Probabilistic Sufficient Conditions for Prediction-based Stabilization of Linear Systems with Random Input Delay

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***Abstract*—This paper focuses on the prediction-based stabilization of a linear system subject to a random input delay. Modeling the delay as a finite-state Markov process, it proves that a constant time-horizon prediction enables robust compensation of the delay, provided the horizon prediction is sufficiently close to the delay values in average. Simulation results emphasize the practical relevance of this condition.**

***Keywords*—Delay systems, prediction-based controller, distributed parameter systems.**

I. INTRODUCTION

Delays are among the most common phenomena in engineering practice [15] as they range from control or sensing processing time, to transport delays. Lately, with the rise of communication and information technologies, communication delays have become a major concern for multiple research areas, such as multi-agent systems coordination or traffic estimation and control based on Vehicle-to-Vehicle transmissions. In such network controlled systems, transmitted information often suffers from lag, data reordering, packets dropouts or quantization [19]. These phenomena can be accounted for by a random delay model.

This paper considers the case where such a random delay affects the input of a dynamical system and investigates the design of a prediction-based controller to compensate for this delay. This control technique is well-known for constant delays [11], [13], [20], and has since then been extended to various contexts including time-varying delays [1], uncertain input delays or disturbances [14], and nonlinear systems [2], [5]. It consists in computing a state prediction over a time window of the length of the (current or future) delay, and using this prediction in the feedback loop to eliminate or mitigate the effect of the delay in the closed-loop dynamics. Yet, while this technique has been recently applied to linear random Delay Differential Equations in [3] and for a deterministic delay term multiplied by a random variable in [12], its extension to the case where the delay itself is a random variable remained to be carried out.

Recently, in our preliminary work [8], we studied for the first time the problem of prediction-based control of dynamical systems subject to random input delays and proposed to use a constant time-horizon prediction. The present paper pursues this study and extends significantly its scope of application. Modeling the delay as a Markov process with a finite number of states as in [7], we reformulate the dynamics as a Partial Differential Equation-Ordinary Differential Equation (PDE-ODE) system as in the now standard methodology for stability analysis of input-delay systems proposed by Krstic and coworkers [2], [10], but applied to our random context. Using the so-called technique of probabilistic delay averaging [7] to a new Lyapunov functional, we prove that mean-square exponential stabilization of the closed-loop system is obtained provided the prediction horizon is sufficiently close to the delay values in average, in the sense of the expected value. This considerably generalizes the condition of [8] which was deterministic and thus quite restrictive, and constitutes the main contribution of the paper.

The paper is organized as follows. In Section II, we formulate the problem under consideration, design a constant-horizon prediction-based controller and formulate our stabilization result. In Section III, we propose a backstepping reformulation of the system, which then enables us to analyze the stability of the closed-loop system in Section IV and prove the paper main result. Simulation results are then provided in Section V to illustrate the practical relevance of the proposed sufficient condition for stabilization.

Notations. In the following sections, for a signal $v:(x,t)\in[0,1]\times\mathbb{R}\rightarrow v(x,t)\in\mathbb{R}$, we denote $\|v(t)\|$ its spatial $\mathcal{L}_2$-norm with respect to x.

$\lambda(A)$ denotes the spectrum of a square matrix A, while $\min(\lambda(A))$ and $\max(\lambda(A))$ are its minimum and maximum eigenvalues, respectively. Additionally, $|A|$ denotes its Euclidean norm $|A|=\sqrt{\max(\lambda(A^TA))}$ in which A^T denotes the transpose of A.

$\mathbb{E}(x)$ denotes the expectation of a random variable x. For a random signal $x(t)$ ($t\in\mathcal{T}\subset\mathbb{R}$), the conditional expectation of $x(t)$ at the instant t knowing that $x(s)=x_0$ at the instant $s\leq t$ is denoted $\mathbb{E}_{[s,x_0]}(x(t))$. Finally, e_i denotes the ith standard basis vector of $\mathbb{R}^r$ ($r\in\mathbb{N}_+$ and $i\in\{1,...,r\}$).

II. PROBLEM STATEMENT AND MAIN RESULT

We consider a controllable linear system with random input delay of the form

$$\dot{X}(t)=AX(t)+BU(t-D(t)),\qquad(1)$$

in which $X\in\mathbb{R}^n$ and $U\in\mathbb{R}$ are the state and control input, respectively. The random delay D is a Markov process with the following properties:

(P1) $D(t)\in\{D_i, i\in\{1,\ldots,r\}\}$, $r\in\mathbb{N}$ with $0<\underline{D}\leq D_1<D_2<\cdots<D_r\leq\overline{D}$.

(P2) The transition probabilities $P_{ij}(t_1,t_2)$, which quantify the probability to switch from D_i at time t_1 to D_j at time t_2 ($(i,j)\in\{1,\ldots,r\}^2$, $t_2\geq t_1\geq 0$) satisfy

a) $P_{ij}:\mathbb{R}^2\rightarrow[0,1]$ with $\sum_{j=1}^{r}P_{ij}(t_1,t_2)=1$.

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b) P_{ij} is a differentiable function which, for $s < t$, follows the Kolmogorov equation

$$\frac{\partial P_{ij}(s,t)}{\partial t} = -c_j(t)P_{ij}(s,t) + \sum_{k=1}^{r} P_{ik}(s,t)\tau_{kj}(t)\,,$$
$$P_{ii}(s,s) = 1 \text{ and } P_{ij}(s,s) = 0 \text{ for } i \neq j\,, \quad (2)$$

in which τ_{ij} and $c_j = \sum_{k=1}^{r} \tau_{jk}$ are positive-valued functions such that $\tau_{ii}(t) = 0$. In addition, we assume that the functions τ_{ij} are bounded by a constant $\tau^\star > 0$.

(P3) The realizations of D are right-continuous.

Considering a finite number of delay values in (P1) is a common assumption considered, e.g., in [7], [18]. In the case of network systems, these discrete values can be seen as a measure of the congestion state of the network. Likewise, Property (P3) is standard for the modeling of continuous-time Markov Chain and to assess the system well-posedness.

Furthermore, it is important to emphasize that the two properties (P1) and (P3), along with the Markov property, guarantee that P_{ij} satisfies the Kolmogorov Equation (2) for certain positive-valued functions τ_{ij}, c_j (see [16], [17]). In that sense, Property (P2) is only requiring the functions τ_{ij} to be bounded, which constitutes a mild modeling assumption.

Finally, its is worth observing that the parameter $\tau_{ij}\Delta t$ is approximately the probability of transition from D_i to D_j on the interval $[t, t+\Delta t)$. Similarly, $1 - c_j(t)\Delta t$ represents the probability of staying at D_j during the time interval $[t, t+\Delta t)$.

We aim at controlling the random-delay system (1) with a prediction-based controller. As the delay may not be known[1] and, in any case, varies abruptly, using the current delay value as prediction horizon would in all likelihood result into a chattering control law and an inaccurate prediction[2]. Thus, we propose to use the following prediction-based controller with constant time horizon D_0 ($D_0 \in [\underline{D}, \overline{D}]$)

$$U(t) = K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-s)}BU(s)ds\right]\,, \quad (3)$$

in which K is a feedback gain such that $A + BK$ is Hurwitz. Obviously, contrary to the deterministic case [1], such a predictor can only *robustly* compensate for the random input delay. We now provide the main result of the paper: a sufficient condition for such a robust compensation.

Theorem 1: Consider the closed-loop system consisting of the system (1) and the control law (3). There exists a positive constant $\epsilon^\star(K)$, such that if, for all time $t \geq 0$,

$$\mathbb{E}_{[0,D(0)]}(|D_0 - D(t)|) \leq \epsilon^\star(K)\,, \quad (4)$$

then, the closed-loop system is mean-square exponentially stable, that is,

$$\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(t)) \leq R\Upsilon(0)e^{-\gamma t}\,, \quad (5)$$

for certain positive constants R and γ and with

$$\Upsilon(t) = |X(t)|^2 + \int_{t-3\overline{D}}^{t} U(s)^2 ds\,. \quad (6)$$

Theorem 1 requires the prediction horizon to be sufficiently close in average to the delay values. This is in accordance with the constant-horizon feature of the prediction used in (3) and generalizes the restrictive deterministic condition obtained in our previous work [8] and bearing on $|D_0 - D_l|$ for all $l \in \{1, \ldots, r\}$. Indeed, requiring $|D_0 - D(t)|$ to be small enough for all time implies that the delay values are themselves close enough, otherwise such a choice of D_0 cannot be achieved. In that sense, Condition (4) represents a considerable relaxation, by distinguishing among the delay distributions.

Besides, it is worth mentioning that an expression for the positive constant $\epsilon^\star$ is provided[3] in the proof of Theorem 1 detailed in the sequel. However, this value is likely to be of very little practical use, due to the conservativeness of the Lyapunov analysis carried out. Nevertheless, thanks to this expression, one can observe that the positive constant $\epsilon^\star$ depends on the feedback gain and that this dependence is likely to be considerable. Providing a quantitatively meaningful bound and studying its relation with K in view of increasing the closed-loop robustness is out of the scope of the present paper, but should be the focus of future works.

Finally, note that the interval of definition of the integral in (6) is $[t-3\overline{D}, t]$ for a technical reason, namely, for Lemma 3 used in the stability analysis to hold. We now provide the proof of this theorem in the following sections.

III. Backstepping transformation

In order to prove Theorem 1, we follow the now standard stability analysis methodology for input-delay systems of [2], [10]. Consequently, we reformulate (1) as a PDE-ODE cascade and introduce additional distributed variables to better account for the effect of the prediction-based control law (3).

We denote $\mathbf{v}(x,t) = \big(v_1(x,t) \ldots v_k(x,t) \ldots v_r(x,t)\big)^T$ the vector of distributed actuators $v_k(x,t) = U(t + D_k(x-1))$. Then, one can rewrite the linear system (1) as the following PDE-ODE cascade with random parameter δ

$$\begin{cases} \dot{X}(t) = AX(t) + B\delta(t)^T\mathbf{v}(0,t) \\ \Lambda_D \mathbf{v}_t(x,t) = \mathbf{v}_x(x,t) \\ \mathbf{v}(1,t) = \mathbf{1}U(t)\,, \end{cases} \quad (7)$$

in which $\Lambda_D = diag(D_1, \ldots, D_r)$, $\mathbf{1}$ is a r-by-1 all-ones vector and $\delta(t) \in \mathbb{R}^r$ is such that, if $D(t) = D_j$, $\delta(t) = e_j$, the j^{th} vector of the standard basis of $\mathbb{R}^r$. Hence, $\delta(t)$ is a Markov process with the same transition probabilities as the process $D(t)$, but with the finite number of states (e_i) instead of (D_i).

Now, we introduce several distributed variables

$$\hat{v}(x,t) = U(t + D_0(x-1))\,, \quad (8)$$
$$\tilde{\mathbf{v}}(x,t) = \mathbf{v}(x,t) - \mathbf{1}\hat{v}(x,t)\,, \quad (9)$$
$$\mu(x,t) = U(t - D_0 + (3\overline{D} - D_0)(x-1))\,. \quad (10)$$

[1] Time stamping of the exchange data can be used, but requires the controller internal clock and the system one to be synchronized, which could be difficult to guarantee in practice.

[2] For instance, if the current delay value is much larger than its average, leading to an over-estimation of the prediction horizon.

[3] Namely, in Equation (31), involving itself various other parameters such as the intermediate constants chosen in (a)–(e) or introduced in Lemmas 3 and 4.

In details, $\hat{v}$ represents the control input $U(t)$ within the interval $[t-D_0, t]$, $\tilde{\mathbf{v}}$ the corresponding input estimation error while μ represents the controller within the interval $[t-3\overline{D}, t-D_0]$. The extended state $(X(t), \hat{v}(x,t), \tilde{\mathbf{v}}(x,t), \mu(x,t))$ then satisfies

$$\begin{cases} \dot{X}(t) = AX(t) + B\hat{v}(0,t) + B\delta(t)^T\tilde{\mathbf{v}}(0,t) \\ D_0\hat{v}_t(x,t) = \hat{v}_x(x,t) \\ \hat{v}(1,t) = U(t) \\ \Lambda_D\tilde{\mathbf{v}}_t(x,t) = \tilde{\mathbf{v}}_x(x,t) - \Sigma_D\hat{v}_x(x,t) \\ \tilde{\mathbf{v}}(1,t) = \mathbf{0} \\ (3\overline{D} - D_0)\mu_t(x,t) = \mu_x(x,t) \\ \mu(1,t) = \hat{v}(0,t)\,, \end{cases} \tag{11}$$

in which $\Sigma_D = (\frac{D_1 - D_0}{D_0}, \ldots, \frac{D_r - D_0}{D_0})^T$ and $\mathbf{0}$ is a r-by-1 all-zeros vector.

Finally, in view of stability analysis, we introduce the backstepping transformation (see [2], [10])

$$w(x,t) = \hat{v}(x,t) - K\left[e^{AD_0x}X(t) + D_0\int_0^x e^{AD_0(x-y)}B\hat{v}(y,t)dy\right]. \tag{12}$$

Lemma 1: The backstepping transformation (12), jointly with the control law (3), transform the plant (11) into the target system $(X, w, \tilde{\mathbf{v}}, \mu)$

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + B\left[\delta(t)^T\tilde{\mathbf{v}}(0,t) + w(0,t)\right] \\ D_0w_t(x,t) = w_x(x,t) - D_0Ke^{AD_0x}B\delta(t)^T\tilde{\mathbf{v}}(0,t) \\ w(1,t) = 0 \\ \Lambda_D\tilde{\mathbf{v}}_t(x,t) = \tilde{\mathbf{v}}_x(x,t) - \Sigma_Dh(t + D_0(x-1)) \\ \tilde{\mathbf{v}}(1,t) = \mathbf{0} \\ \mu_t(x,t) = \dfrac{1}{(3\overline{D} - D_0)}\mu_x(x,t) \\ \mu(1,t) = KX(t) + w(0,t)\,, \end{cases} \tag{13}$$

in which, h is defined for $t \geq 0$ as

$$h(t) = D_0K\big[(A+BK)e^{AD_0}X(t) + e^{AD_0}B\delta(t)^T\tilde{\mathbf{v}}(0,t) + D_0(A+BK)\int_0^1 e^{AD_0(1-x)}B\big(w(x,t) + Ke^{(A+BK)D_0x} \times X(t) + \int_0^x KD_0e^{(A+BK)D_0(x-y)}Bw(y,t)dy\big)dx\big]. \tag{14}$$

Proof: The proof is similar to the one of [8, Lemma 2]. ■

With this new set of coordinates, we are now ready to analyze the exponential stabilization of the closed-loop system.

IV. Stability Analysis

A. Definition of the infinitesimal generator

Let us define the state of the target system (13) as $\Psi = (X, w, \tilde{\mathbf{v}}, \mu) \in \mathbb{R}^n \times \mathcal{L}_2([0,1],\mathbb{R}) \times \mathcal{L}_2([0,1],\mathbb{R}^r) \times \mathcal{L}_2([0,1],\mathbb{R}) \triangleq \mathcal{D}_\Psi$. As the solution to (13) is unique (see [8] for further details), (Ψ, δ) defines a continuous-time Markov process and we can therefore introduce the following elements for stability analysis.

Define the infinitesimal generator L (see [6] and [7]) acting on a functional $V : \mathcal{D}_\Psi \times \{e_1, \ldots, e_r\} \to \mathbb{R}$ as

$$LV(\Psi,\delta) = \limsup_{\Delta t \to 0^+} \frac{1}{\Delta t}\Big(\mathbb{E}_{[t,(\Psi,\delta)]}(V(\Psi(t+\Delta t), \delta(t+\Delta t)) - V(\Psi,\delta)\Big). \tag{15}$$

We also define L_j, the infinitesimal generator of the Markov process (Ψ, δ) obtained from the target system (13) by fixing $\delta(t) = e_j$, as

$$L_jV(\Psi) = \frac{dV}{d\Psi}(\Psi, e_j)f_j(\Psi) + \sum_{l=1}^r (V_l(\Psi) - V_j(\Psi))\,\tau_{jl}(t)\,, \tag{16}$$

in which $V_l(\Psi) = V(\psi, e_l)$ and f_j denotes the operator corresponding to the dynamics of the target system (13) with the fixed value $\delta(t) = e_j$, that is, for $\Psi = (X, w, \tilde{\mathbf{v}}, \mu)$,

$$f_j(\Psi)(x) = \begin{pmatrix} (A+BK)X + Be_j^T\tilde{\mathbf{v}}(0) + Bw(0) \\ \frac{1}{D_0}\big[w_x(x) - D_0Ke^{AD_0x}Be_j^T\tilde{\mathbf{v}}(0)\big] \\ \Lambda_D^{-1}\big[\tilde{\mathbf{v}}_x(x) - \Sigma_Dh(\cdot + D_0(x-1))\big] \\ \mu_x(x)/\overline{D} \end{pmatrix}. \tag{17}$$

For the sake of conciseness, in the sequel, we denote $V(t)$, $LV(t)$, $V_j(t)$ and $L_jV(t)$, for short, instead of $V(\Psi(t), \delta(t))$, $LV(\Psi(t), \delta(t))$, $V(\Psi(t), e_j)$ and $L_jV(\Psi(t))$, respectively.

Due to the dynamics (2) of the transition probabilities, the infinitesimal generators (15) and (16) are related as follows

$$\sum_{j=1}^r P_{ij}(0,t)\frac{dV_j}{d\Psi}(\Psi(t))f_j(\Psi(t)) + \sum_{j=1}^r \frac{\partial P_{ij}}{\partial t}(0,t)V_j(t) = \sum_{j=1}^r P_{ij}(0,t)L_jV(t) = LV(t)\,. \tag{18}$$

Therefore, for stability analysis, one can first focus on the constant delay functional L_jV. This is the probabilistic delay averaging approach [7], which we follow in the sequel.

B. Lyapunov analysis

Consider the following Lyapunov functional candidate

$$V(\Psi,\delta) = X^TPX + c\int_0^1 (1+x)(\delta\cdot\mathbf{D})^T\tilde{\mathbf{v}}(x)^2dx + bD_0\int_0^1(1+x)w(x)^2dx + d(3\overline{D} - D_0)\int_0^1(1+x)\mu(x)^2dx\,, \tag{19}$$

with $b, c, d > 0$, P the symmetric positive definite solution of the equation $P(A+BK) + (A+BK)^TP = -Q$, for a given symmetric positive definite matrix Q, and $\mathbf{D} = (D_1 \ldots D_r)^T$ and where $\cdot$ denotes the Hadamard multiplication and the square in $\tilde{\mathbf{v}}(x)^2$ should be understood component-wise. Note that, contrary to [8], the functional (19) explicitly depends on δ. We can then get the following result.

Lemma 2: There exist $(b,c,d) \in (\mathbb{R}_+^*)^3$ such that the Lyapunov functional V defined in (19) satisfies, for $t \geq 2\overline{D}$,

$$LV(t) \leq -\big(\eta - M\mathbb{E}_{[0,D(0)]}(|D_0 - D(t)|) - Ng(t)\big)\,V(t)\,, \tag{20}$$

with $\eta, M, N > 0$ positive constants and the function g defined as $g(t) \triangleq \sum_{j=1}^r |D_j - D_0|^2\left(\frac{\partial P_{ij}(0,t)}{\partial t} + c_j(t)P_{ij}(0,t)\right)$.

Proof: According to (18), we first consider L_jV defined in (16). For the first term in (16), from (17), applying integration by parts and Young's inequality, one obtains

$$\begin{aligned}\frac{dV_j}{d\Psi}(\Psi)f_j(\Psi) \leq &-\left(\frac{\min(\lambda(Q))}{2} - 4d|K|^2\right)|X(t)|^2 \quad (21)\\ &- b(1-2D_0|K||B|e^{|A|D_0}\gamma_1)\|w(t)\|^2\\ &- c\left(1-\frac{2}{D_0}|D_0-D_j|\gamma_2\right)\|\tilde{v}_j(t)\|^2 - d\|\mu(t)\|^2\\ &- \left(b - 4d - \frac{4|PB|^2}{\min(\lambda(Q))}\right)w(0,t)^2\\ &- \left(c - \frac{4|PB|^2}{\min(\lambda(Q))} - 2bD_0|K||B|e^{|A|D_0}\frac{1}{\gamma_1}\right)\tilde{v}_j(0,t)^2\\ &- d\mu(0,t)^2 + \frac{2c}{D_0\gamma_2}|D_0-D_j|\|h(t+D_0(\cdot-1))\|^2,\end{aligned}$$

for any $\gamma_1, \gamma_2 \geq 0$. Observing that $D_0 \in [\underline{D}, \overline{D}]$, let us choose $(b, c, d, \gamma_1, \gamma_2) \in (\mathbb{R}_+^*)^5$ as follows

(a) $d < \frac{\min(\lambda(Q))}{8|K|^2}$, (b) $b \geq 4d + \frac{4|PB|^2}{min(\lambda(Q))}$,
(c) $\gamma_1 < \frac{1}{2\overline{D}|K|e^{|A|\overline{D}}|B|}$, (d) $\gamma_2 < \frac{1}{4}\min\left\{\frac{\underline{D}}{\overline{D}-D_1}, \frac{\underline{D}}{D^r-\underline{D}}\right\}$,
(e) $c \geq \frac{4|PB|^2}{\min(\lambda(Q))} + 2b\overline{D}|K||B|e^{|A|\overline{D}}\frac{1}{\gamma_1}$,

and define $\eta_0 = \min\{\min(\lambda(Q))/2 - 4d|K|^2, b(1 - 2D_0|K||B|e^{|A|D_0}\gamma_1), d\}$, which implies

$$\begin{aligned}\frac{dV_j}{d\Psi}(\Psi)f_j(\Psi) \leq &-\eta_0\left(|X(t)|^2 + \|w(t)\|^2 + \|\mu(t)\|^2\right)\\ &+ \frac{2c}{\gamma_2 D_0}|D_0-D_j|\|h(t+D_0(\cdot-1))\|^2. \quad (22)\end{aligned}$$

Using Lemmas 3 and 4 for the index $j_0 \in \{1, \ldots, r\}$ such that $e_{j_0} = \delta(t)$, this finally gives

$$\frac{dV_j}{d\Psi}(\Psi)f_j(\Psi) \leq -\eta V(t) + \frac{2cM_1}{\gamma_2 D_0}|D_0-D_j|V(t), \quad (23)$$

with $\eta = \frac{\eta_0}{2\max\left\{\max(\lambda(P)), 2b\overline{D}, 2cD_r\max\{N_X, N_w, N_\mu\}, 2d(3\overline{D}-\underline{D})\right\}}$, in which N_X, N_w, N_μ are defined in Lemma 4.

In addition, for the second term in (16), by definition of the Lyapunov function (19), one obtains

$$\begin{aligned}&\sum_{l=1}^{r}(V_l(\Psi) - V_j(\Psi))\tau_{jl}(t) \quad (24)\\ &= \sum_{l=1}^{r} c\int_0^1 (1+x)\tau_{jl}(t)\left(D_l\tilde{v}_l(x,t)^2 - D_j\tilde{v}_j(x,t)^2\right)dx,\end{aligned}$$

in which, from the definition (9) of the input estimation error,

$$\begin{aligned}&D_l\tilde{v}_l(x,t)^2 - D_j\tilde{v}_j(x,t)^2 \quad (25)\\ &= \left(\sqrt{D_l}\int_{t+D_0(x-1)}^{t+D_l(x-1)}\dot{U}(s)ds - \sqrt{D_j}\int_{t+D_0(x-1)}^{t+D_j(x-1)}\dot{U}(s)ds\right)\\ &\times\left(\sqrt{D_l}\int_{t+D_0(x-1)}^{t+D_l(x-1)}\dot{U}(s)ds + \sqrt{D_j}\int_{t+D_0(x-1)}^{t+D_j(x-1)}\dot{U}(s)ds\right)\\ &\leq (1-x)^2\left(|\sqrt{D_l}-\sqrt{D_j}||D_j-D_0| + \sqrt{\overline{D}}|D_l-D_j|\right)\\ &\times\sqrt{\overline{D}}(|D_l-D_0| + |D_j-D_0|)\max_{s\in[-\overline{D},0]}\dot{U}(t+s)^2\\ &\leq M_2M_3\Big(|D_j-D_0| + |D_l-D_j||D_l-D_0|\Big)V(t),\end{aligned}$$

in which we used Lemma 3 in the last inequality and with $M_3 = \max\left\{\left(3\overline{D} - 2(\overline{D}\underline{D})^{1/2}\right)|\overline{D}-\underline{D}|, \overline{D}\right\}$. Therefore, gathering (23), (24) and (25), one gets

$$\begin{aligned}L_jV(t) \leq &-\eta V(t) + M_4|D_0-D_j|V(t)\\ &+ N\sum_{l=1}^{r}\tau_{jl}(t)|D_l-D_j||D_l-D_0|V(t), \quad (26)\end{aligned}$$

with $M_4 = 2c\left(\frac{M_1}{\gamma_2 D_0} + M_2M_3r\tau^\star\right)$ and $N = 2cM_2M_3$. Then, from (18) and as $\sum_{j=1}^{r}P_{ij}(0,t)|D_0 - D_j| = \mathbb{E}_{[0,D(0)]}(|D_0 - D(t)|)$, the following inequality holds

$$\begin{aligned}LV(t) \leq &-\Big(\eta - M_4\mathbb{E}_{[0,D(0)]}(|D_0-D(t)|)\Big)V(t)\\ &+ N\sum_{j=1}^{r}P_{ij}(0,t)\sum_{l=1}^{r}\tau_{jl}(t)|D_l-D_j||D_l-D_0|V(t).\end{aligned}$$

Hence, applying the triangle inequality and using (2), one finally gets

$$\begin{aligned}LV(t) \leq &-(\eta - M_4\mathbb{E}_{[0,D(0)]}(|D(t)-D_0|))V(t) \quad (27)\\ &+ N\sum_{l=1}^{r}\sum_{j=1}^{r}P_{ij}(0,t)\tau_{jl}(t)|D_l-D_0|^2V(t)\\ &+ N\tau^\star\mathbb{E}_{[0,D(0)]}(|D(t)-D_0|)\sum_{l=1}^{r}|D_l-D_0|V(t)\\ \leq &-\left(\eta - (M_4 + N\tau^\star r|\overline{D}-\underline{D}|)\mathbb{E}_{[0,D(0)]}(|D(t)-D_0|)\right)V(t)\\ &+ N\sum_{j=1}^{r}|D_j-D_0|^2\left(\frac{\partial P_{ij}(0,t)}{\partial t} + c_j(t)P_{ij}(0,t)\right)V(t).\end{aligned}$$

Lemma 2 is then proved with $M = M_4 + N\tau^\star r|\overline{D}-\underline{D}|$. ■

C. Conclusion of the stability analysis

With Lemma 2, we are now ready to conclude the proof of Theorem 1. Let us denote $\gamma_0(t) = \eta - M\mathbb{E}_{[0,D(0)]}(|D(t) - D_0|) - Ng(t)$, in which η, M, N and g are defined in Lemma 2 and introduce the functional Z as $Z(t) = \exp\left(\int_0^t \gamma_0(s)ds\right)V(t)$. Applying Lemma 2, we obtain $LZ(t) = \gamma_0(t)Z(t) + \exp\left(\int_0^t \gamma_0(s)ds\right)LV(t) \leq 0$. Therefore, for $t \geq 3\overline{D}$, according to Dynkin's formula [4, Theorem 5.1, p. 133],

$$\begin{aligned}&\mathbb{E}_{[3\overline{D},(\Psi,D)(3\overline{D})]}(Z(t)) - Z(3\overline{D}) \quad (28)\\ &= \mathbb{E}_{[3\overline{D},(\Psi,D)(3\overline{D})]}\left(\int_{3\overline{D}}^t LZ(s)ds\right) \leq 0,\end{aligned}$$

from which, using standard conditional expectation properties, one deduces $\mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(3\overline{D}))$. In addition, observe that

$$\begin{aligned}\int_0^t g(s)ds \leq &(\overline{D}-\underline{D})\Big(\mathbb{E}_{[0,D(0)]}(|D_0-D(t)|) \quad (29)\\ &+ c^\star\int_0^t \mathbb{E}_{[0,D(0)]}(|D_0-D(s)|)ds\Big),\end{aligned}$$

as $c_j = \sum_{k=1}^{r}\tau_{jk} \leq r\tau^\star \triangleq c^\star$. Hence, it follows that

$$\mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) \geq \mathbb{E}_{[0,(\Psi,D)(0)]}\Big(V(t) \quad (30)$$

$$\times \exp(-N(\overline{D}-\underline{D})\mathbb{E}_{[0,D(0)]}(|D_0-D(t)|) + \int_0^t (\eta - (M+Nc^\star(\overline{D}-\underline{D}))\mathbb{E}_{[0,D(0)]}(|D_0-D(s)|))ds\Big) .$$

Thus, if (4) holds with

$$\epsilon^\star \triangleq \frac{\eta}{2(M+Nc^\star(\overline{D}-\underline{D}))} , \quad (31)$$

one obtains from (28) and (30)

$$\mathbb{E}_{[0,(\Psi,D)(0)]}\left(e^{-N(\overline{D}-\underline{D})\epsilon^\star+\frac{\eta}{2}t}V(t)\right) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(3\overline{D})) \leq e^{2\overline{D}\eta}\mathbb{E}_{[0,(\Psi,D)(0)]}(V(3\overline{D})) , \quad (32)$$

which implies, with $\gamma = \frac{\eta}{2}$,

$$\mathbb{E}_{[0,(\Psi,D)(0)]}(V(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(V(3\overline{D}))e^{3\overline{D}\eta+N(\overline{D}-\underline{D})\epsilon^\star-\gamma t} . \quad (33)$$

Notice that V and Υ are equivalent, that is, there exist positive constants q_1 and q_2 such that for $\forall t \geq 0$, $q_1V(t) \leq \Upsilon(t) \leq q_2V(t)$ (see [9, Lemma 4] for a proof of this fact in a similar case). It thus follows that $\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(t)) \leq \frac{q_2}{q_1}e^{3\overline{D}\eta+N(\overline{D}-\underline{D})\epsilon^\star}e^{-\gamma t}$. In addition, as the dynamics (1) is linear, there exists a constant $R_0 > 0$ (see [8, Lemma 5]) such that $\Upsilon(t) \leq R_0\Upsilon(0)$, $t \in [0, 3\overline{D}]$. Consequently, (5) follows with $R = R_0e^{3\overline{D}\eta+N(\overline{D}-\underline{D})\epsilon^\star}q_2/q_1$.

V. Simulations

To illustrate Theorem 1 and in particular the role played by the condition (4), we consider the following toy example

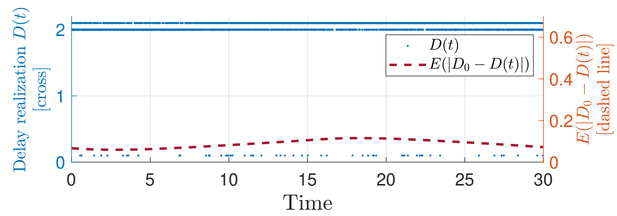
$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t-D(t)) . \quad (34)$$

The control law (3) is applied with the feedback gain $K = -\begin{bmatrix} 1 & 2 \end{bmatrix}$ resulting in conjugate closed-loop eigenvalues $\lambda(A+BK) = \{-0.5000+1.3229i, -0.5000-1.3229i\}$. The initial conditions are chosen as $X(0) = [1 \quad 0]^T$ and $U(t) = 0$, for $t \leq 0$. Simulations are carried out with a fixed-step solver in Matlab-Simulink and a sampling time $\Delta t = 0.01$ s. Finally, the integral in (3) is discretized using its zero-order hold approximation, in line with a suggestion in [13].

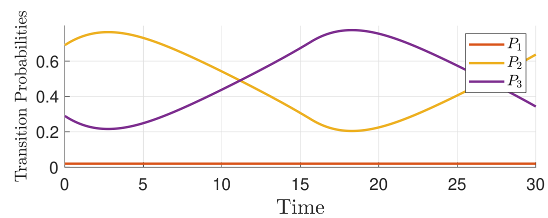
We consider 3 different delay values $(D_1, D_2, D_3) = (0.1, 2.0, 2.1)$. The initial transition probabilities are taken as[4] $P_1(0, 0^+) = 0.02$, $P_2(0, 0^+) = 0.69$ and $P_3(0, 0^+) = 0.29$, which means that the delay values are initially concentrated in D_2, and D_3. We pick the prediction horizon as $D_0 = 2$. Notice that the delay margin of the closed-loop system (34) and (3) with constant delay D_0 is $\Delta D = 0.096$ (see [9] for details on the computation of this quantity). Thus, the realizations of both D_1 and D_3 lead to a delay difference which is beyond the robustness margin of the closed-loop system, resulting in a challenging set-up as prediction-based controllers are well-known to be sensitive to delay mismatch [14].

Simulations performed for constant transition probabilities equal to the above initial conditions (i.e., $\tau_{ij} = 0$) resulted

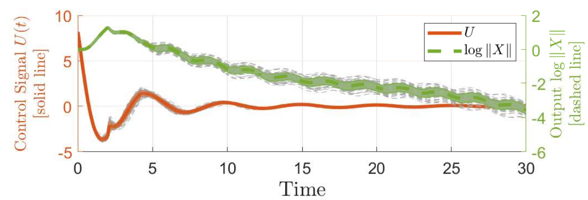
[4]The subscript i is omitted in this section, as the probability transitions do not depend on the initial delay value. This is consistent with the fact that the expectation in (5) is conditioned by the initial delay value. Besides, to avoid a conflict between the initial condition in (2) and their discretized version used in simulation, we denote their initial conditions as $P_j(0, 0^+)$.



(a) Example of a realization of the random delay D



(b) Dynamic of the transition probabilities P_1, P_2 and P_3.



(c) Monte Carlo simulation of $\log \|X\|$ and the closed-loop input U (100 trials), in which the means and the standard deviations are highlighted by the colored lines.

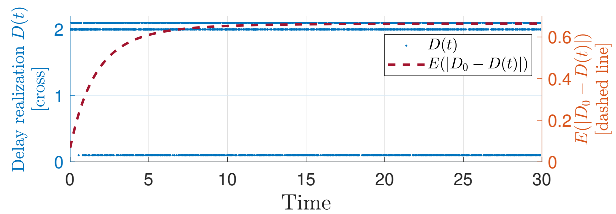
Fig. 1: Simulation results of the closed-loop system (34) and (3) for $\mathbf{D} = (0.1, 2.0, 2.1)^T$, $X(0) = [1 \quad 0]^T$ and $U(t) = 0$ for $t \leq 0$. The prediction horizon is $D_0 = 2.0$. The transition probabilities follow (2) with τ_{ij} defined in (35).

in exponentially stable trajectories (and are not reported here, due to space limitation). Indeed, even though D_0 is picked quite distant from the value D_1, thus largely exceeding the corresponding delay margin, the value D_1 has a weak enough occurrence for the prediction horizon to be close in average to the delay realizations and to enable stabilization. This is in accordance with condition (4).

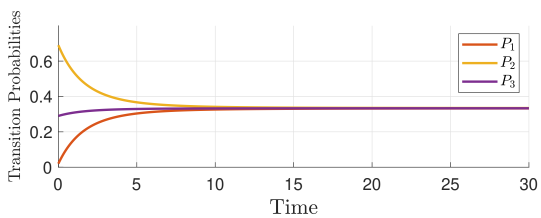
Secondly, we operate the simulation with the transition rates τ_{ij} defined on the one hand as

$$\tau(t) = \left(\{\tau_{ij}(t)\}\right)_{1\leq i,j\leq 3} = \tau^\star \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & |\sin(kt)| \\ 0 & |\cos(kt)| & 0 \end{pmatrix} , \quad (35)$$

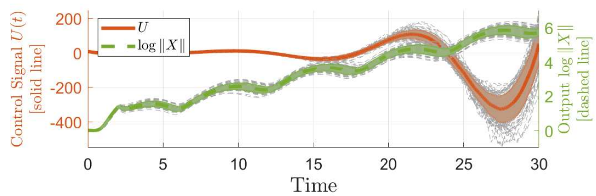
and, on the other hand, as $\tau(t) = \tau^\star e^{-kt}(\mathbf{1}_{3\times 3} - I_3)$ with constants $\tau^\star = 0.2$ and $k = 0.1$. This corresponds, respectively, to the case of a probability transition P_1 close to zero with P_2 and P_3 oscillating (see Fig. 1(b)) and to the case of convergence to a uniform delay distribution (see Fig. 2(b)). As could be expected, the choice $D_0 = 2$ of the prediction horizon results in Fig. 1(c) into a stable closed-loop behavior in the first case as D_0 remains close to the delay realizations. Notice that the example of realization of $E(|D_0 - D(t)|)$ depicted in Fig. 1(a) is in the same range as the delay margin, but is sometimes higher. Yet, in the second case, the realizations of both D_1 and D_3 are too frequent, resulting into a diverging behavior pictured in Fig. 2(c).



(a) Example of a realization of the random delay D



(b) Dynamic of transition probabilities P_1, P_2 and P_3.



(c) Monte Carlo simulation of $\log\|X\|$ and the closed-loop input U (100 trials), in which the means and the standard deviations are highlighted by the colored lines.

Fig. 2: Simulation results of the closed-loop system (34) and (3) for $\mathbf{D} = (0.1, 2.0, 2.1)^T$, $X(0) = [1 \quad 0]^T$ and $U(t) = 0$ for $t \leq 0$. The prediction horizon is $D_0 = 2.0$. The transition probabilities follow (2) with $\tau(t) = \tau^\star e^{-kt} \left(\mathbf{1}_{3\times 3} - I_3\right)$ ($\tau^\star = 0.2$ and $k = 0.1$).

VI. Conclusion

In this paper, we proposed a constant horizon prediction-based controller to compensate for a random input delay modeled as a Markov process with a finite number of values. We proved the exponential mean-square stability of the closed-loop control system provided that the chosen prediction horizon is in average close enough to the delay value. Simulations illustrated the relevance of this condition and the interest of this prediction-based control law.

Future works will focus on the adaptation of the prediction-horizon to the current delay distribution, as it is likely to increase the closed-loop delay robustness, and thus represents an interesting design feature to explore.

Appendix

Lemma 3: Consider the control law defined in (3) and the function h defined in (14), there exist $M_1, M_2 > 0$ such that

$$\|h(t + D_0(\cdot - 1))\|^2 \leq M_1 V(t), \quad t \geq D_0\,, \tag{36}$$

$$\max_{s\in[-\overline{D},0]} \dot{U}(t+s)^2 \leq M_2 V(t), \quad t \geq 2\overline{D}\,. \tag{37}$$

Proof: (36) is proved in [8]. Observing that $h(t) = D_0\dot{U}(t)$, (37) is obtained with similar arguments. ■

Lemma 4: There exist $N_X, N_w, N_\mu > 0$ such that, for all $j \in \{1, \ldots, r\}$ and $t \geq \overline{D}$,

$$\|\tilde{v}_j(t)\|^2 \leq N_X|X(t)|^2 + N_w\|w(t)\|^2 + N_\mu\|\mu(t)\|^2\,. \tag{38}$$

Proof: From the definition of the input estimation error (9), it follows that

$$\begin{aligned}\|\tilde{v}_j(t)\|^2 &= \int_0^1 \left(U(t + D_j(x-1)) - U(t + D_0(x-1))\right)^2 dx \\ &\leq \frac{4}{\underline{D}}\int_{t-D_0-\overline{D}}^{t} U(s)^2 ds \leq 4\frac{\overline{D}+D_0}{\underline{D}}\left(\|\hat{v}(t)\|^2 + \|\mu(t)\|^2\right).\end{aligned} \tag{39}$$

Besides, from the inverse of the backstepping transformation (12), which is

$$\begin{aligned}\hat{v}(x,t) =& w(x,t) + Ke^{(A+BK)D_0 x}X(t) \\ &+ \int_0^x KD_0 e^{(A+BK)D_0(x-y)}Bw(y,t)dy\,,\end{aligned} \tag{40}$$

it follows, using Young's and Cauchy-Schwarz inequalities that $\|\hat{v}(t)\|^2 \leq N_1|X(t)|^2 + N_2\|w(t)\|^2$ with the positive constants $N_1 = 3|K|^2 e^{2|A+BK|D_0}$ and $N_2 = 3(1 + |K|^2 D_0^2 e^{2|A+BK|D_0}|B|^2)$. Hence, (38) follows with $N_X = 4N_1(\overline{D} + D_0)/\underline{D}$, $N_w = 4N_2(\overline{D} + D_0)/\underline{D}$ and $N_\mu = 4(\overline{D} + D_0)/\underline{D}$. ■

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