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Explicit Prediction-based Control for Linear Difference Equations with Distributed Delays

Jean Auriol[1], SiJia Kong[2] and Delphine Bresch-Pietri[2]

***Abstract*—This paper presents a prediction-based control law for linear difference equations subject to a distributed state delay and a pointwise input delay. We propose to use a prediction-based control to overcome the instability potentially related to the distributed delay. We obtain an explicit formulation of the controller, depending only on the state and input history and involving integral kernels, which are the solutions to recursive Volterra equations. In view of future delay-sensitivity analysis, we develop an alternative approach to prove closed-loop stability, recasting the input delay as a transport Partial Differential Equation. In an analog manner to the stability analysis methodology developed for linear Delay Differential Equations, we propose a backstepping transformation to map the closed-loop system to a distributed-delay free target system. Simulation results underline the efficiency of the proposed control design.**

***Index Terms*—difference systems, prediction, backstepping, stabilization**

I. Introduction

Linear Difference Equations appear in various contexts, such as sampled-data systems [10], or for a large number of hyperbolic Partial Differential Equations (PDEs). These include conservation laws [5] and wave equations, which are widely used in the modeling of transport phenomena occurring, for instance, in flow systems, catalysts equipping thermal engines or process plants, or also acoustics systems, to name a few. This property has been long noticed, with the earliest link probably going back to the D'Alembert formula, which enables to rewrite a wave equation as a difference equation. Since then, this equivalence has often been used, grounding on Riemann invariants. Recently, the exact relation between Linear First-Order Hyperbolic PDEs and Linear Difference Equations has been comprehensively studied in [4].

In this paper, we consider such a Linear Difference Equation, with both pointwise and distributed delays, and subjected to a pointwise input delay. This specific type of difference system arises, for example, when controlling a network of hyperbolic PDEs, such as mining ventilation systems [22] or oil production systems consisting of networks of pipes [23]. When control is applied at the boundary of one of the PDEs subsystems, it will act on distal PDE subsystems through proximal ones, resulting in a pointwise input delay. This is the situation we encountered in our previous work [3] investigating robust feedback for an underactuated network of interconnected PDEs, and for which we proposed to rely on an implicit prediction-based control.

Indeed, prediction-based control [1], [24], also known as Finite-Spectrum Assignment, is a standard control strategy to cope with input delay, for delay differential equations. It compensates for the input delay and recovers nominal closed-loop finite-dimensional performances at the expense of relying on a memory-based control law. Lately, its scope of application has been remarkably widened, to cover nonlinear systems [6] and more complex delay dependences [7], address implementation issues [12], [19], counteract disturbances [15], apply to neutral delay systems [13] or even tackle diffusion processes [16]. Yet, up to our knowledge, its extension to difference equations has not been investigated.

In this paper, we present an explicit formulation for prediction-based control of a linear difference equation. Assuming that the system's principal part, consisting of pointwise delay terms, is exponentially stable, we propose compensating for the destabilizing distributed state delay term with the controller. We prove that the resulting control law can be explicitly written in terms of integrals of the state and input history over a time window of fixed length. Contrary to standard prediction approaches relying on Cauchy formula and involving the fundamental matrix (see [13], [21]), our method does not ground on an explicit expression of the predictor. Instead, we directly study integral kernels of the controller and show that they satisfy Volterra equations, which can be solved offline. This is the main contribution of the paper.

Besides, in view of future delay-sensitivity analysis, we extend the backstepping design originally proposed in [14] for Ordinary Differential Equations (ODEs) to the present case of Difference Equations. With this aim in view, we rewrite the input delay as a transport PDE cascading into the Difference Equation and then propose a backstepping transformation to map the closed-loop system into a target system, the stability analysis of which is then easier to analyze.

The paper is organized as follows. Section II is devoted to the problem statement. Section III then presents our explicit formulation of the predictor, while Section IV details the corresponding backstepping transformation. Finally, Section V illustrates the merits of this prediction-based control design with a numerical example.

Notations: Inspired by [11], for any fixed $\tau > 0$, we denote

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$D_\tau = L^2([-\tau, 0], \mathbb{R}^n)$ the Banach space of L^2 functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$. For a function $\phi : [-\tau, \infty) \mapsto \mathbb{R}^n$, we define its partial trajectory $\phi_{[t]} \in D_\tau$ by $\phi_{[t]}(\theta) = \phi(t+\theta), -\tau \leq \theta \leq 0$. The associated norm is given by $||\phi_{[t]}||_{L^2} = \left(\int_{-\tau}^{0} \phi^T(t+\theta)\phi(t+\theta)d\theta\right)^{\frac{1}{2}}$. Similarly, for a function $\psi \in L^2([0,1]; \mathbb{R}^n)$, we denote $||\psi||_{L^2}$ its spatial L^2 norm. For all real a, b, for all $\nu \in \mathbb{R}$, we define the *characteristic function* $\mathbb{1}_{[a,b]}(\nu)$, as the function equal to 1 if $\nu \in [a, b]$, and equal to 0 elsewhere. For all positive integers p and q, we denote $\mathfrak{M}_{p\times q}(\mathbb{R})$ the set of real matrices with p rows and q columns. The identity matrix of size $n \in \mathbb{N}$ is denoted Id_n. The variable s denotes the Laplace variable. Provided it is defined, the Laplace transform of a function $f(t)$ will be denoted $\hat{f}(s)$ and a transfer function $G(s)$ is called strictly proper if, for sufficiently large ρ, $\sup_{\text{Re}(s)\geq 0, |s|>\rho} |G(s)| < \infty$ and if the limit of $G(s)$ at infinity exists and is 0. A function f is said to be piecewise continuous on an interval $[a, b] \subset \mathbb{R}$ if the interval can be partitioned by a finite number of points $(t_i)_{0\leq i\leq n}$ so that f is continuous on each subinterval (t_{i-1}, t_i) and admits finite right-hand and left-hand limits at each t_i. Two piecewise continuous functions f, g defined on $[a, b]$ are said to be equal if they differ only on a finite number of points. Finally, we denote $C_\tau^{pw} = C^{pw}([-\tau, 0], \mathbb{R}^n)$ the Banach space of piecewise continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ and denote its associated norm as $||\phi_{[t]}||_{C_\tau^{pw}} = \sup_{s\in[-\tau,0]} \sqrt{\phi^T(t+s)\phi(t+s)}$.

II. Problem under consideration

Consider $M \in \mathbb{N}\backslash\{0\}$ and positive time-delays $\tau_k > 0$ $(1 \leq k \leq M)$ ordered as $0 < \tau_1 \leq \tau_2 \leq ... \leq \tau_M$. Consider the following difference system with distributed delays and delayed actuation

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(-\nu)X(t+\nu)d\nu + U(t-\delta), \quad t \geq 0 \qquad (1)$$

where $A_k \in \mathfrak{M}_{n\times n}(\mathbb{R})$, and where $N(\cdot)$ is a piecewise continuous function. The initial data are given by $X_0 = X^0 \in C_{\tau_M}^{pw}$. The function U is the input function, which has values in $\mathbb{R}^n$. Its initial condition belongs to C_δ^{pw}. The delay $\delta > 0$ is a positive constant. A function $X : [-\tau_M, \infty) \to \mathbb{R}^n$ is called a solution of the initial value problem (1) if $X_0 = X^0$ and if equation (1) is satisfied for $t \geq 0$. It has been shown in [17] that having an open-loop transfer function with an infinite number of poles on the closed right half-plane (RHP) implies no (delay-) robustness margins in closed-loop for any control law[1]. Consequently, if the open-loop transfer function of the system has an infinite number of poles in the RHP, no feedback law could be implementable for practical applications (where there always are small delays in the actuation path). It can be shown that if the system $X(t) = \sum_{k=1} A_K X(t-\tau_k)$ is unstable, then it has an infinite number of unstable poles [11]. If this is the case, then so does the open-loop equation (1) (see e.g. [2, Lemma 3]). To avoid such a situation, we make the following assumption

Assumption 1: In the absence of the distributed delay term (i.e. $N \equiv 0$), the open-loop system (1) is exponentially stable. In other words, the *principal part* of the system (1) has to be exponentially stable in the sense of the $C_{\tau_M}^{pw}$-norm.

Assumption 1 requires the exponential stability of the principal part of the system (1). Thus, it is slightly stronger than the necessary condition to guarantee the possibility to design a delay-robust controller. It constitutes a reasonable assumption since it prevents system (1) from having an asymptotic chain of eigenvalues with non-negative real parts [4], [11]. Note that, if the delays are rationally independent[2], Assumption 1 is equivalent to the following condition [11] $\sup_{\theta_k\in[0,2\pi]^M} \text{Sp}\left(\sum_{k=1}^{M} A_k \exp(i\theta_k)\right) < 1$, where Sp denotes the spectral radius. Furthermore, easy to compute sufficient conditions for this spectral radius condition to hold can be derived using different norms of the involved matrices, at the cost of increased conservatism [18], [20]. In presence of the distributed delay term $\int_{-\tau_M}^{0} N(-\nu)X(t+\nu)d\nu$, Assumption 1 is not sufficient to guarantee the open-loop stability of (1). The objective of this paper is to design a control law $U(t)$ that exponentially stabilizes the system.

Remark 1: We consider in this paper that all the components of the control input U act on the system with the same delay δ. This assumption may appear very restrictive at first sight, as each component of the actuator may a priori be subject to different physical delay δ_i. However it can always be satisfied by artificially delaying[3] the components of U. Yet, this may worsen transient performances, and the case of different input delays should be considered in future works.

Remark 2: We have assumed that the bound of the integral term in (1) is equal to τ_M. This condition is not restrictive. Indeed, if the distributed delay term was defined on a time-horizon $\tau < \tau_M$, it would be possible to extend the function N by 0 on $[\tau, \tau_M]$. The resulting function would still be piecewise continuous. Conversely if $\tau > \tau_M$, we can add an artificial term $A_\tau X(t-\tau)$ (with $A_\tau = 0$) to preserve the same structure.

Using Remark 2, we can consider that $\delta = \tau_M$. Indeed, if $\delta < \tau_M$, we can still redefine the input signal such that a part of it compensates for the state terms affected by a (pointwise or distributed) delay larger than δ. Conversely, if $\delta > \tau_M$, one can add artificial terms to rewrite equation (1) with $\delta = \tau_M$.

Remark 3: Having the control input dimension equal to the state dimension is a current limitation of our approach. However, underactuated neutral systems have not been well-studied in the literature and the design of stabilizing control laws is an open question (even for the undelayed case).

Our control objective is to exactly compensate for the input delay. Without input delay ($\delta = 0$), due to Assumption 1, a possible control approach would be to let the principal

[1]i.e., the introduction of any arbitrarily small delay in the actuation will destabilize the closed-loop system

[2]Extending the variable X, it is always possible to rewrite the system in a situation where the delays are rationally independent

[3]That is, one can deliberately pick $U_i(t) = \hat{U}_i(t - \delta + \delta_i)$ in which $\delta = \max_i \delta_i$ and then define $U(t) = \hat{U}(t)$, which corresponds to equation (1)

part of the system untouched but to eliminate the state distributed delay term. In other words, one would pick $U(t) = -\int_{-\tau_M}^{0} N(-\nu)X(t+\nu)d\nu \triangleq \kappa(X_{[t]})$. This choice would lead to a strictly proper controller, which implies important consequences in terms of robustness properties [2] (in particular robustness to small delays is granted). Hence, we propose to use this control law κ, but to apply it to the prediction $P_{[t]} = X_{[t+\delta]}$ of the system state, to take the input-delay into account. Namely, we wish to apply

$$U_{\text{pred}}(t) = -\int_{-\delta}^{0} N(-\nu)P_{[t]}(\nu)d\nu, \tag{2}$$

in which the prediction $P_{[t]}$ is implicitly defined [3], [6] as

$$\begin{aligned} P_{[t]}(s) = \sum_{k=1}^{M} A_K P_{[t]}(s-\tau_k) + \int_{-\delta}^{0} N(-\nu)P_{[t]}(s+\nu)d\nu \\ + U_{[t]}(s)\,, \qquad t \geq -\delta,\ s \in [-\delta, 0] \end{aligned} \tag{3}$$

with initial condition $P_{[-\delta]} = X^0$. Though its definition is implicit, through an integral relation of Volterra type, this prediction is well-defined, as the solution to the difference equation (1). Nevertheless, its online computation could reveal troublesome. Indeed, implicit expressions of prediction-based feedback are known [12] to sometimes lead to burdensome numerical procedures, in the ODE case. This is why, in the following, we focus on an explicit formulation.

III. Explicit realization of the predictor

In this section, we propose to look for the desired control law under the form

$$U(t) = \int_{-\delta}^{0} \left[f(-\nu)X(t+\nu) + g(-\nu)U(t+\nu) \right] d\nu, \tag{4}$$

with f and g piecewise continuous matrix-valued functions to be defined (we recall that we made the simplification assumption $\delta = \tau_M$). We will then rigorously show that the resulting feedback law corresponds to the predictor (2).

To design f and g, let us first compute the quantity $X(t) - \int_{-\delta}^{0} g(-\nu)X(t+\nu)d\nu$. Using equation (1), we get for $t \geq \delta$

$$\begin{aligned} X(t) - \int_{-\delta}^{0} g(-\nu)X(t+\nu)d\nu = \sum_{k=1}^{M} A_k X(t-\tau_k) \\ + \int_{-\delta}^{0} N(-\nu)X(t+\nu)d\nu - \int_{-\delta}^{0} g(-\nu)U(t+\nu-\delta)d\nu \\ + U(t-\delta) - \sum_{k=1}^{M} \int_{-\delta}^{0} g(-\nu)A_k X(t+\nu-\tau_k)d\nu \\ - \int_{-\delta}^{0}\int_{-\delta}^{0} g(-\nu)N(-\eta)X(t+\nu+\eta)d\eta d\nu\,, \end{aligned} \tag{5}$$

in which, using the control law (4), $U(t-\delta) - \int_{-\delta}^{0} g(-\nu) U(t+\nu-\delta)d\nu = \int_{-2\delta}^{-\delta} f(-\nu-\delta)X(t+\nu)d\nu$. Besides, due to Fubini's theorem, the last integral in (5) rewrites as

$$\begin{aligned} \int_{-\delta}^{0}\int_{-\delta}^{0} g(-\nu)N(-\eta)X(t+\nu+\eta)d\eta d\nu = \\ \int_{-\delta}^{0}\int_{s}^{0} g(-s+\eta)N(-\eta)d\eta X(t+s)ds \\ + \int_{-2\delta}^{-\delta}\int_{-\delta}^{s+\delta} g(-s+\eta)N(-\eta)d\eta X(t+s)ds. \end{aligned} \tag{6}$$

Thus, equation (5) now reads $X(t) - \sum_{k=1}^{M} A_k X(t-\tau_k) = \int_{-2\delta}^{0} (\mathbb{1}_{[-\delta,0]}(\nu)I_1(-\nu) + \mathbb{1}_{[-2\delta,-\delta]}(\nu)I_2(-\nu))X(t+\nu)d\nu$, where the functions I_1 and I_2 are defined on $[0,\delta]$ and $[\delta, 2\delta]$, respectively, by

$$\begin{aligned} I_1(\nu) = g(\nu) + N(\nu) - \int_{0}^{\nu} g(\nu-\eta)N(\eta)d\eta \\ - \sum_{k=1}^{M} \mathbb{1}_{[\tau_k,\delta]}(\nu)g(\nu-\tau_k)A_k, \end{aligned} \tag{7}$$

$$\begin{aligned} I_2(\nu) = f(\nu-\delta) - \int_{\nu-\delta}^{\delta} g(\nu-\eta)N(\eta)d\eta \\ - \sum_{k=1}^{M} \mathbb{1}_{[\delta,\tau_k+\delta]}(\nu)g(\nu-\tau_k)A_k. \end{aligned} \tag{8}$$

Hence, our control objective corresponds to finding f and g such that $I_1 \equiv 0$ and $I_2 \equiv 0$. The following lemma states that these solutions are uniquely defined.

Lemma 1: Consider the functions I_1 and I_2 defined in equations (7)-(8). There exist two unique piecewise continuous functions (f, g) such that $I_1(\nu) = 0$ for $\nu \in [0,\delta]$, and $I_2(\nu) = 0$ for $\nu \in [\delta, 2\delta]$.

Proof: Consider first (7) and the equation $I_1(\nu) = 0$, $\nu \in [0,\delta]$. For $\nu < \tau_1$, we obtain $g(\nu) = -N(\nu) + \int_0^{\nu} g(\eta)N(\nu-\eta)d\eta$, which is a Volterra equation of the second kind and thus admits a unique continuous solution on $[0, \tau_1)$ (see [25]). Consider now $\nu \in [\tau_1, \min\{\tau_2, 2\tau_1\})$. Using (7), the equation $I_1(\nu) = 0$ now rewrites as the Volterra equation $g(\nu) = h(\nu) + \int_{\tau_1}^{\nu} g(\eta)N(\nu-\eta)d\eta$, in which $h(\nu) \triangleq -N(\nu) + \int_0^{\tau_1} g(\eta)N(\nu-\eta)d\eta + g(\nu-\tau_1)A_1$ is given, as $g(\nu)$ is known for $\nu \in [0, \tau_1)$. It thus admits a unique solution on $[\tau_1, \min\{\tau_2, 2\tau_1\})$. Iterating the process, g is uniquely defined on $[\tau_1, \tau_2)$. Thus, by a straightforward induction, there exists a unique piecewise continuous function g defined on $[0,\delta]$. Given this function g, equation (8) then provides an explicit expression of f that leads to $I_2 \equiv 0$. ∎

We can now write the following theorem.

Theorem 1: Consider the functions I_1 and I_2 defined in (7)-(8) and let f and g be the unique piecewise continuous functions that leads to $I_1(\nu) = 0$ for all $\nu \in [0,\delta]$ and to $I_2(\nu) = 0$ for all $\nu \in [\delta, 2\delta]$ (as stated in Lemma 1). Then, the closed-loop system consisting of the plant (1) and the control law in (4) rewrites, for $t \geq \delta$, as $X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k)$ and is consequently exponentially stable in the sense of the $C_{\tau_M}^{pw}$-norm under Assumption 1. Moreover, the control law (4) is strictly proper and exponentially converges to zero.

Proof: We start this proof by showing that, as desired, U defined in (4) satisfies $U(t) = \kappa(X_{[t+\delta]})$. Using the definitions of the controller in (4) and of the function f in (8), we have

$$U(t) = \int_{-2\delta}^{-\delta} f(-\nu-\delta)X(t+\nu+\delta)d\nu + \int_{-\delta}^{0} g(-\nu)U(t+\nu)d\nu$$

$$= \int_{-2\delta}^{-\delta}\int_{-\nu-\delta}^{\delta} g(-\nu-\eta)N(\eta)d\eta X(t+\nu+\delta)d\nu$$
$$+ \int_{-\delta}^{0} g(-\nu)U(t+\nu)d\nu$$
$$+ \sum_{k=1}^{M}\int_{-\delta}^{-\delta+\tau_k} g(-\nu)A_k X(t+\nu+\delta-\tau_k)d\nu \tag{9}$$

Using the plant equation (1), the last term of this expression can be replaced by

$$\int_{-\delta}^{0} g(-\nu)X(t+\nu+\delta)d\nu - \int_{-\delta}^{0} g(-\nu)U(t+\nu)d\nu$$
$$- \int_{-\delta}^{0}\int_{-\delta}^{0} g(-\nu)N(-\eta)X(t+\nu+\delta+\eta)d\eta d\nu$$
$$- \sum_{k=1}^{M}\int_{-\delta+\tau_k}^{0} g(-\nu)A_k X(t+\nu+\delta-\tau_k)d\nu.$$

Hence, injecting this expression into (9) and using (6), we get the desired result

$$U(t) = -\int_{-\delta}^{0} N(-\nu)X(t+\nu+\delta)d\nu, \tag{10}$$

where the last equation is obtained using the definition of g given by (7). This shows that the control law (4) is an explicit realization of the predictor (2) and thus, replacing the controller in (1), that the closed-loop system rewrites as the principal part of (1) for $t \geq \delta$. Due to Assumption 1, it is exponentially stable and, since $X(t)$ exponentially converges to zero, so does $U(t)$ due to equation (1). Finally, taking the Laplace transform of equation (4), we obtain $\hat{U}(s) = \frac{\int_{-\delta}^{0} f(-\nu)e^{\nu s}d\nu}{1-\int_{-\delta}^{0} g(-\nu)e^{\nu s}d\nu}\hat{X}(s)$, which, due to Riemann-Lebesgues lemma, defines a strictly proper transfer function. ■

Having a strictly proper control law guarantees the w-stability of the closed-loop system (see [3, Theorem 17] and [9, Theorem 9.5.4]) and consequently the robustness to delays and uncertainties on the parameters.

IV. Transport PDE reformulation

In the previous section, we designed an explicit representation of a prediction-based controller cancelling the distributed delays terms from (1), thus resulting in an exponentially stable system. An important direction of work would then be to evaluate the robustness of this controller to an uncertain input delay[4], as prediction-based controller are known to be sensitive to this feature [12]. This robustness analysis has been carried out for prediction-based control of ODEs with an input delay (see [8] for instance) with a backstepping design originally proposed in [14]. Thus, in this section, we propose to follow this line of research and design a backstepping framework for the present case of Difference Equations.

With this aim in view, let us define the distributed actuator vector $v(t,x) = (v_1(t,x),...,v_n(t,x)) = U(t+\delta(x-1))$ corresponding to the input delay. The time-delay equation (1) then rewrites as the following PDE-Difference Equation cascade

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\delta}^{0} N(-\nu)X(t+\nu)d\nu + v(t,0) \tag{11}$$
$$v_t(t,x) = \Lambda v_x(t,x), \tag{12}$$
$$v(t,1) = U(t), \tag{13}$$

where $\Lambda = \frac{1}{\delta}\mathrm{Id}_n$, and where $x \in [0,1]$. The state v belongs to $L^2([0,1],\mathbb{R}^n)$, as well as its initial condition $v(0,x)$.

A. Bacsktepping transformation

We want to map system (11)-(13) to the target system

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \gamma(t,0), \tag{14}$$
$$\gamma_t(t,x) = \Lambda\gamma_x(t,x), \tag{15}$$
$$\gamma(t,1) = 0, \tag{16}$$

To this end, we introduce the backstepping transformation

$$\gamma(t,x) = v(t,x) - \int_{0}^{x} \delta g(\delta(x-y))v(t,y)dy$$
$$- \sum_{k=1}^{M}\int_{-\tau_k}^{\min(0,\delta x-\tau_k)} g(-y+\delta x-\tau_k)A_k X(t+y)dy$$
$$- \int_{-\delta}^{\delta(x-1)}\int_{-\delta}^{y} g(\delta x+\eta-y)N(-\eta)d\eta X(t+y)dy$$
$$+ \int_{\delta(x-1)}^{0}\left(N(-y+\delta x) - \int_{y-\delta x}^{y} g(\delta x+\eta-y)N(-\eta)d\eta\right)$$
$$\times X(t+y)dy. \tag{17}$$

We then have the following lemma.

Lemma 2: Consider the system (11)-(13) with the initial condition $(v(0,\cdot),X_0) \in L^2([0,1],\mathbb{R}^n) \times L^2([-\delta,0],\mathbb{R}^n)$. The backstepping transformation (17) along with the control law (4) transform this original system into the target system (14)-(16).

Proof: We only give a sketch of proof due to space restriction. Computing (17) for $x=0$, we obtain $\gamma(t,0) = v(t,0) + \int_{-\delta}^{0} N(-y)X(t+y)dy$, which gives equation (14). Computing (17) for $x=1$, we obtain

$$\gamma(t,1) = v(t,1) - \int_{0}^{1} \delta g(\delta(1-y))v(t,y)dy$$
$$- \int_{-\delta}^{0}\left(\sum_{k=1}^{M} \mathbb{1}_{[-\tau_k,0]}(y)g(-y+\delta-\tau_k)A_k\right.$$
$$\left. + \int_{-\delta}^{y} g(\delta+\eta-y)N(-\eta)d\eta\right)X(t+y)dy$$

Since $v(t,x) = U(t+\delta(x-1)))$, the first integral reads as $\int_{-\delta}^{0} g(-\eta)U(t+\eta)d\eta$. To deal with the second integral term, we use equation (8) and obtain $\gamma(t,1) = U(t) - \int_{-\delta}^{0} g(-\eta)U(t+\eta)d\eta - \int_{-\delta}^{0} f(-\eta)X(t+\eta)d\eta = 0$. To show that the PDE (15) is verified, we differentiate equation (17) with respect to time and space and integrate by parts. To deal with the term $\min(0,\delta x-\tau_k)$ that appears in the bound of the second integral, we consider the cases $\delta x > \tau_k$ and $\delta x \leq \tau_k$,

[4] Notice that the robustness to state delays is somehow inferred by the strong stability assumption of Assumption 1, see [11].

perform the computations in each case, and use a characteristic function to obtain a single common expression (as imposing $\delta x > \tau_k$ corresponds to the multiplication by $\mathbb{1}_{[\tau_k,\delta]}(\delta x)$). After simplification, one gets

$$\gamma_t(t,x) - \Lambda\gamma_x(t,x) = \Bigg(g(\delta x) - \int_0^{\delta x} g(\delta x-\eta)N(\eta)d\eta + N(\delta x) - \sum_{k=1}^{M}\mathbb{1}_{[\tau_k,\delta]}(\delta x)g(\delta x-\tau_k)A_k\Bigg)X(t).$$

Using the definition of the function g through equation (7), we finally obtain (15), which concludes the proof. ∎

B. Lyapunov analysis

In this section, we now propose a Lyapunov functional to put the finishing touch to the stability analysis of (14)–(16) via backstepping. From now on and without any loss of generality[5], we will assume that $n = M$. Inspired by [5], we introduce the function $\rho_2 : \mathfrak{M}_{p\times p}(\mathbb{R}) \to \mathbb{R}$ ($p \in \mathbb{N}$) as $\rho_2(M) = \inf\{||\Delta M\Delta^{-1}||_2,\ \Delta \in \mathcal{D}_p^+\}$, where $\mathcal{D}_p^+$ denotes the set of diagonal $p\times p$ real matrices with positive diagonal entries. We now define the matrices $R \in \mathfrak{M}_{n^2\times n^2}$ and $B \in \mathfrak{M}_{n^2\times n}$ as

$$R = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ \mathrm{Id}_n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Id}_n & 0 & \cdots & 0 \end{pmatrix},\ B = \begin{pmatrix} \mathrm{Id}_n \\ \vdots \\ \mathrm{Id}_n \end{pmatrix}, \tag{18}$$

and the following candidate Lyapunov functional, inspired from [5, Chapter 3]

$$\begin{aligned} V(t) &= \int_0^1 \sum_{j=1}^n p_1^j \tau_1 X_j^2(t-\tau_1 x)\exp(-\nu\tau_1 x)dx \\ &+ \int_0^1 \sum_{i=2}^n\sum_{j=1}^n p_i^j(\tau_i-\tau_1)X_j^2(t-\tau_1-(\tau_i-\tau_1)x) \\ &\times \exp(-\nu(\tau_i-\tau_1)x)dx + \int_0^1 \sum_{i=1}^n q_i\delta\gamma_i^2(t,x)\exp(\nu\delta x)dx\,, \end{aligned} \tag{19}$$

where the coefficients p_i^j, q_i and ν are positive.

Lemma 3: If $\rho_2(R) < 1$, then V is a strict Lyapunov function, that is there exists $\nu > 0$ such that $\dot V(t) \le -\nu V(t)$ for $t \ge 0$. Consequently the solution of (14)-(16) exponentially converges to zero.

Proof: The proof is inspired by the one of [5, Theorem 2]. Let us introduce $K = \begin{pmatrix} R & B \\ 0_{n\times n^2} & 0_{n\times n}\end{pmatrix}$. Observe that since $\rho_2(R) < 1$, then $\rho_2(K) < 1$. Indeed, consider $\Delta \in \mathcal{D}^+_{n^2+n}$. The matrix Δ rewrites $\Delta = \mathrm{diag}(\Delta_1\Delta_2)$, where $\Delta_1 \in \mathcal{D}^+_{n^2}$ and $\Delta_2 \in \mathcal{D}^+_n$. We have $\Delta K\Delta^{-1} = \begin{pmatrix} \Delta_1 R\Delta_1^{-1} & \Delta_1 B\Delta_2^{-1} \\ 0_{n\times n^2} & 0_{n\times n}\end{pmatrix}$. For any $\xi = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \in \mathbb{R}^{n^2+n}$ (with $\xi_1 \in \mathbb{R}^{n^2}$ and $\xi_2 \in \mathbb{R}^n$) such that $||\xi|| = 1$, one gets $||\Delta K\Delta^{-1}\xi||_2 \le ||\Delta_1 R\Delta_1^{-1}\xi_1||_2 + ||\Delta_1 B\Delta_2^{-1}\xi_2||_2$. Choosing the coefficients of Δ_2 large enough, the second term can be made arbitrarily small and the desired result follows. Besides, the time derivative of V along the solutions of (14)-(16) is

$$\begin{aligned} \dot V(t) &= -\nu V(t) - \sum_{i=1}^n X^T(t-\tau_i)P_i(\nu)X(t-\tau_i) \\ &+ X^T(t)P_1(0)X(t) + \sum_{i=2}^n X^T(t-\tau_1)P_i(0)X(t-\tau_1) \\ &+ \gamma(t,1)^TQ(\nu)\gamma(t,1) - \gamma(t,0)^TQ(0)\gamma(t,0) \end{aligned} \tag{20}$$

where the matrices $P_i(\nu x)$ and $Q(\nu x)$ are defined by

$$\begin{aligned} P_1(\nu x) &= e^{-\tau_1\nu x}\mathrm{diag}\begin{pmatrix} p_1^1 & \cdots & p_1^n\end{pmatrix}, \\ P_i(\nu x) &= e^{-(\tau_i-\tau_1)\nu x}\mathrm{diag}\begin{pmatrix} p_i^1 & \cdots & p_i^n\end{pmatrix}, \\ Q(\nu x) &= e^{\delta\nu x}\mathrm{diag}\begin{pmatrix} q_1 & \cdots & q_n\end{pmatrix} \end{aligned}$$

Let us denote $P(\nu x) = \mathrm{diag}\left(P_1(\nu x),\cdots,P_n(\nu x)\right)$ and $Y = (X(t-\tau_1)^T,\ldots,X(t-\tau_n)^T)^T$. Using equation (14) and the boundary condition (16), equation (20) rewrites as

$$\begin{aligned} \dot V(t) &= -\nu V(t) - Y^T(t)P(\nu)Y(t) - \gamma(t,0)^TQ(0)\gamma(t,0) \\ &+ (Y^T(t)R^T + \gamma(t,0)B^T)P(0)(RY(t) + B\gamma(t,0)). \end{aligned} \tag{21}$$

Since $\rho_2(K) < 1$, there exists $\Delta = \mathrm{diag}(D_0, D_1)$ such that $||\Delta K\Delta^{-1}|| < 1$. Let us choose the parameters p_i^j such that $P(0) = D_0^2$ and $Q(0) = D_1^2$. Define the matrix Ω and W as

$$\Omega(\nu) = \begin{pmatrix} P(\nu)D_0^{-2} - D_0^{-1}R^TD_0^2RD_0^{-1} & -D_0^{-1}R^TD_0^2BD_1^{-1} \\ -D_1^{-1}B^TD_0^2RD_0^{-1} & \mathrm{Id}_n - D_1^{-1}B^TD_0^2BD_1^{-1}\end{pmatrix}$$

$$W(\nu) = -\begin{pmatrix} Y^T(t)D_0 & \gamma^T(t,0)D_1\end{pmatrix}\Omega(\nu)\begin{pmatrix} D_0Y(t) \\ D_1\gamma(t,0)\end{pmatrix}.$$

Equation (20) thus rewrites as $\dot V(t) = -\nu V(t) + W(\nu)$. Since $||\Delta K\Delta^{-1}|| < 1$, we have that $W(0)$ is a strictly negative quadratic form. By continuity, for ν small enough, we obtain $\dot V(t) \le -\nu V(t)$. To conclude the proof, we use the fact that V is equivalent to the norm $\|X_{[t]}\|_{D_\delta} + \|\gamma(t,\cdot)\|_{L_2}$. ∎

We can now formulate an alternative version to Theorem 1.

Theorem 2: Consider the functions I_1 and I_2 defined in (7)-(8) and let f and g be the unique piecewise continuous functions that lead to $I_1(\nu) = 0$ for all $\nu \in [0,\delta]$ and to $I_2(\nu) = 0$ for all $\nu \in [\delta, 2\delta]$ (as stated in Lemma 1). Then, the closed-loop consisting of (11)-(13) satisfying Assumption 1 and $\rho_2(R) < 1$, and with the initial condition $(v(0,\cdot), X_0) \in L^2([0,1],\mathbb{R}^n) \times L^2([-\delta,0],\mathbb{R}^n)$, and the control law (4) can be mapped into the target system (14)-(16). Furthermore, it is exponentially stable in the sense of the norm $\|X_{[t]}\|_{D_\delta} + \|v(t,\cdot)\|_{L_2}$.

Proof: The result is a direct consequence of Lemmas 2 and 3, and the inverse backstepping transformation of (17) (which exists due to Volterra integral equation theory). ∎

This backstepping design opens the path for further analysis, such as sensitivity to delay uncertainties for instance. Notice that, compared to Theorem 1, exponential stability only holds in the L_2 sense in Theorem 2, due to the quadratic nature of the Lyapunov functional. Also, it is worth underlining that Theorem 2 requires the coupling matrix R to satisfy $\rho_2(R) < 1$, which translates into a requirement on the pointwise state delay matrix A_i which is stronger than Assumption 1. These

[5]Indeed, if $n \neq M$, we can artificially extend the state X and the matrices A_k, completing them with zero coefficients.

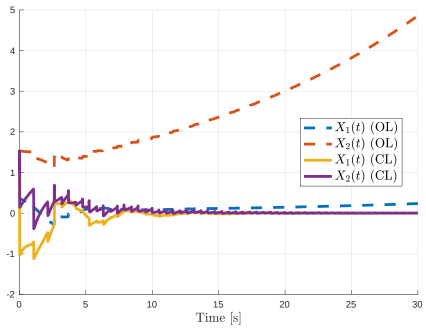

Fig. 1. Evolution of the states $\boldsymbol{X_1}$, $\boldsymbol{X_2}$ in open loop (OL) and in closed loop (CL) using the control law (9). The initial state is $\boldsymbol{X}_{[0]} \equiv \mathbf{1}$.

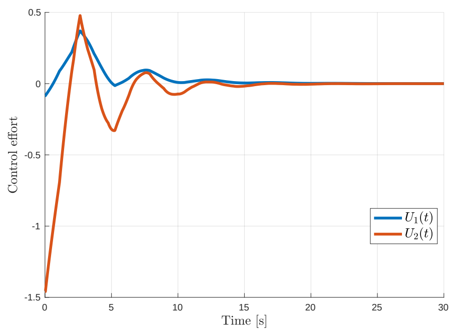

Fig. 2. Evolution of the control input $\boldsymbol{U(t)}$. The initial input is $\boldsymbol{U}_{[0]} \equiv \mathbf{0}$.

two points should thus be relaxed in future works. Recently, a quadratic Lyapunov functional has been proposed in [21] for exponentially stable linear difference systems. Further study of this functional to apply it to the case of additional source terms in the difference equation is thus an interesting direction to explore.

V. Simulation results

In this section, we illustrate the performances of our controller. We implemented the proposed approach using Matlab. The integral equations (7)-(8) are solved using a fixed point procedure. We consider $n = M = 2$, and define the coupling matrices as $A_1 = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -0.3 & 0.1 \\ -0.2 & 0.5 \end{pmatrix}$. The different delays are defined by $\tau_1 = 1$, $\tau_2 = 2.5$. The input delay is $\delta = 2$. Finally, the integral coupling term N is defined by $N(\nu) = \begin{pmatrix} \sin(\nu)\mathrm{e}^{-\nu} & -0.1 \\ 0.1 & 0.2\sin(\nu) \end{pmatrix}$, where the argument $\nu \in [0, \tau_2]$. Assumption 1 and the condition $\rho_2(R) < 1$ can be numerically verified. We have plotted in Fig. 1 the time evolution of the state X in open loop and in closed loop using the control law (9). To show the robustness of the proposed control strategy, the control input is subject to a 0.02s delay and the delays τ_i are subject to a 10% uncertainties. The control input is also subject to a small additive disturbance. The system is unstable in open-loop due to the presence of the integral coupling N, but exponentially converges to zero in closed loop even in the presence of the delay and uncertainties. The control effort is shown in Fig. 2.

VI. Conclusion

This work proposed an explicit formulation of a prediction-based controller for linear difference systems. It also presented the preliminary steps toward generic backstepping analysis of such a control law. Future works should investigate alternative Lyapunov functional for stability analysis, such as the one proposed recently in [21]. Extension of the proposed framework to the case of multiple input delays is also an important path to explore. Finally, a crucial question to address is the extension of this technique in the case of uncertain or time-varying delays.

References

[1] Z. Artstein. Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic control*, 27(4):869–879, 1982.
[2] J. Auriol, U. J. F. Aarsnes, P. Martin, and F. Di Meglio. Delay-robust control design for heterodirectional linear coupled hyperbolic PDEs. *IEEE Transactions on Automatic Control*, 2018.
[3] J. Auriol and D. Bresch-Pietri. Robust state-feedback stabilization of an underactuated network of interconnected $n+m$ hyperbolic pde systems. *Automatica*, 136:110040, 2022.
[4] J. Auriol and F. Di Meglio. An explicit mapping from linear first order hyperbolic PDEs to difference systems. *Systems & Control Letters*, 123:144–150, 2019.
[5] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*. Springer, 2016.
[6] N. Bekiaris-Liberis and M. Krstic. *Nonlinear control under nonconstant delays*. SIAM, 2013.
[7] N. Bekiaris-Liberis and M. Krstic. Robustness of nonlinear predictor feedback laws to time-and state-dependent delay perturbations. *Automatica*, 49(6):1576–1590, 2013.
[8] D. Bresch-Pietri, J. Chauvin, and N. Petit. Adaptive control scheme for uncertain time-delay systems. *Automatica*, 48(8):1536–1552, 2012.
[9] R.F Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*. Springer Science & Business Media, 2012.
[10] E. Fridman. *Introduction to time-delay systems: Analysis and control*. Springer, 2014.
[11] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional differential equations*. Springer-Verlag, 1993.
[12] I. Karafyllis and M. Krstic. *Predictor feedback for delay systems: Implementations and approximations*. Springer, 2017.
[13] V. Kharitonov. Predictor based stabilization of neutral type systems with input delay. *Automatica*, 52:125–134, 2015.
[14] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*, volume 16. Siam, 2008.
[15] V. Léchappé, E. Moulay, F. Plestan, A. Glumineau, and A. Chriette. New predictive scheme for the control of lti systems with input delay and unknown disturbances. *Automatica*, 52:179–184, 2015.
[16] H. Lhachemi and C. Prieur. Predictor-based output feedback stabilization of an input delayed parabolic pde with boundary measurement. *Automatica*, 137:110115, 2022.
[17] H. Logemann, R. Rebarber, and G. Weiss. Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. *SIAM Journal on Control and Optimization*, 34(2):572–600, 1996.
[18] D. Melchor-Aguilar. Exponential stability of linear continuous time difference systems with multiple delays. *Systems & Control Letters*, 62(10):811–818, 2013.
[19] S. Mondié and W. Michiels. Finite spectrum assignment of unstable time-delay systems with a safe implementation. *IEEE Transactions on Automatic Control*, 48(12):2207–2212, 2003.
[20] S.-I. Niculescu. *Delay effects on stability: a robust control approach*, volume 269. Springer Science & Business Media, 2001.
[21] E. Rocha Campos, S. Mondié, and M. Di Loreto. Necessary stability conditions for linear difference equations in continuous time. *IEEE Transactions on Automatic Control*, 63(12):4405–4412, 2018.
[22] O. Rodriguez-Diaz, D. Novella-Rodriguez, E. Witrant, and E. Franco-Mejia. Control strategies for ventilation networks in small-scale mines using an experimental benchmark. *Asian Journal of Control*, 23(1):72–81, 2021.
[23] B. Saldivar, S. Mondié, S.-I. Niculescu, H. Mounier, and I. Boussaada. A control oriented guided tour in oilwell drilling vibration modeling. *Annual Reviews in Control*, 42:100 – 113, 2016.
[24] O. Smith. A controller to overcome dead time. *ISA J.*, 6:28–33, 1959.
[25] K. Yoshida. *Lectures on differential and integral equations*, volume 10. Interscience Publishers, 1960.