}(t) = h(t)\times(A_iX(t)+B_iu^r(\hat{\theta})) \quad (50)$$

$$h(t) = \frac{\tilde{X}(t)^TP(\hat{\theta})}{b_2} - \hat{D}K(\hat{\theta})\int_0^1(1+x)\Big[\hat{w}(x,t) + A(\hat{\theta})\hat{D}\hat{w}_x(x,t)\Big]e^{A(\hat{\theta})\hat{D}x}dx \quad (51)$$

with $\gamma_\theta>0$, $1\le i\le p$ and where the transformed estimate state of the actuator satisfies the following Volterra integral equation of the second kind

$$\hat{w}(x,t) = \hat{e}(x,t) - \hat{D}\int_0^x K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(x-y)}B(\hat{\theta})\hat{e}(y,t)dy - K(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}\tilde{X}(t). \quad (52)$$

In (51), the matrix P is the one considered in Assumption 3, the constant b_2 is chosen such that $b_2 \ge 8\sup_{\theta\in\Pi}|PB(\theta)|^2\underline{\lambda}$ and Proj_Π is the standard projector operator onto the convex set Π

$$\mathrm{Proj}_\Pi\{\tau_\theta\} = \tau_\theta\begin{cases} I, & \hat{\theta}\in\mathring{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^T\tau_\theta\le 0 \\ I - \dfrac{\nabla_{\hat{\theta}}\mathscr{P}\nabla_{\hat{\theta}}\mathscr{P}^T}{\nabla_{\hat{\theta}}\mathscr{P}^T\nabla_{\hat{\theta}}\mathscr{P}}, & \hat{\theta}\in\partial\Pi \text{ and } \nabla_{\hat{\theta}}\mathscr{P}^T\tau_\theta>0. \end{cases} \quad (53)$$

Theorem 3. *Consider the closed-loop system consisting of (6)–(9), the control law (48) and the update law defined through (49)–(51). Define*

$$\Gamma_3(t) = |\tilde{X}(t)| + \|e(t)\|^2 + \left\|\hat{e}(t)\right\|^2 + \left\|\hat{e}_x(t)\right\|^2 + \tilde{\theta}(t)^2. \quad (54)$$

Then, there exists $\gamma^*>0$, $\delta^*>0$, $R>0$ and $\rho>0$ such that, provided the initial state $(\tilde{X}(0), e_0, \hat{e}_0, \hat{e}_{x,0}, \tilde{\theta}(0))$ is such that $\Gamma_3(0)<\rho$, if $|\tilde{D}|<\delta^*$ and if $\gamma_\theta<\gamma^*$, then

$$\forall t\ge 0 \quad \Gamma_3(t)\le R\Gamma_3(0), \quad (55)$$

$$\lim_{t\to\infty} Y(t) = Y^r, \qquad \lim_{t\to\infty}\tilde{X}(t)=0 \quad \text{and} \quad \lim_{t\to\infty}\tilde{U}(t)=0. \quad (56)$$

Comparing this result to Theorem 1 (resp. Theorem 2), one can observe that the parameter estimation error has been included in the functional Γ_3, instead of the delay estimation error in Γ_1 (resp. the state one in Γ_2), which could have been expected.

Besides, both Theorems 1 and 3 state asymptotic and local results (i.e. they require that, initially, each of the state variables is sufficiently close to its corresponding set-point) and require to upper bound the update gain (respectively of the delay estimate and the parameter estimate). Finally, as in Theorem 2, the delay estimate must be sufficiently close to the uncertain delay, which can be interpreted as a robustness property to delay mismatch.

The main difference with the above statements lies in the introduction of a parameter update law, based on the projector (49)–(53). This operator, commonly found in adaptive schemes (see Ioannou & Sun, 1996 or Ioannou & Fidan, 2006), is typical of a Lyapunov adaptive design, which is here enabled thanks to the backstepping transformation (52), as is shown in the following.

7.2. Error variable dynamics and Lyapunov Analysis

As previously, to take advantage on the introduced backstepping transformation, we use an alternative functional to Γ_3, which is the Lyapunov–Krasovskii functional we consider,

$$V_3(t) = \tilde{X}(t)^TP(\hat{\theta})\tilde{X}(t) + b_1D\int_0^1(1+x)\tilde{e}(x,t)^2dx + b_2\hat{D}\int_0^1(1+x)\hat{w}(x,t)dx + b_2\hat{D}\int_0^1(1+x)\hat{w}_x(x,t)dx + b_2|\tilde{\theta}(t)|^2/\gamma_\theta$$

where b_1 and b_2 are positive constants. Before working with this functional, we consider the dynamics of the involved variables, using (52) and its inverse transformation which yields

$$\dot{\tilde{X}}(t) = (A+BK)(\hat{\theta})\tilde{X}(t) + B(\hat{\theta})\hat{w}(0,t) + B(\hat{\theta})\tilde{e}(0,t) + \tilde{A}X(t) + \tilde{B}u(0,t) - \frac{\partial X^r}{\partial\hat{\theta}}\dot{\hat{\theta}}(t) \quad (57)$$

$$D\tilde{e}_t(x,t) = \tilde{e}_x(x,t) - \tilde{D}(t)f(x,t)$$

$$\tilde{e}(1,t) = 0$$

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$$\hat{D}\hat{w}_t(x,t) = \hat{w}_x(x,t) - \hat{D}\dot{\hat{\theta}}(t)^T g(x,t) - \hat{D}\tilde{\theta}(t)^T g_0(x,t) - \hat{D}K(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}B(\hat{\theta})\tilde{e}(0,t) \quad (58)$$

$$\hat{w}(1,t) = 0$$

$$\hat{D}\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t) - \hat{D}\dot{\hat{\theta}}(t)^T g_x(x,t) - \hat{D}\tilde{\theta}(t)^T g_{0,x}(x,t) - \hat{D}^2 KA(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}B(\hat{\theta})\tilde{e}(0,t) \quad (59)$$

$$\hat{w}_x(1,t) = \hat{D}\dot{\hat{\theta}}(t)^T g(1,t) + \hat{D}\tilde{\theta}(t)^T g_0(1,t) + \hat{D}K(\hat{\theta})e^{A(\hat{\theta})\hat{D}}B(\hat{\theta})\tilde{e}(0,t) \quad (60)$$

where $\tilde{A} = \sum_{i=1}^p A_i\tilde{\theta}_i(t)$, $\tilde{B} = \sum_{i=1}^p B_i\tilde{\theta}_i(t)$ and f, g and g_0 are defined in Appendix C.

Taking a time-derivative of V_3, after suitable integration by parts and using the update law (49)–(51), one can obtain

$$\begin{aligned}\dot{V}_3(t) \le\ & -(\underline{\lambda}|\tilde{X}(t)| + b_1\tilde{e}(0,t)^2 + b_1\left\|\tilde{e}(t)\right\|^2 + b_2\hat{w}(0,t)^2 \\ & + b_2\left\|\hat{w}(t)\right\|^2 + b_2\left\|\hat{w}_x(t)\right\|^2) + 2|\dot{\hat{\theta}}(t)|\left|P(\hat{\theta})\frac{\partial X^r}{\partial\hat{\theta}}\right||\tilde{X}(t)| \\ & + 2b_2|h(t)\|\tilde{B}\|\tilde{e}(0,t) + \hat{w}(0,t) + K(\hat{\theta})\tilde{X}(t)| \\ & + 2|\tilde{X}(t)^T PB(\hat{\theta})(\hat{w}(0,t) + \tilde{e}(0,t))| \\ & + 2b_1|\tilde{D}|\int_0^1(1+x)|\tilde{e}(x,t)\|f(x,t)|dx \\ & + b_2\Bigg(2\hat{D}|\dot{\hat{\theta}}(t)|\int_0^1(1+x)|\hat{w}(x,t)||g(x,t)|dx \\ & + 2\hat{D}|\tilde{e}(0,t)|\int_0^1(1+x)|\hat{w}(x,t)\|K(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}B(\hat{\theta})|dx\Bigg) \\ & + b_2\Bigg(2\hat{D}|\dot{\hat{\theta}}(t)|\int_0^1(1+x)|\hat{w}_x(x,t)\|g_x(x,t)|dx \\ & + 2\hat{D}^2|\tilde{e}(0,t)|\int_0^1(1+x)|\hat{w}_x(x,t)\|KA(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}B(\hat{\theta})|dx\Bigg) \\ & + 2b_2\hat{w}_x(1,t)^2 + \sum_{i=1}^p|\dot{\hat{\theta}}_i(t)|\left\|\frac{\partial P}{\partial\hat{\theta}_i}\right\|_\infty|\tilde{X}(t)|^2.\end{aligned}$$

Further, with Young inequality, Cauchy–Schwarz's inequality and Agmon's inequality $\hat{w}(0,t)^2 \le 4\left\|\hat{w}_x(t)\right\|^2$ (with the help of the fact that $\hat{w}(1,t)^2 = 0$), one can obtain the inequalities below. The positive constants $M_1,\ldots,M_{10}$ are independent on initial conditions and the functional V_0 is defined as $V_0(t) = |\tilde{X}(t)|^2 + \left\|\tilde{e}(t)\right\|^2 + \left\|\hat{w}(t)\right\|^2 + \left\|\hat{w}_x(t)\right\|^2$.

$$2|h(t)\|\tilde{B}\|\tilde{e}(0,t) + \hat{w}(0,t) + K(\hat{\theta})\tilde{X}(t)| \le M_1|\tilde{\theta}(t)|\left(V_0(t) + \tilde{e}(0,t)^2\right)$$

$$2|\tilde{X}(t)^T PB(\hat{\theta})(\hat{w}(0,t) + \tilde{e}(0,t))| \le \frac{\lambda}{2}|\tilde{X}(t)|^2 + \frac{4\left\|PB\right\|_\infty^2}{\underline{\lambda}}(\hat{w}(0,t)^2 + \tilde{e}(0,t)^2)$$

$$2\int_0^1(1+x)|\tilde{e}(x,t)\ \|\ f(x,t)|dx \le M_2V_0(t)$$

$$2\hat{D}\int_0^1(1+x)|\hat{w}(x,t)\ \|\ g(x,t)|dx \le M_3\left(V_0(t) + \left\|\hat{w}(t)\right\|\right)$$

$$2\hat{D}|\tilde{e}(0,t)|\int_0^1(1+x)|\hat{w}(x,t)\ \|\ K(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}B(\hat{\theta})|dx \le M_4\tilde{e}(0,t)^2 + \left\|\hat{w}(t)\right\|^2/2$$

$$2\hat{D}\int_0^1(1+x)|\hat{w}(x,t)\ \|\ g_x(x,t)|dx \le M_5V_0(t)$$

$$2\hat{D}^2|\tilde{e}(0,t)|\int_0^1(1+x)|\hat{w}_x(x,t)||KA(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}B(\hat{\theta})|dx \le M_6\tilde{e}(0,t)^2 + \left\|\hat{w}_x(t)\right\|^2/2$$

$$2\hat{w}_x(1,t)^2 \le M_7|\dot{\hat{\theta}}(t)|^2\left(V_0(t)+1\right) + M_8\tilde{e}(0,t)^2 + M_9|\tilde{\theta}(t)|^2\left(|\tilde{X}(t)|^2 + \left\|\hat{w}_x(t)\right\|^2\right) \quad (61)$$

$$|\dot{\hat{\theta}}(t)| \le \gamma_\theta M_{10}\left(V_0(t) + |\tilde{X}(t)| + \left\|\hat{w}(t)\right\| + \left\|\hat{w}_x(t)\right\|\right). \quad (62)$$

With these inequalities, the previous inequality yields, by choosing $b_{2,2} \ge \frac{8\|PB\|_\infty^2}{\underline{\lambda}}$ and defining $M_{11} = 2\left\|P\partial X^r/\partial\hat{\theta}\right\|_\infty$ and $M_0 = p\max_{1\le i\le p}\left\|\partial P/\partial\hat{\theta}_i\right\|_\infty$,

$$\begin{aligned}\dot{V}_3(t) \le\ & -\frac{\lambda}{2}|\tilde{X}(t)|^2 - b_1\left\|\tilde{e}(t)\right\|^2 - \frac{b_2}{2}\hat{w}(0,t)^2 - \frac{b_2}{2}\left\|\hat{w}(t)\right\|^2 \\ & - \frac{b_2}{2}\left\|\hat{w}_x(t)\right\|^2 + M_0|\dot{\hat{\theta}}(t)||\tilde{X}(t)|^2 + M_{11}|\dot{\hat{\theta}}||\tilde{X}(t)| \\ & - \left(b_1 - b_2\left(\frac{1}{2} + M_1|\tilde{\theta}(t)| + M_4 + M_6 + M_8\right)\right)\tilde{e}(0,t)^2 \\ & + [b_2M_1|\tilde{\theta}(t)| + b_1|\tilde{D}|M_2]V_0(t) \\ & + b_2|\dot{\hat{\theta}}(t)|\left[M_3\left(V_0(t) + \left\|\hat{w}(t)\right\|\right) + M_5\left(V_0(t) + \left\|\hat{w}_x(t)\right\|\right)\right] \\ & + b_2M_7|\dot{\hat{\theta}}(t)|^2\left(V_0(t)+1\right) + b_2M_9|\tilde{\theta}(t)|^2V_0(t).\end{aligned}$$

To obtain a negative definite expression, we choose $b_1 > b_2(1/2 + 2M_1\left\|\theta\right\|_\infty + M_4 + M_6 + M_8)$ and define $\eta = \min\left\{\underline{\lambda}/2, b_1, b_2/2\right\}$. Then, with the help of (62), the Young inequality $|\tilde{\theta}(t)| \le \frac{\varepsilon_2}{2} + \frac{1}{2\varepsilon_2}(V_2(t) - \eta_2V_0(t))$, involving $\varepsilon_2 > 0$, yields

$$\begin{aligned}\dot{V}_3(t) \le & -\Bigg[\eta - b_1M_2|\tilde{D}| - \gamma_\theta n_1(\gamma_\theta) - b_2(M_1 + 2M_9\left\|\theta\right\|_\infty) \\ & \times\left(\frac{\varepsilon_2}{2} + \frac{1}{2\varepsilon_2}V_2(t)\right)\Bigg]V_0(t) - \Bigg[\frac{\eta_2b_{2,2}}{2\varepsilon_2}(M_1 + 2M_9\left\|\theta\right\|_\infty) \\ & - \gamma_\theta n_2(\gamma_\theta) - 5M_7\gamma_\theta M_{10}V_0(t)\Bigg]V_0(t)^2\end{aligned} \quad (63)$$

where the function n_1 and n_2 are defined as $n_1(\gamma_\theta) = 2M_{10}(M_0 + 3M_{11} + 4M_3 + 4M_5 + 4\gamma_\theta M_7M_{10})$ and $n_2(\gamma_\theta) = M_{10}(M0 + 2M_{11} + 5M_3 + 5M_5 + 13M_7\gamma_\theta M_{10})$. Consequently, if the delay estimate error satisfies $|\tilde{D}(t)| < \frac{\eta}{b_1M_2}$, choosing the update gain γ_θ and the parameter ε_2 such that

$$\gamma_\theta < \gamma^* = \min\left\{1, \frac{\eta - b_1M_2|\tilde{D}|}{n_1(1)}\right\} \quad (64)$$

$$\varepsilon_2 < \min\left\{\frac{2(\eta - b_1M_2|\tilde{D}| - \gamma_\theta n_1(\gamma_\theta))}{b_2(M_1 + 2M_9\left\|\theta\right\|_\infty^2)}, \times\frac{\eta_2b_2}{2\gamma_\theta n_2(\gamma_\theta)}(M_1 + 2M_9\left\|\theta\right\|_\infty)\right\}$$

and restricting the initial condition as

$$V_3(0) \le \min\left\{\varepsilon_2\left[2\frac{\eta - b_1M_2|\tilde{D}| - \gamma_\theta n_1(\gamma_\theta)}{b_2(M_1 + 2M_9\left\|\theta\right\|_\infty^2)} - \varepsilon_2\right], \times\ \eta_2\frac{\eta_2b_2(M_1 + 2M_9\left\|\theta\right\|_\infty) - 2\varepsilon_2\gamma_\theta n_2(\gamma_\theta)}{10\varepsilon_2M_7\gamma_\theta M_{10}}\right\} \quad (65)$$

one finally obtains the existence of two non-negative functions μ_1 and μ_2 such that

$$\dot{V}_3(t) \leq -\mu_1(t)V_0(t) - \mu_2(t)V_0(t)^2 \tag{66}$$

and consequently

$$\forall t \geq 0, \quad V_3(t) \leq V_3(0). \tag{67}$$

7.2.1. Equivalence and convergence result

To obtain the stability result stated in Theorem 3, similarly to what was done in Sections 4.2.3 and 6.2.1, one can prove the equivalence of the two functionals V_2 and Γ_2. This gives the desired stability property (55).

Then, again, we conclude using Barbalat's Lemma on the variables $|\tilde{X}(t)|$ and $\tilde{U}(t)$. Integrating (66) from 0 to $+\infty$, it is straightforward to get that both signals are square integrable. Then, considering the following equations

$$\frac{d|\tilde{X}(t)|^2}{dt} = 2\tilde{X}(t)\left(A(\theta)X(t) + B(\theta)u(0,t) - \frac{\partial X^r}{\partial \hat{\theta}}\dot{\hat{\theta}}\right)$$

$$\frac{d\tilde{U}(t)^2}{dt} = 2\tilde{U}(t)\left(K(\hat{\theta})e^{A(\hat{\theta})\hat{D}}\dot{\tilde{X}}(t) + \sum_{i=1}^{p}\dot{\hat{\theta}}_i(t)G_i(t) + H_0(t)\right)$$

where

$$\begin{aligned} G_i(t) = {} & \frac{\partial K}{\partial \hat{\theta}_i}\left[e^{A(\hat{\theta})\hat{D}}\tilde{X}(t) + \hat{D}\int_0^1 e^{A(\hat{\theta})\hat{D}(1-y)}B(\hat{\theta}) \right. \\ & \left. \times\ \hat{e}(y,t)dy\right] + K(\hat{\theta})\left[\hat{D}A_i e^{A(\hat{\theta})\hat{D}}\tilde{X}(t)\right. \\ & + \hat{D}\int_0^1 e^{A(\hat{\theta})\hat{D}(1-y)}(A_i\hat{D}(1-y)B(\hat{\theta}) + B_i) \\ & \left. \times\ \hat{e}(y,t)dy - \hat{D}\int_0^1 e^{A(\hat{\theta})\hat{D}(1-y)}B(\hat{\theta})\frac{du^r}{d\hat{\theta}_i}(\hat{\theta})dy\right] \end{aligned}$$

$$H_0(t) = K(\hat{\theta})\int_0^1 e^{A(\hat{\theta})\hat{D}(1-y)}B(\hat{\theta})\hat{e}_x(y,t)dy$$

jointly with (67), one can conclude that both $d|\tilde{X}(t)|^2/dt$ and $d\tilde{U}(t)^2/dt$ are uniformly bounded for $t \geq \max\left\{D, \hat{D}\right\}$. Consequently, one obtains, with Barbalat's Lemma, that $\tilde{X}(t) \to 0$ and $\tilde{U}(t) \to 0$ as $t \to \infty$.

7.2.2. Main specificity and other comments

As previously, the essence of the proof is based on the backstepping transformation of the actuator state. Nevertheless, contrary to the proof of Theorem 1, the present proof uses an actual Lyapunov design of the parameter update law. In details, the $(\tilde{X}, \hat{w}, \hat{w}_x)$-dynamics (57)–(59) has introduced bilinear parameterizations of the form $\tilde{\theta}_i(t)(A_iX(t) + B_iu(0,t))$ (for $1 \leq i \leq p$). Because of the presence of the unknown term $u(0,t)$, it is impossible to exactly cancel these terms via the parameter update law. Yet, one can create vanishing terms, namely $e(0,t)$ by incorporating the control reference $u^r(\hat{\theta})$ in the parameter update law, as is done in (50). Because these terms arise in the three dynamics, in turn $\tilde{X}$, $\hat{w}(.,t)$ and $\hat{w}_x(.,t)$ appear in (51).

One important point to notice is that this creation of vanishing terms cannot be directly applied to a non-constant trajectory. Indeed, in this context, the reference distributed input $u^r(x,t,\hat{\theta})$ does depend explicitly on time and space. Then, the quantity $u^r(0,t,\hat{\theta})$ is unknown and cannot be used in the parameter update law like is done above.

One parameter estimation error term is also present inside $\hat{w}_x(1,t)^2$, as (60) points it out. Its quadratic form is inconsistent with a Lyapunov design. Consequently, its treatment requires the introduction of the intermediate function V_0 and the reduction on the initial condition. The treatment of the delay estimation error $\tilde{D}$ directly yields the condition stated in Theorem 3.

The main calculus difficulty is generated by the appearance of cubic terms, due again to the quantity $\hat{w}_x(1,t)^2$. Different bounds can be used to express them in a polynomial form in the V_0 variables. The bounds presented in (63) have been relatively roughly chosen. Consequently, the proposed expression of the bound for the update gain γ^* (64) and for the initial condition (65) are not the least conservative ones.

8. Input disturbance rejection

In this section, we focus on the compensation of a constant unknown bias acting on the system input. Recently, a certain number of papers have dealt with complex external disturbances for linear disturbances on delay systems (Pyrkin, Smyshlyaev, Bekiaris-Liberis, & Krstic, 2010a,b), even in a nonlinear context (Bobtsov, Kolyubin, & Pyrkin, 2010). Nevertheless, these works do consider the delay value as known. We aim here at giving some directions for filling this gap, considering the simple case of a constant disturbance.

Besides, we assume here that the system state X is measured and that θ is known. Further, the delay estimate is again kept constant. For sake of conciseness, we denote $A = A(\theta)$, $B = B(\theta)$ and so on for every quantity depending on θ.

8.1. Controller and convergence result

To reject the disturbance d, we introduce a dedicated estimate in the control law

$$\begin{cases} U(t) = U_0(t) - \hat{d}(t) \\ U_0(t) = U^r - KX^r + Ke^{A\hat{D}}X(t) \\ \qquad + K\hat{D}\displaystyle\int_0^1 e^{A\hat{D}(1-x)}B\hat{u}_0(x,t)dx \end{cases} \tag{68}$$

and define the corresponding distributed actuator corresponding to the control prediction part U_0, namely, for $x \in [0,1]$ and $t \geq 0$, $u_0(x,t) = U_0(t + D(x-1))$, $\hat{u}_0(x,t) = U_0(t + \hat{D}(x-1))$, $\hat{e}_0(x,t) = \hat{u}_0(x,t) - U^r$ and $\tilde{e}_0(x,t) = u_0(x,t) - \hat{u}_0(x,t)$. The estimate $\hat{d}$ is chosen as

$$\dot{\hat{d}}(t) = \gamma_d\tau_d(t) \tag{69}$$

$$\begin{aligned} \tau_d(t) = {} & \frac{\tilde{X}(t)^TPB}{b_2} - \hat{D}\int_0^1 (1+x)[\hat{w}_0(x,t) \\ & + A\hat{D}\hat{w}_{0,x}(x,t)]Ke^{A\hat{D}x}Bdx. \end{aligned} \tag{70}$$

Theorem 4. *Consider the closed-loop system consisting of (6)–(9) and the control law (68) with (69)–(70). Define*

$$\begin{aligned} \Gamma_4(t) = {} & |\tilde{X}(t)|^2 + \|e_0(t)\|^2 + \left\|\hat{e}_0(t)\right\|^2 + \left\|\hat{e}_{0,x}(t)\right\|^2 + \tilde{d}(t)^2 \\ & + \int_{t-D}^{t}\int_s^t \left[|\tilde{X}(r)|^2 + \left\|\hat{e}_0(r)\right\|^2 + \left\|\hat{e}_{0,x}(r)\right\|^2\right]drds. \end{aligned}$$

Then, there exists $\delta^* > 0$ and $\gamma^* > 0$ such that, provided that $|\gamma_d| < \gamma^*$ and $|\tilde{D}| < \delta^*$,

$$\forall t \geq 0, \quad \Gamma_4(t) \leq R\Gamma_4(0),$$

$$\lim_{t\to\infty} X(t) = X^r \quad \text{and} \quad \lim_{t\to\infty} U(t) = U^r.$$

The previous theorem gives a global but asymptotic convergence (except from the delay estimation error which is required to be sufficiently small). The disturbance estimate update law is chosen through a Lyapunov design, as in the previous section. It is consistent with the well-known result that, for a linear system, an integral stabilizing controller rejects any disturbance (see Kailath, 1980). In other words, the delay-free form of the system is stabilized by the $\tilde{X}$-part of the update law (69)–(70), whereas the rest of the update law accounts for the delay existence. One can observe the similarity between this update law and the one proposed in Section 7 in (49)–(51), as in the two cases the estimate errors have the same impacts in the dynamics. The main difference is the addition of a double integral term in the functional Γ_4. This term does not help to characterize the system state, as it is more or less redundant with the first one, but reveals necessary in the Lyapunov analysis, as will appear below. Besides, an update gain limitation is still present and impacts on the integrator gain.

8.2. Error variable dynamics and Lyapunov Analysis

Before working with a Lyapunov–Krasovskii functional, we introduce again the backstepping transformation of the actuator state

$$\hat{w}_0(x,t) = \hat{e}_0(x,t) - \hat{D}\int_0^x Ke^{A\hat{D}(x-y)}B\hat{e}_0(y,t)dy - Ke^{A\hat{D}x}\tilde{X}(t). \tag{71}$$

Using this transformation, (6) can now be expressed as

$$\dot{\tilde{X}}(t) = (A+BK)\tilde{X}(t) + B\left[\tilde{e}_0(0,t) + \hat{w}_0(0,t)\right] + B\tilde{d}(t) + B\left[\hat{d}(t) - \hat{d}(t-D)\right]. \tag{72}$$

In details, (72) can now be viewed as the result of four distinct factors: (i) the stabilized dynamics (the $A+BK$-term), (ii) the mismatch between the delay and its estimate (namely $B\tilde{e}(0,t)$), (iii) the mismatch between the disturbance and its estimate, (iv) the delay effects over the disturbance rejection (the non-synchronization between the estimate and the plant). Now, define the following Lyapunov–Krasovskii functional

$$\begin{aligned} V_4(t) &= \tilde{X}(t)^T P\tilde{X}(t) + \frac{b_2}{\gamma_d}\tilde{d}(t)^2 + b_1 D \\ &\quad\times \int_0^1 (1+x)\tilde{e}_0(x,t)^2dx + b_2\hat{D}\int_0^1 (1+x)\hat{w}_0(x,t)^2dx \\ &\quad + b_2\hat{D}\int_0^1 (1+x)\hat{w}_{0,x}(x,t)^2dx \\ &\quad + b_3\int_{t-D}^t\int_s^t \left[|\tilde{X}(r)|^2 + \left\|\hat{w}_0(r)\right\|^2 + \left\|\hat{w}_{0,x}(r)\right\|^2\right]drds. \end{aligned}$$

Considering (71) and its inverse transformation

$$\hat{e}_0(x,t) = \hat{w}_0(x,t) + \hat{D}\int_0^x Ke^{(A+BK)\hat{D}(x-y)}B\hat{w}_0(y,t)dy + Ke^{(A+BK)\hat{D}x}\tilde{X}(t)$$

the actuators dynamics can be written as

$$\begin{aligned} &D\tilde{e}_{0,t}(x,t) = \tilde{e}_{0,x}(x,t) - \tilde{D}f(x,t) \\ &\tilde{e}(1,t) = 0 \\ &\hat{D}\hat{w}_{0,t}(x,t) = \hat{w}_{0,x}(x,t) - \hat{D}Ke^{A\hat{D}x}B\left[\tilde{e}_0(0,t) + \tilde{d}(t) + \hat{d}(t) - \hat{d}(t-D)\right] \\ &\hat{w}(1,t) = 0 \\ &\hat{D}\hat{w}_{0,xt}(x,t) = \hat{w}_{0,xx}(x,t) - \hat{D}^2KAe^{A\hat{D}x}B[\tilde{e}_0(0,t) + \tilde{d}(t) + \hat{d}(t) - \hat{d}(t-D)] \\ &\hat{w}_x(1,t) = \hat{D}Ke^{A\hat{D}}B\left[\tilde{e}(0,t) + \tilde{d}(t) + \hat{d}(t) - \hat{d}(t-D)\right] \end{aligned}$$

where the function f is given in Appendix D. Then, taking a time-derivative of V_4 and using suitable integrations by parts, one can get

$$\begin{aligned} \dot{V}_4(t) &= -\tilde{X}(t)^T Q\tilde{X}(t) + 2\tilde{X}(t)^T PB[\tilde{e}_0(0,t) + \hat{w}_0(0,t)] \\ &\quad + 2\tilde{X}(t)^T PB\left[\hat{d}(t) - \hat{d}(t-D)\right] + \frac{2b_2}{\gamma_D}\tilde{d}(t)\left(\tau_d(t) - \dot{\hat{d}}(t)\right) \\ &\quad + b_1\Bigg(-\tilde{e}_0(0,t)^2 - \left\|\tilde{e}_0(t)\right\|^2 \\ &\quad - 2\tilde{D}\int_0^1 (1+x)\tilde{e}_0(x,t)f(x,t)dx\Bigg) + b_2\Bigg(-\hat{w}_0(0,t)^2 \\ &\quad - \left\|\hat{w}_0(t)\right\|^2 - 2\hat{D}\int_0^1 (1+x)Ke^{A\hat{D}x}B[\tilde{e}_0(0,t) \\ &\quad + \hat{d}(t) - \hat{d}(t-D)]\hat{w}_0(x,t)dx\Bigg) + b_2\Bigg(2\hat{w}_{0,x}(1,t)^2 \\ &\quad - \hat{w}_{0,x}(0,t)^2 - \left\|\hat{w}_{0,x}(t)\right\|^2 - 2\hat{D}^2\int_0^1 (1+x)KAe^{A\hat{D}x}B \\ &\quad \times[\tilde{e}_0(0,t) + \hat{d}(t) - \hat{d}(t-D)]\hat{w}_{0,x}(x,t)dx\Bigg) \\ &\quad + b_3\Bigg(-\int_{t-D}^t\left[|\tilde{X}(r)|^2 + \left\|\hat{w}_0(r)\right\|^2 + \left\|\hat{w}_{0,x}(r)\right\|^2\right]dr \\ &\quad + D\left[|\tilde{X}(t)|^2 + \left\|\hat{w}_0(t)\right\|^2 + \left\|\hat{w}_{0,x}(t)\right\|^2\right]\Bigg). \end{aligned} \tag{73}$$

Further, observing that $\hat{d}(t) - \hat{d}(t-D) = \gamma_d\int_{t-D}^t \tau_d(s)ds$ jointly with the definition of τ_d (70) and using Cauchy–Schwarz inequality and Young inequality, one can obtain the following inequalities on the non-negative terms of (73)

$$\begin{aligned} &2|\tilde{X}(t)^T PB[\tilde{e}_0(0,t) + \hat{w}_0(0,t) + \hat{d}(t) - \hat{d}(t-D)]| \\ &\quad\le \frac{\lambda_{\min}(Q)}{2}|\tilde{X}(t)|^2 + \frac{4|PB|^2}{\lambda_{\min}(Q)}[\tilde{e}_0(0,t)^2 + \hat{w}_0(0,t)^2] \\ &\qquad + \gamma_d M_1\int_{t-D}^t\left(|\tilde{X}(r)|^2 + \left\|\hat{w}_0(r)\right\|^2 + \left\|\hat{w}_{0,x}(r)\right\|^2\right)dr \\ &2\int_0^1 (1+x)|\tilde{e}_0(x,t)||f(x,t)|dx \\ &\quad\le M_2\left(|\tilde{X}(t)|^2 + \left\|\tilde{e}_0(t)\right\|^2 + \left\|\hat{w}_0(t)\right\|^2 + \left\|\hat{w}_{0,x}(t)\right\|^2\right) \\ &2\hat{D}\left|\int_0^1 (1+x)Ke^{A\hat{D}x}B[\tilde{e}_0(0,t) + \hat{d}(t) - \hat{d}(t-D)]\hat{w}_0(x,t)dx\right| \\ &\quad\le \left\|\hat{w}_0(t)\right\|^2/2 + M_3\tilde{e}_0(0,t)^2 \\ &\qquad + \gamma_d M_3\int_{t-D}^t\left(|\tilde{X}(r)|^2 + \left\|\hat{w}_0(r)\right\|^2 + \left\|\hat{w}_{0,x}(r)\right\|^2\right)dr \\ &2\hat{w}_{0,x}(1,t)^2 - \hat{w}_{0,x}(0,t)^2 \\ &\quad\le M_4\tilde{e}_0(0,t)^2 \\ &\qquad + \gamma_d M_5\int_{t-D}^t\left(|\tilde{X}(r)|^2 + \left\|\hat{w}_0(r)\right\|^2 + \left\|\hat{w}_{0,x}(r)\right\|^2\right)dr \end{aligned}$$

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$$2\hat{D}^2\left|\int_0^1(1+x)KAe^{A\hat{D}x}[\tilde{e}_0(0,t)+\hat{d}(t)-\hat{d}(t-D)]\hat{w}_{0,x}(x,t)dx\right| \leq \left\|\hat{w}_{0,x}(t)\right\|^2/2+M_6\tilde{e}_0(0,t)^2 + \gamma_d M_6\int_{t-D}^t\left(|\tilde{X}(r)|^2+\left\|\hat{w}_0(r)\right\|^2+\left\|\hat{w}_{0,x}(r)\right\|^2\right)dr.$$

With these inequalities and choosing $b_2 \geq \frac{8|PB|^2}{\lambda_{\min}(Q)}$, then (73) yields

$$\begin{aligned}\dot{V}_4(t) &\leq -\frac{\lambda_{\min}(Q)}{2}|\tilde{X}(t)|^2-\frac{b_2}{2}\hat{w}_0(0,t)^2 \\ &-\left(b_1-\frac{b_2}{2}-b_2(M_3+M_4+M_6)\right)\tilde{e}_0(0,t)^2 \\ &-b_1\left\|\tilde{e}_0(t)\right\|^2-\frac{b_2}{2}\left\|\hat{w}_0(t)\right\|^2-\frac{b_2}{2}\left\|\hat{w}_{0,x}(t)\right\|^2 \\ &+b_1|\tilde{D}|M_2\left(|\tilde{X}(t)|^2+\left\|\tilde{e}_0(t)\right\|^2+\left\|\hat{w}_0(t)\right\|^2+\left\|\hat{w}_{0,x}(t)\right\|^2\right) \\ &+b_3\left(D\left[|\tilde{X}(t)|^2+\left\|\hat{w}_0(t)\right\|^2+\left\|\hat{w}_{0,x}(t)\right\|^2\right]\right. \\ &\left.-\int_{t-D}^t\left[|\tilde{X}(r)|^2+\left\|\hat{w}_0(r)\right\|^2+\left\|\hat{w}_{0,x}(r)\right\|^2\right]dr\right) \\ &+\gamma_d\left(M_1+b_2(M_3+M_5+M_6)\right) \\ &\times\int_{t-D}^t\left(|\tilde{X}(r)|^2+\left\|\hat{w}_0(r)\right\|^2+\left\|\hat{w}_{0,x}(r)\right\|^2\right)dr.\end{aligned}$$

By choosing $b_1 > b_2\,(1/2+M_3+M_4+M_6)$ and defining $M_7 = M_1+b_2(M_3+M_5+M_6)$ jointly with

$$V_{0,1}(t)=|\tilde{X}(t)|^2+\left\|\tilde{e}_0(t)\right\|^2+\left\|\hat{w}_0(t)\right\|^2+\left\|\hat{w}_{0,x}(t)\right\|^2$$
$$V_{0,2}(t)=\int_{t-D}^t\left[|\tilde{X}(r)|^2+\left\|\hat{w}_0(r)\right\|^2+\left\|\hat{w}_{0,x}(r)\right\|^2\right]dr$$
$$\eta=\min\{\lambda_{\min}(Q)/2,b_1,b_2/2\}$$

one gets

$$\dot{V}_4(t) \leq -(\eta-b_3D)V_{0,1}(t)+b_1|\tilde{D}(t)|M_2V_{0,1}(t) - (b_3-\gamma_dM_7)V_{0,2}(t).$$

Consequently, if $b_3 = \gamma_d M_7$, if the update gain γ_d is chosen such that $\gamma_d = \frac{\eta}{2\bar{D}M_7}$ and if $|\tilde{D}| < \delta^* = \frac{\eta}{4b_1M_2}$, one concludes

$$\forall t\geq 0,\quad \dot{V}_4(t)\leq -\frac{\eta}{4}V_{0,1}(t) \tag{74}$$

and, finally, that $\forall t \geq 0$, $V_4(t) \leq V_4(0)$.

8.2.1. Convergence result

From there, to obtain the stability result stated in Theorem 4, we apply the same arguments as in Sections 4 and 7. This proof is omitted.

8.2.2. Main specificity and other comments

Details of the proof are more or less similar to the ones given in previous sections. Here, as in Section 7, the disturbance estimate is chosen via a Lyapunov design. The main difference relies on the appearance of a desynchronized term $\hat{d}(t) - \hat{d}(t-D)$ due to the additive form of the controller (68), which implies the introduction of a double integral term to treat this mismatch. Further, one point to notice is that the disturbance estimate converge to the unknown but constant disturbance.

9. Illustrative example

In this section, to illustrate the merits, the practical interest and the feasibility of the proposed adaptive control scheme, we consider an open-loop unstable dynamics, that we wish to regulate around the set-point $Y^r = 1$. It is given under the following state-space realization (see Huang & Lin, 1995 and Huang & Chen, 1997, where the original transfer function is studied)

$$\dot{X}(t)=\begin{pmatrix}0 & \frac{1}{aT}\\ 1 & \frac{a-T}{aT}\end{pmatrix}X(t)+\begin{pmatrix}\frac{k}{aT}\\ 0\end{pmatrix}[U(t-D)+d] \tag{75}$$

$$Y(t)=(0\ 1)X(t) \tag{76}$$

with $a = 5$, $T = 2.07$, $k = 1$, $d = 0.5$ and the delay $D = 0.939$ is highly uncertain. Assuming that k is known (the static gain can be easily identified), the previous system can be expressed under the linearly parametrized form (3) with the parameter $\theta = \frac{1}{aT}(1 \quad a-T)^T \in \mathbb{R}^2$. The references corresponding to the output set-point Y^r are $X^r(\theta) = [-\theta_2\ 1]^T Y^r$ and $U^r(\theta) = -Y^r$. By defining $[\underline{D},\bar{D}] = [0.8, 1.1]$ and $\Pi = [0.05, 0.15] \times [0.2, 0.4]$, one can easily check that the assumptions stated in Section 2 are satisfied.

For tutorial purposes, we illustrate the cases treated in this paper. This yields the control structure reported in Fig. 1.

9.1. Delay adaptation

For now, we consider that (X,θ,d) is measured/known and focus on the design of a delay update law satisfying Condition 2. We define the following cost function

$$\begin{aligned}\phi : [0,+\infty)\times[\underline{D},\bar{D}] &\to \mathbb{R},\\ (t,\hat{D}) &\mapsto |X_P(t,\hat{D})-X(t)|^2\\ &= \left|e^{A(t-\hat{D})}X(\hat{D})+\int_{\hat{D}}^t e^{A(t-s)}BU(s-\hat{D})ds-X(t)\right|^2\end{aligned}$$

where $X_P(t,\hat{D})$ is a t-units of time prediction if the system state, starting from $X(\hat{D})$ as initial condition and assuming that the actual delay value is $\hat{D}(t)$. Then, using a steepest descent algorithm, one can take

$$\tau_D(t)=-\gamma_D(X_P(t,\hat{D})-X(t))\times\frac{\partial X_P}{\partial\hat{D}}(t,\hat{D}) \tag{77}$$
$$\frac{\partial X_P}{\partial\hat{D}}=e^{At}BU(0)-BU(t-\hat{D})-\int_{\hat{D}}^t Ae^{A(t-\tau)}BU(\tau-\hat{D})d\tau$$

where these expressions are directly implementable.[4] This choice is based on the comparison of two versions of a signal, the one corresponding to the unknown delay D and the other to the controlled delay $\hat{D}(t)$. It provides an accurate estimation of the unknown delay provided that the initial delay estimate is sufficiently close to the true value. In details, this condition, which is compliant with the one stated in Theorem 1, guarantees that no extraneous local minimum interferes with the minimization process.

The simulation results are provided in Fig. 2. The tracked trajectory is a 10 s-periodic signal, arbitrarily chosen to highlight transient behaviors. Indeed, one can notice that the delay estimate

[4] The most tricky point arise while implementing the integral term. This problem has been widely studied in the literature (Mondie & Michiels, 2003; Van Assche, Dambrine, Lafay, & Richard, 2002). Here, interestingly, this difficulty has been easily handled with a trapezoidal discretization method and a periodic reset of this update law when convergence is achieved.

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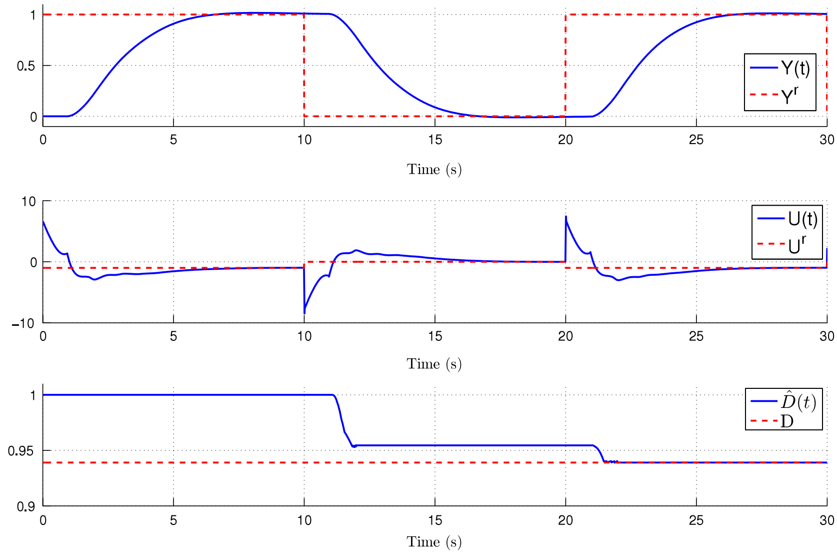

Fig. 2. Simulation results of the control of system (75) with Block $(\hat{D})$, starting from $X(0) = [0 \quad 0]^T$, $u(., 0) = 0$ and $\hat{D}(0) = 1$. The plant is assumed to be fully measured and known. The gradient-based delay update law (77) is employed, with $\gamma_D = 50$ and the controller gain K is chosen thanks to an LQR criterion.

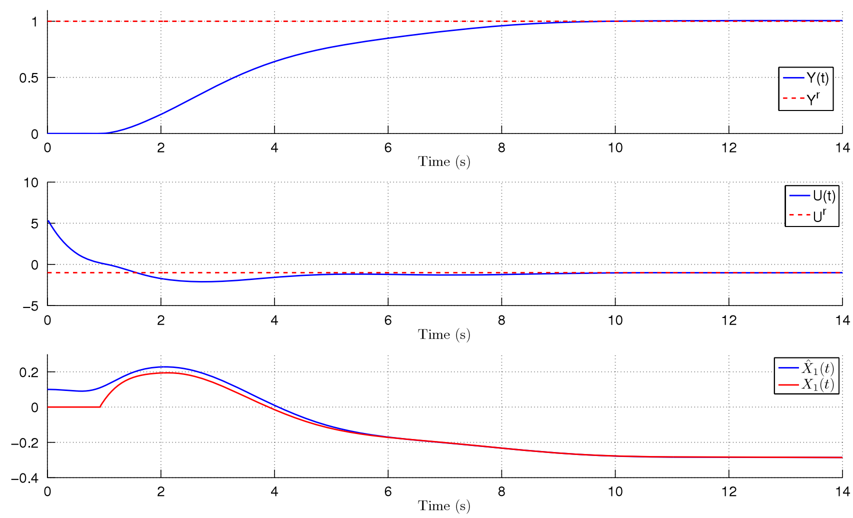

Fig. 3. Simulation results of the control of system (75) with Block $(\hat{X})$, starting from $X(0) = [0 \quad 0]^T$, $u(., 0) = 0$ with $\hat{X}(0) = [0.1 \quad 0]$. The plant is assumed to be known and the delay estimate is $\hat{D} = 1$. The controller gain K is chosen thanks to an LQR criterion, while the observer one is chosen as $L = -[1.1 \quad 2.3]^T$.

provides a good identification of the unknown delay, but also that the most visible improvements in the estimation occur at step changes of the reference signal. This is consistent with the employed update law, as the cost function presents its most important gradient at these instants. This identification extends the possibilities of regulation, as the tracking of any time-varying smooth trajectory is then achievable. Generally speaking, the performance of the controller is consistent with the properties stated in Theorem 1.

9.2. Observer and output feedback

The results of an output feedback strategy, i.e. the Block $(\hat{X})$ assuming that the plant parameter θ and the input disturbance d are constant and known as provided in Fig. 3. The initial system state estimate is chosen as $\hat{X}(0) = [0.10]^T \neq X(0) = [00]$ and the observer gain is taken arbitrarily as $L = -[1.12.3]$. Tis gain does not give the best performance achievable with the scheme, as it has not tuned for speed of convergence. This value has been selected to illustrate the effects of the addition of an observer into the control scheme.

One can observe that the response time of the controller is significantly shortened (namely 8 s instead of 5 s). This is due to the observer behavior, which converges only after a few seconds.

9.3. Parameter adaptation

We now choose a constant delay estimate $\hat{D} = 1$ and address the plant parameter adaptation of (75), considering that the input disturbance d is known. Namely, the control is based on the Block $(\hat{\theta})$: the closed-loop system (6)–(9), the control law (48) and the update law defined through (49)–(51) as

$$\tau_{\theta,1}(t) = h(t) \times (A_1 X(t) - B_1 Y^r)$$

$$\tau_{\theta,2}(t) = h(t) \times A_2 X(t).$$

The simulation results are provided in Fig. 4, with $\hat{\theta}(0) = [0.08 \quad 0.258]$ while the unknown plant parameter value is $\theta = [0.097 \quad 0.28]$ which represents an error of approximately 15%.

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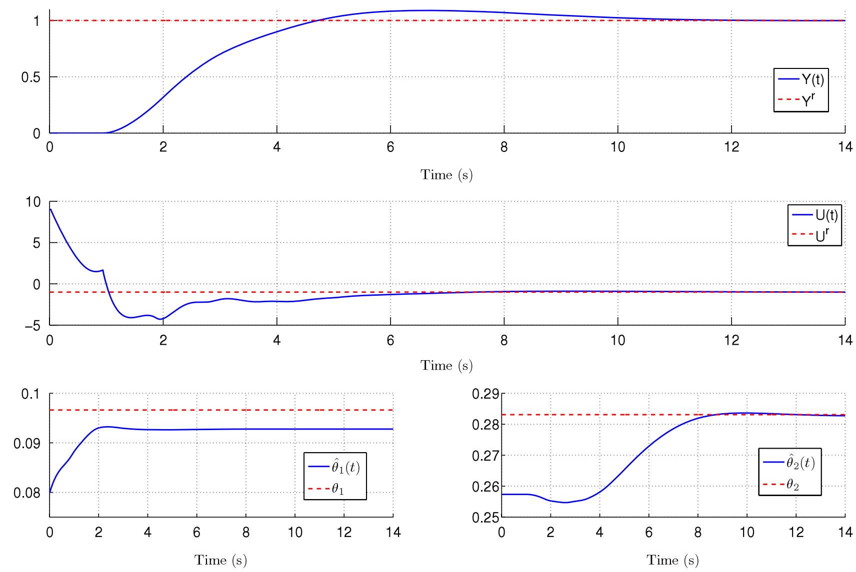

Fig. 4. Simulation results of the control of system (75) with Block $(\hat{\theta})$, starting from $X(0) = [0 \quad 0]^T$, $u(., 0) = 0$ and $\hat{\theta}(0) = [0.08 \quad 0.258]$. The system state is assumed to be measured and the delay estimate is kept as constant $\hat{D} = 1$. The controller gain $K(\hat{\theta})$ is chosen thanks to an LQR criterion and the update gain is chosen as $\gamma_\theta = 10^{-4}$.

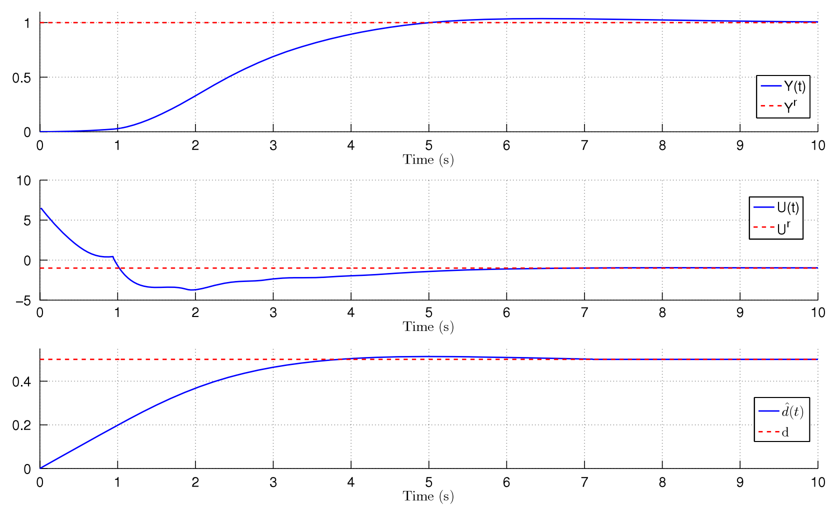

Fig. 5. Simulation results of the control of system (75) with Block $(\hat{d})$, starting from $X(0) = [0 \quad 0]^T$, $u(., 0) = 0$ with $\hat{d}(0) = 0$. The plant is assumed to be fully known and measured and the delay estimate is kept as constant $\hat{D} = 1$. The controller gain K is chosen thanks to an LQR criterion and the integrator one as $k_I = 0.4$.

The output response is similar to the previous one, except a slight oscillation during transient around 4–5 s, due to the parameter adaptation lag.

One can notice that, when the convergence of the system state and control is achieved (around 12 s), the estimate $\hat{\theta}(t)$ also converges, which is consistent with the update law (49)–(51). Nevertheless, the multi-parameters estimation is not obtained, as is well-known in adaptive control (see Ioannou & Sun, 1996). In details, for the considered system, $\tilde{\theta}_2(t)$ converges to zero. This is consistent with the dynamics given above: by examining (4)–(5) in Assumption 2 and comparing it to the plant (75) jointly with the convergence result of Theorem 3, one can infer the convergence of $\hat{\theta}_2$ to θ_2, but not of $\hat{\theta}_1$.

9.4. Integrator and disturbance rejection

The corresponding results are given in Fig. 5. One can observe that the disturbance bias has been perfectly estimated and that the same general comments on performance as before hold.

10. Conclusion

In this paper, we have presented a general adaptive control scheme (see Fig. 1), based on a backstepping transformation for linear uncertain time-delay systems, capable of addressing a collection of classic problems of equilibrium regulation. A simulation example with a second order linear plant from the literature has been provided to illustrate, for each case, the effectiveness of the design and its implementability, without knowledge of the actuator state. From the various Lyapunov analysis presented in this paper, it has been shown that the proposed technique is compliant with numerous contexts and practical difficulties (see e.g. Bresch-Pietri et al., 2011b for one possible combination) and that its implementation difficulties are limited (see Bresch-Pietri et al., 2010 for experimental applications of the technique).

One path that remains to explore is the relevance of this approach for time-varying delays, since the presence of time variation may induce more complex behaviors (see for instance the quenching phenomenon described in Louisell, 1999).

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Appendix A. Expression of the functions of f and g in (24)–(29)

$$\begin{aligned} f(x,t) &= \frac{\hat{w}(x,t)}{\hat{D}(t)} + KB\hat{w}(x,t) + K(A+BK)e^{(A+BK)\hat{D}(t)x}\tilde{X}(t) \\ &\quad + \hat{D}(t)\int_0^x K(A+BK)e^{(A+BK)\hat{D}(t)(x-y)}B\hat{w}(y,t)dy \\ g(x,t) &= (1-x)f(x,t) + \hat{D}(t)K\int_0^x e^{A\hat{D}(t)(x-y)}B(y-1) \\ &\quad \times f(y,t)dy + \int_0^x K(I + A\hat{D}(t)(x-y))e^{A\hat{D}(t)(x-y)}B \\ &\quad \times \left[\hat{w}(y,t) + \hat{D}(t)\int_0^y Ke^{(A+BK)\hat{D}(t)(y-\xi)}B\hat{w}(\xi,t)d\xi \right. \\ &\quad \left. + \ Ke^{(A+BK)\hat{D}(t)y}\tilde{X}(t)\right] + KAxe^{A\hat{D}(t)x}\tilde{X}(t). \end{aligned}$$

Appendix B. Expression of the dynamics function f in Section 6

$$\begin{aligned} f(x,t) &= \frac{\hat{w}_x(x,t)}{\hat{D}} + KB\hat{w}(x,t) + K(A+BK)e^{(A+BK)\hat{D}x}\Delta\hat{X}(t) \\ &\quad + \hat{D}\int_0^x K(A+BK)e^{(A+BK)\hat{D}(x-y)}B\hat{w}(y,t)dy. \end{aligned}$$

Appendix C. Expression of the dynamics functions f, g and g_0 in Section 7

$$\begin{aligned} f(x,t) &= \frac{\hat{w}_x(x,t)}{\hat{D}} + KB(\hat{\theta})\hat{w}(x,t) + \hat{D}\int_0^x K(A+BK)(\hat{\theta}) \\ &\quad \times e^{(A+BK)(\hat{\theta})\hat{D}(x-y)}B(\hat{\theta})\hat{w}(y,t)dy \\ &\quad + K(A+BK)(\hat{\theta})e^{(A+BK)(\hat{\theta})\hat{D}x}\tilde{X}(t) \\ g_i(x,t) &= \hat{D}\int_0^x \hat{w}(y,t)\left[\left(\frac{\partial K}{\partial\hat{\theta}_i} + K(\hat{\theta})A_i\hat{D}(x-y)\right)\right. \\ &\quad \times e^{A(\hat{\theta})\hat{D}(x-y)}B(\hat{\theta}) + K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(x-y)}B_i \\ &\quad + \hat{D}\int_y^x\left[\left(\frac{\partial K}{\partial\hat{\theta}_i} + K(\hat{\theta})A_i\hat{D}(x-\xi)\right)\right. \\ &\quad \left. \times e^{A(\hat{\theta})\hat{D}(x-\xi)}B(\hat{\theta}) + K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(x-\xi)}B_i\right] \\ &\quad \left. \times K(\hat{\theta})e^{(A+BK)(\hat{\theta})\hat{D}(\xi-y)}B(\hat{\theta})d\xi\right]dy \\ &\quad + \left(\hat{D}\int_0^x\left(\left[\frac{\partial K}{\partial\hat{\theta}_i} + K(\hat{\theta})A_i\hat{D}(x-y)\right]\right.\right. \\ &\quad \left. \times e^{A(\hat{\theta})\hat{D}(x-y)}B(\hat{\theta}) + K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(x-y)}B_i\right) \\ &\quad \times K(\hat{\theta})e^{(A+BK)(\hat{\theta})\hat{D}y}dy \\ &\quad \left. + \left[\frac{\partial K}{\partial\hat{\theta}_i} + K(\hat{\theta})A_i\hat{D}x\right]e^{A(\hat{\theta})\hat{D}x}\right)\tilde{X}(t) - K(\hat{\theta}) \\ &\quad \times e^{A(\hat{\theta})\hat{D}x}\frac{\partial X^r}{\partial\hat{\theta}_i} \\ &\quad - \hat{D}\int_0^x K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(x-y)}B(\hat{\theta})\frac{du^r}{d\hat{\theta}_i}(\hat{\theta})dy + \frac{du^r}{d\hat{\theta}_i}(\hat{\theta}) \end{aligned}$$

$$g_{0,i}(x,t) = K(\hat{\theta})e^{A(\hat{\theta})\hat{D}x}(A_iX(t) + B_iu(0,t)).$$

Appendix D. Expression of the dynamics function f in Section 8

$$\begin{aligned} f(x,t) &= \frac{\hat{w}_{0,x}(x,t)}{\hat{D}} + KB\hat{w}_0(x,t) + K(A+BK) \\ &\quad \times e^{(A+BK)\hat{D}x}\tilde{X}(t) + \hat{D}\int_0^x K(A+BK) \\ &\quad \times e^{(A+BK)\hat{D}(x-y)}B\hat{w}_0(y,t)dy. \end{aligned}$$

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