Stability analysis of a 2×2 linear hyperbolic system with a sampled-data controller via backstepping method and looped-functionals

M. A. Davó, D. Bresch-Pietri, C. Prieur, and F. Di Meglio

***Abstract*—This work is concerned with the global exponential stability of a 2×2 linear hyperbolic system with a sampled-data boundary feedback control designed by means of the backstepping method for a nominal continuous input. We show that there exists a sufficiently small inter-sampling time (that encompasses both periodic and aperiodic sampling) for which the global exponential stability of the closed-loop system is guaranteed. In addition, we provide easily tractable sufficient stability conditions which can be used to find an upper bound of the maximum inter-sampling time. The results rely on the combination of the Lyapunov method and looped-functionals. The effectiveness of the proposed results is illustrated with a numerical example.**

I. Introduction

Nowadays, control systems are usually implemented using digital technologies, which leads to the interaction of continuous dynamics, usually coming from the plant, and discrete dynamics, coming from the sampling mechanism, controllers, or digital communication networks. The problems arising in this context have been deeply studied for finite dimensional systems (see, e.g., [8, 17]) under different areas such as *sampled-data control*, *networked control systems*, *event-based control*, and *hybrid control systems*. However, considerably less attention has been paid to infinite-dimensional systems in presence of sampling, mainly due to the complexity of the systems. The majority and the first results were developed for general infinite-dimensional systems based mainly on the semigroups approach [16, 19, 24]. These results include sufficient and necessary conditions for the stabilizability of an infinite-dimensional system, under some assumptions, with a static sampled-data controller, see e.g., [14, 15]. For particular infinite-dimensional systems, most of the results focused on delay systems, see e.g., [10, 18, 23] and references therein, and only few contributions in the last decade deal with systems described by Partial Differential Equations (PDEs). Besides, the majority of the results are focused on parabolic PDEs [6, 12, 13, 20, 21]. There are only three results recently published on hyperbolic PDEs: the work [11] analyzes the exponential stability of a linear 1-D PDE under sampled-data boundary feedback control, the works [5] and [4] propose event-triggered control strategies for 2×2 linear hyperbolic PDEs of balance laws and linear hyperbolic PDEs of conservation laws, respectively.

This work analyzes the global exponential stability of 2×2 linear hyperbolic systems under boundary feedback control in presence of sampling. There are two main reasons for considering this class of systems: first, it is a good model of many physical systems having a direct engineering interest, for instance transmission lines, heat exchangers, etc. (see e.g., [2]); the second reason is that it is simple enough to perform an explicit mathematical analysis, that will illustrate how the proposed method can be applied to more general systems. We consider a full state feedback law designed by the backstepping approach, proposed in [1] based on the results in [25], that render the closed-loop system exponentially stable. It is then proved that there exists a small enough inter-sampling time (encompassing periodic and aperiodic sampling), such that the exponential stability is preserved. This is the main contribution of the paper. In addition, we provide sufficient conditions in the form of matrix inequalities to check the exponential stability of the closed-loop system, which can be used as a procedure to numerically compute an appropriate sampling period. On the other hand, the results in [5] indirectly prove that such small enough inter-sampling time exists. However, there are several differences between [5] and this work: we consider a more general system than [5], where the distributed coupling term are constants with some constraints; second the results in [5] consider a particular boundary condition where only one boundary interconnection of the two PDEs is considered; third we consider the modified control law proposed in [1]; finally, our stability analysis is based on the Lyapunov method and the looped-functional approach (see, e.g., [3, 22]), that is applied to PDEs for the first time.

The rest of the paper is organized as follows. First, the problem under consideration is introduced in Section 2. Section 3 contains the main results: first, the existence of an upper-bound of the inter-sampling times and the sampling period, and second, the sufficient conditions for its numerical computation. Section 4 illustrates the results by a numerical example. The proof of the results is provided in the appendix.

Notation: $\mathbb{R}_{\geqslant}$ and $\mathbb{R}_{>}$ are the set of non-negative real numbers and positive real numbers, respectively. Given a topological set S and a interval $I \subseteq \mathbb{R}$, $C^0(I,S)$ denotes the class of continuous functions $f : I \to S$. By $C^k(I,S)$, we denote the class of continuous functions $f : I \to S$, which have continuous derivatives of order k. Let us define the vectors $e_j^n = [0_{1,j-1}, 1, 0_{1,n-j}]^\top$ with $0 < j \leqslant n$.

M. A. Davó and C. Prieur are with Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France. D. Bresch-Pietri and F. Di Meglio are with the MINES Paristech, PSL Research University, CAS – Centre Automatique et Systèmes, 60 bd St Michel 75006 Paris, France. This work has been supported by ANR project MACS-Drill ANR-15-CE23-0008-01. Corresponding author: Miguel.Davo-Navarro@gipsa-lab.fr

II. Preliminaries and problem statement

This work focuses on the following linear hyperbolic system

$$\begin{aligned} u_t(t,x)+\lambda_1 u_x(t,x) &= \sigma_1(x)v(t,x) \\ v_t(t,x)-\lambda_2 v_x(t,x) &= \sigma_2(x)u(t,x) \end{aligned} \tag{1}$$

with boundary conditions

$$\begin{aligned} &u(t,0)=qv(t,0), \\ &v(t,1)=\rho u(t,1)+U(t), \end{aligned} \tag{2}$$

where $u,\ v : \mathbb{R}_{\geqslant} \times [0,1] \to \mathbb{R}$ are the system states, λ_1, $\lambda_2 \in \mathbb{R}_{>}$, $\sigma_1,\ \sigma_2 \in C^0([0,1],\mathbb{R})$, $\rho, q \in \mathbb{R}$ are such that $|\rho q|<1$, and $U:\mathbb{R}_{\geqslant} \to \mathbb{R}$ is the control signal.

A. Preliminaries

The results in [25] guarantee that there exist functions K^{uu}, K^{uv}, K^{vu}, K^{vv}, $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\beta\beta} \in C^1(\mathcal{J},\mathbb{R})$ with domain $\mathcal{J}=\{(x,y): 0 \leqslant y \leqslant x \leqslant 1\}$ such that the Volterra transformation

$$\begin{aligned} \alpha(t,x) &= u(t,x) - \int_0^x \left[K^{uu}(x,y)\ K^{uv}(x,y)\right] \begin{bmatrix} u(t,y) \\ v(t,y) \end{bmatrix} dy \\ \beta(t,x) &= v(t,x) - \int_0^x \left[K^{vu}(x,y)\ K^{vv}(x,y)\right] \begin{bmatrix} u(t,y) \\ v(t,y) \end{bmatrix} dy \end{aligned} \tag{3}$$

with inverse transformation

$$\begin{aligned} u(t,x) &= \alpha(t,x) - \int_0^x \left[L^{\alpha\alpha}(x,y)\ L^{\alpha\beta}(x,y)\right] \begin{bmatrix} \alpha(t,y) \\ \beta(t,y) \end{bmatrix} dy \\ v(t,x) &= \beta(t,x) - \int_0^x \left[L^{\beta\alpha}(x,y)\ L^{\beta\beta}(x,y)\right] \begin{bmatrix} \alpha(t,y) \\ \beta(t,y) \end{bmatrix} dy \end{aligned} \tag{4}$$

maps the original system (1)-(2) into the target system

$$\begin{aligned} \alpha_t(t,x)+\lambda_1\alpha_x(t,x) &= 0 \\ \beta_t(t,x)-\lambda_2\beta_x(t,x) &= 0 \end{aligned} \tag{5}$$

with boundary conditions

$$\begin{aligned} &\alpha(t,0)=q\beta(t,0), \\ &\beta(t,1)=\rho\alpha(t,1)-\textstyle\int_0^1 \left[L^{\alpha}(x)\ L^{\beta}(x)\right] \begin{bmatrix} \alpha(t,x) \\ \beta(t,x) \end{bmatrix} dx + U(t), \end{aligned} \tag{6}$$

where $L^{\alpha}(x) = L^{\beta\alpha}(1,x)+\rho L^{\alpha\alpha}(1,x)$ and $L^{\beta}(x) = L^{\beta\beta}(1,x)+\rho L^{\alpha\beta}(1,x)$. In addition, it has been already proved (see [1] and [2]) that for all[1] $k_0 \in [0,1]$ there exist constants C and η such that the solution to system (5)-(6) with $U(t)=\bar{\kappa}((\alpha(t,\cdot),\beta(t,\cdot))^\top)$, where

$$\begin{aligned} \bar{\kappa}((\alpha(t,\cdot),\beta(t,\cdot))^\top) &= -k_0\rho\alpha(t,1) \\ &\quad + \int_0^1 \left[L^{\alpha}(x)\ L^{\beta}(x)\right] \begin{bmatrix} \alpha(t,x) \\ \beta(t,x) \end{bmatrix} dx \end{aligned} \tag{7}$$

satisfies

$$\|(\alpha(t,\cdot),\beta(t,\cdot))^\top\|_{L_2([0,1],\mathbb{R}^2)} \leqslant Ce^{-\eta t}\|(\alpha^0,\beta^0)^\top\|_{L_2([0,1],\mathbb{R}^2)} \tag{8}$$

for all initial condition $(\alpha^0,\beta^0)^\top \in L_2([0,1],\mathbb{R}^2)$. From the inverse transformation (4), the same exponential estimation is obtained for the solution to system (1)-(2) with $U(t)=\kappa((u(t,\cdot),v(t,\cdot))^\top)$, where

$$\begin{aligned} \kappa((u(t,\cdot),v(t,\cdot))^\top) &= -k_0\rho u(t,1) \\ &\quad + \int_0^1 \left[K^{u}(x)\ K^{v}(x)\right] \begin{bmatrix} u(t,x) \\ v(t,x) \end{bmatrix} dx \end{aligned} \tag{9}$$

with $K^u(x)=K^{vu}(1,x)+\rho(1-k_0)K^{uu}(1,x)$ and $K^v(x)=K^{vv}(1,x)+\rho(1-k_0)K^{uv}(1,x)$.

B. Problem statement

In this work, we investigate the effect of sampling the control signal on the exponential convergence of the solution to (1)-(2). Let us assume that there exists a strictly increasing sequence of instants $\{t_k\}_{k\in\mathbb{N}}$ satisfying

$$t_0=0, \quad T_k=t_{k+1}-t_k \in (0,T^*], \quad \lim_{k\to\infty} t_k=\infty \tag{10}$$

with $T^*>0$, such that the control signal U is sampled at those instants, that is

$$U(t)=\kappa((u(t_k,\cdot),v(t_k,\cdot))^\top), \quad t\in[t_k,t_{k+1}). \tag{11}$$

Note that the sampling process introduces discontinuities in the solutions u and v, which will propagate along the characteristic lines. For the sake of clarity of the following results, we define the following two families of sets, that contain all the discontinuities in the interval $[0,x]$ of the solutions u and v at the instant t:

$$\begin{aligned} \widehat{\mathbb{X}}(t,x) &= \{\bar{x}\in[0,x] : \exists(k,j)\in\mathbb{N}^2, \bar{x}=\hat{\chi}(t-t_k,j)\} \\ \check{\mathbb{X}}(t,x) &= \{\bar{x}\in[0,x] : \exists(k,j)\in\mathbb{N}^2, \bar{x}=\check{\chi}(t-t_k,j)\} \end{aligned} \tag{12}$$

where $t \geqslant 0$, $x\in[0,1]$, and

$$\hat{\chi}(\tau,j)=\lambda_1(\tau-jT_p-\tfrac{1}{\lambda_2}), \quad \check{\chi}(\tau,j)=1-\lambda_2(\tau-jT_p), \tag{13}$$

$$T_p=\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}. \tag{14}$$

For the case of $\rho=0$, the above definitions are used with $j=0$. In what follows, we restrict the analysis to the class of solutions given by Definition 1 below. Note that this mild restriction serves purely technical purposes and does not affect the controller design.

Definition 1 (Solution to system (1)-(2)). *For an initial condition $(u^0,v^0)^\top \in C([0,1],\mathbb{R}^2)$, functions $u:\mathbb{R}_{\geqslant}\times[0,1]\to\mathbb{R}$ and $v:\mathbb{R}_{\geqslant}\times[0,1]\to\mathbb{R}$ are a solution to system (1)-(2) with control signal* (11) *if*

a) the mappings $t\mapsto u(t,x)$ and $t\mapsto v(t,x)$ are piecewise continuous and right differentiable for all $x\in[0,1]$,

b) the mappings $x\mapsto u(t,x)$ and $x\mapsto v(t,x)$ are piecewise left continuous and piecewise right continuous, respectively, for all $t\in\mathbb{R}_{\geqslant}$,

c) the left derivative $\partial_x^- u(t,x)=\lim_{\epsilon\to 0^+}\frac{u(t,x)-u(t,x-\epsilon)}{\epsilon}$ exists and is finite for all $(t,x)\in\mathbb{R}_{\geqslant}\times(0,1]$,

d) the right derivative $\partial_x^+ v(t,x)=\lim_{\epsilon\to 0^+}\frac{v(t,x)-v(t,x+\epsilon)}{\epsilon}$ exists and is finite for all $(t,x)\in\mathbb{R}_{\geqslant}\times[0,1)$,

[1] The results in [2] only consider the case $k_0=0$, but the extension to $k_0\in[0,1]$ is straightforward. The work [1] shows that the coefficient k_0 can be interpreted as a tuning parameter, enabling a trade-off between convergence rate and robustness with respect to delays.

$$
\begin{aligned}
&N_0 = [\lambda_2 L^\beta(0) - \lambda_1 L^\alpha(0) q \quad -\lambda_2 L^\beta(1) \quad \lambda_1 L^\alpha(1) \quad 0 \quad -\rho\lambda_1 \quad -1], \qquad N_1 = \begin{bmatrix} 1 & -\rho(1-k_0) \\ -\rho(1-k_0) & \rho^2(1-k_0)^2 \end{bmatrix}, \\
&\Pi_1 = -\mathrm{diag}\left(\left(\tfrac{p_2}{\lambda_2} - \tfrac{p_1 q^2}{\lambda_1}\right), -\tfrac{p_2}{\lambda_2} e^{\frac{\gamma}{\lambda_2}}, \tfrac{p_1}{\lambda_1} e^{\frac{-\gamma}{\lambda_1}}, \left(\tfrac{p_4}{\lambda_2} - \tfrac{p_3 q^2 \lambda_2^2}{\lambda_1^3}\right), \tfrac{p_3}{\lambda_1} e^{\frac{-\gamma}{\lambda_1}}, 0\right) + \tfrac{p_4}{\lambda_2^3} e^{\frac{\gamma}{\lambda_2}} N_0^\top N_0, \\
&\Pi_2 = p_5 \mathrm{diag}(0_{1,1}, N_1, 0_{3,3}), \qquad \Pi_3 = p_5 \begin{bmatrix} e_2^{6\top} \\ e_3^{6\top} \\ N_0 \\ -\lambda_1 e_5^{6\top} \end{bmatrix}^\top \begin{bmatrix} 0_{2,2} & N_1 \\ N_1^\top & 0_{2,2} \end{bmatrix} \begin{bmatrix} e_2^n \\ e_3^6 \\ N_0 \\ -\lambda_1 e_5^{6\top} \end{bmatrix}. \\
&M = 2 \max_{x\in[0,1]} \left(\tfrac{\lambda_1^3 L_x^\alpha(x)^2 e^{\frac{\gamma}{\lambda_1}}}{p_1}, \tfrac{\lambda_2^3 L_x^\beta(x)^2}{p_2} \right),
\end{aligned}
\tag{15}
$$

e) functions u and v are continuous on $\mathcal{D}^u = \{(t,x) \in \mathbb{R}_\geqslant \times [0,1] : x \notin \widehat{\mathbb{X}}(t,1)\}$ and $\mathcal{D}^v = \{(t,x) \in \mathbb{R}_\geqslant \times [0,1] : x \notin \check{\mathbb{X}}(t,1)\}$, respectively, and they are of class C^1 on $\mathcal{D}_1 = \mathcal{D}^u \cap \mathcal{D}^v \cap (\mathbb{R}_> \times [0,1])$,

f) equation (1) *holds for all $(t,x) \in \mathcal{D}_1$, equation (2) holds, $u(0,x) = u^0(x)$ holds for all $x \in (0,1]$, and $v(0,x) = v^0(x)$ holds for all $x \in [0,1)$.*

For the case $(u^0, v^0) \in C^1([0,1], \mathbb{R}^2)$, the functions are differentiable on $\mathcal{D}_1 = \mathcal{D}^u \cap \mathcal{D}^v$. In addition, similar definition applies to the solution to system (5)-(6) where functions α and β are of class C^1 on $\mathcal{D}^u \cap (\mathbb{R}_> \times [0,1])$ and $\mathcal{D}^v \cap (\mathbb{R}_> \times [0,1])$, respectively.

III. Main results

In this section, we analyze two different cases; firstly, we consider periodic sampling where $T_k = T$ for some $T \in (0, T^*]$, secondly, we focus on aperiodic sampling with $T_k \in (0, T^*]$. In both cases, we prove that there exists a small enough $T^* > 0$ such that the solution to system (1)-(2) with a sampled controller exponentially converges. Before presenting the main results, let us define the following set $\mathcal{T} = \{T \in \mathbb{R}_> : \exists m \in \mathbb{N},\ T_p = mT\}$ with T_p given in (14), which is used in the following theorems and lemma.

Theorem 1 (System with boundary reflection, $\rho \neq 0$). *There exist constants η, $C > 0$, and*

$$
T^* \in \begin{cases} \mathbb{R}_>, & \text{if } k_0 = 0, \\ \mathcal{T}, & \text{if } k_0 \in (0,1], \end{cases} \tag{16}
$$

such that for all constant $T \leqslant T^*$ and for all initial condition $(u^0, v^0)^\top \in \mathcal{C}^1([0,1], \mathbb{R}^2)$, the solution to system (1)-(2) *satisfies*

$$
\|s(t,\cdot)\|^2_{L_2([0,1],\mathbb{R}^4)} \leqslant C e^{-\eta t} \|(u^0, v^0, u^0_x, v^0_x)^\top\|^2_{L_2([0,1],\mathbb{R}^4)} \tag{17}
$$

with $s(t,\cdot) = (u(t,\cdot), v(t,\cdot), \partial_x^- u(t,\cdot), \partial_x^+ v(t,\cdot))^\top$, when the control law (11) *is applied with*

- *$T_k \in (0, T]$, $k \in \mathbb{N}$ for $k_0 = 0$ (aperiodic sampling);*
- *$T_k = T \in \mathcal{T}$, $k \in \mathbb{N}$ for $k_0 = (0,1]$ (periodic sampling).*

Theorem 2 (System without boundary reflection, $\rho = 0$). *There exist constants T^*, η, $C > 0$ such that for all $(u^0, v^0) \in C([0,1], \mathbb{R}^2)$ the solution to system* (1)-(2) *with $\rho = 0$ and control law* (11) *with $T_k \in (0, T^*]$ for all $k \in \mathbb{N}$, satisfies*

$$
\|(u(t,\cdot), v(t,\cdot))^\top\|_{L_2([0,1],\mathbb{R}^2)} \leqslant C e^{-\eta t} \|(u^0, v^0)^\top\|_{L_2([0,1],\mathbb{R}^2)}. \tag{18}
$$

The above results guarantee the existence of a sufficiently small inter-sampling time for which the global exponential stability of the closed-loop system is guaranteed. Moreover we provide sufficient conditions in the form of matrix inequalities to efficiently compute an upper-bound of the inter-sampling times.

Lemma 1. *Suppose there exist T^*, $\gamma > 0$, and $p_i > 0$, $i \in \{1, \ldots, 6\}$ such that*

$$
\gamma - p_6 > 0, \tag{19}
$$

$$
\Pi_1 - \Pi_2 - \frac{p_6}{M} e_6^6 e_6^{6\top} \leqslant 0, \tag{20}
$$

$$
\Pi_1 - \Pi_2 + T^*((\gamma - p_6)\Pi_2 + \Pi_3) - \frac{p_6}{M} e_6^6 e_6^{6\top} \leqslant 0, \tag{21}
$$

where the different matrices and constants are defined in (15). *Then the following statements hold*

- ***Periodic sampling and boundary reflection:*** *if $T^* \in \mathcal{T}$ and $k_0 \in (0,1]$, then for all initial conditions $(u^0, v^0)^\top \in \mathcal{C}^1([0,1], \mathbb{R}^2)$ the solution to system* (1)-(2) *with control law* (11) *and $T_k = T$ for all $k \in \mathbb{N}$, satisfies* (17) *with $\eta = \gamma - p_6$ for all $T \in \mathcal{T}$, $T \leqslant T^*$.*
- ***Aperiodic sampling and boundary reflection:*** *if $T^* \in \mathbb{R}_>$ and $k_0 = 0$, then for all initial conditions $(u^0, v^0)^\top \in \mathcal{C}^1([0,1], \mathbb{R}^2)$ the solution to system* (1)-(2) *with control law* (11) *and $T_k \in (0, T^*]$ for all $k \in \mathbb{N}$, satisfies* (17) *with $\eta = \gamma - p_6$.*
- ***Aperiodic sampling and no boundary reflection:*** *if $\rho = 0$, then for all initial conditions $(u^0, v^0)^\top \in \mathcal{C}([0,1], \mathbb{R}^2)$ the solution to system* (1)-(2) *with control law* (11) *with $T_k \in (0, T^*]$ for all $k \in \mathbb{N}$, satisfies* (18) *with $\eta = \gamma - p_6$.*

A feasible solution to conditions (19)-(21) can be found by a line search on several parameters and solving the resulted Linear Matrix Inequalities. Note that Theorems 1 and 2 guarantee that the feasible region is not empty.

Remark 1. *The existence of a dwell-time (minimum inter-event time) for the event-triggering mechanism proposed in [5], suggests an alternative proof of Theorem 2 for the particular case of $k_0 = 0$ and σ_1, $\sigma_2 \in \mathbb{R}$.*

IV. Proof of the main results

This section focuses on the proof of the main results of this work. Since there are several steps in common for all the results, we provide a unique proof divided into several steps.

Following, we assume that the initial conditions u^0 and v^0 satisfy equation (2) for the sake of clarity[2]. The steps are as follows:

1) Existence of the solution to system (1)-(2) with a piecewise constant input $U : \mathbb{R}_{\geqslant} \to \mathbb{R}$;
2) Proof that transformation (3) maps (1)-(2) to (5)-(6) despite the discontinuities introduced by a piecewise constant input U.
3) Construction of a Lyapunov function V and looped-functionals $\mathcal{V}_k$ with an exponential convergence for system (5)-(6), and proof of Lemma 1.
4) Proof of Theorems 1 and 2.

For the sake of clarity, we provide in this section the proof of Steps 3 and 4, and the proof of the two first steps are given in Appendix A.

3) Exponential convergence of the solution to system (5)**-**(6) **and proof of Lemma 1:** First, let us assume periodic sampling with $T_k = T \in \mathcal{T}$ (later this assumption will be removed in order to prove the second and third items of Lemma 1). Consider the following Lyapunov functional $V(t) = V_1(t) + V_2(t)$ with

$$V_1(t) = \frac{p_1}{\lambda_1}\int_0^1 e^{\frac{-\gamma}{\lambda_1}x}\alpha(t,x)^2dx + \frac{p_2}{\lambda_2}\int_0^1 e^{\frac{\gamma}{\lambda_2}x}\beta(t,x)^2dx \tag{22}$$

$$V_2(t) = \frac{p_3}{\lambda_1}\int_0^1 e^{\frac{-\gamma}{\lambda_1}x}\partial_x^-\alpha(t,x)^2dx + \frac{p_4}{\lambda_2}\int_0^1 e^{\frac{\gamma}{\lambda_2}x}\partial_x^+\beta(t,x)^2dx, \tag{23}$$

where $p_i > 0$, $i \in \{1, \ldots, 4\}$.

Consider the sets given in (12). First, note that $T_k \in (0, T^*]$ guarantees that there is a finite number of elements in $\widehat{\mathbb{X}}(t,1)$ and $\check{\mathbb{X}}(t,1)$ for all $t \geqslant 0$. In addition, the elements in $\widehat{\mathbb{X}}(t,1)$ and $\check{\mathbb{X}}(t,1)$ can be ordered, and thus, we define the sequences $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{\hat{N}(t,1)})$ and $(\check{x}_1, \check{x}_2, \ldots, \check{x}_{\check{N}(t,1)})$ for any instant $t \in \mathbb{R}_{\geqslant}$, where $\hat{N}(t,1)$ and $\check{N}(t,1)$ are the number of elements in the sets $\widehat{\mathbb{X}}(t,1)$ and $\check{\mathbb{X}}(t,1)$, respectively. Note that the elements in the above sequences evolve with time and the dynamics is given by equation (13). Considering these sequences, the functional V_1 and V_2 are rewritten as follows:

$$\begin{aligned} V_1(t) = {} & \frac{p_1}{\lambda_1}\sum_{i=0}^{\hat{N}(t,1)}\int_{\hat{x}_i^+}^{\hat{x}_{i+1}^-} e^{\frac{-\gamma}{\lambda_1}x}\alpha(t,x)^2dx \\ & + \frac{p_2}{\lambda_2}\sum_{i=0}^{\check{N}(t,1)}\int_{\check{x}_i^+}^{\check{x}_{i+1}^-} e^{\frac{\gamma}{\lambda_2}x}\beta(t,x)^2dx \end{aligned} \tag{24}$$

$$\begin{aligned} V_2(t) = {} & \frac{p_3}{\lambda_1}\sum_{i=0}^{\hat{N}(t,1)}\int_{\hat{x}_i^+}^{\hat{x}_{i+1}^-} e^{\frac{-\gamma}{\lambda_1}x}\partial_x^-\alpha(t,x)^2dx \\ & + \frac{p_4}{\lambda_2}\sum_{i=0}^{\check{N}(t,1)}\int_{\check{x}_i^+}^{\check{x}_{i+1}^-} e^{\frac{\gamma}{\lambda_2}x}\partial_x^+\beta(t,x)^2dx, \end{aligned} \tag{25}$$

[2]Note that when u^0 and v^0 does not satisfy equation (2), the initial condition will introduce discontinuities into the solution of the system. These discontinuities can be treated as the discontinuities due to the sampling, and the proof follows similarly.

where we define $\hat{x}_0 = 0$, $\check{x}_0 = 0$, $\hat{x}_{\hat{N}(t,1)+1} = 1$ and $\check{x}_{\check{N}(t,1)+1} = 1$. Every interval $[t_k, t_{k+1})$ can be divided into subintervals in which $\hat{N}(t,1)$ and $\check{N}(t,1)$ are constants and the elements $\hat{x}_i$, $\check{x}_i$ evolve linearly and described by (13). In these intervals the time-derivative of V_1 along the trajectories of the system is given by

$$\begin{aligned} \dot{V}_1(t) = {} & p_1\sum_{i=1}^{\hat{N}(t,1)-1} e^{\frac{-\gamma}{\lambda_1}\hat{x}_i}\left(\alpha(t,\hat{x}_i^-)^2 - \alpha(t,\hat{x}_i^+)^2\right) \\ & + \frac{p_1}{\lambda_1}\sum_{i=0}^{\hat{N}(t,1)}\int_{\hat{x}_i^+}^{\hat{x}_{i+1}} 2e^{\frac{-\gamma}{\lambda_1}x}\alpha_t(t,x)\alpha(t,x)dx \\ & - p_2\sum_{i=1}^{\check{N}(t,1)-1} e^{\frac{\gamma}{\lambda_2}\check{x}_i}\left(\beta(t,\check{x}_i^-)^2 - \beta(t,\check{x}_i^+)^2\right) \\ & + \frac{p_2}{\lambda_2}\sum_{i=0}^{\check{N}(t,1)}\int_{\check{x}_i}^{\check{x}_{i+1}^-} 2e^{\frac{\gamma}{\lambda_2}x}\beta_t(t,x)\beta(t,x)dx. \end{aligned} \tag{26}$$

Using (5) and integration by parts, we get

$$\begin{aligned} \dot{V}_1(t) = {} & -\frac{p_1}{\lambda_1}e^{\frac{-\gamma}{\lambda_1}}\alpha(t,1)^2 + \frac{p_1}{\lambda_1}\alpha(t,0)^2 \\ & - \frac{p_1\gamma}{\lambda_1}\int_0^1 e^{\frac{-\gamma}{\lambda_1}x}\alpha(t,x)^2dx \\ & + \frac{p_2}{\lambda_2}e^{\frac{\gamma}{\lambda_2}}\beta(t,1)^2 \\ & - \frac{p_2}{\lambda_2}\beta(t,0)^2 - \frac{p_2\gamma}{\lambda_2}\int_0^1 e^{\frac{\gamma}{\lambda_2}x}\beta(t,x)^2dx. \end{aligned} \tag{27}$$

Following the same procedure for V_2 and from the fact that α and β satisfy

$$\begin{aligned} \alpha_{tx}(t,x) + \lambda_1\alpha_{xx}(t,x) &= 0 \\ \beta_{tx}(t,x) - \lambda_2\beta_{xx}(t,x) &= 0 \end{aligned} \tag{28}$$

with the boundary conditions

$$\begin{aligned} \partial_x^-\alpha(t,0) &= -q\tfrac{\lambda_2}{\lambda_1}\partial_x^+\beta(t,0) \\ \partial_x^+\beta(t,1) &= \tfrac{1}{\lambda_2}\beta_t(t,1) \end{aligned} \tag{29}$$

for almost all $t \geqslant 0$ and $x \in [0,1]$, we can also obtain

$$\begin{aligned} \dot{V}_2(t) = {} & -\frac{p_3}{\lambda_1}e^{-\frac{\gamma}{\lambda_1}}\partial_x^-\alpha(t,1)^2 + \frac{p_3}{\lambda_1}\partial_x^-\alpha(t,0)^2 \\ & - \frac{p_3\gamma}{\lambda_1}\int_0^1 e^{\frac{-\gamma}{\lambda_1}x}\partial_x^+\alpha(t,1)^2dx \\ & + \frac{p_4}{\lambda_2}e^{\frac{\gamma}{\lambda_2}}\partial_x^+\beta(t,1)^2 \\ & - \frac{p_4}{\lambda_2}\partial_x^+\beta(t,0)^2 - \frac{p_4\gamma}{\lambda_2}\int_0^1 e^{\frac{\gamma}{\lambda_2}x}\partial_x^+\beta(t,x)^2dx. \end{aligned} \tag{30}$$

Using the boundary conditions, it follows

$$\begin{aligned} \dot{V}_1(t) \leqslant {} & -\left(\frac{p_2}{\lambda_2} - \frac{p_1q^2}{\lambda_1}\right)\beta(t,0)^2 - \frac{p_1}{\lambda_1}e^{-\frac{\gamma}{\lambda_1}}\alpha(t,1)^2 \\ & + \frac{p_2}{\lambda_2}e^{\frac{\gamma}{\lambda_2}}\beta(t,1)^2 - \gamma V_1(t) \end{aligned} \tag{31}$$

$$\begin{aligned} \dot{V}_2(t) \leqslant {} & -\left(\frac{p_4}{\lambda_2} - \frac{p_3q^2\lambda_2^2}{\lambda_1^3}\right)\partial_x^+\beta_x(t,0)^2 \\ & - \frac{p_3}{\lambda_1}e^{\frac{-\gamma}{\lambda_1}}\partial_x^-\alpha_x(t,1)^2 + \frac{p_4}{\lambda_2^3}e^{\frac{\gamma}{\lambda_2}}\beta_t(t,1)^2 - \gamma V_2(t). \end{aligned} \tag{32}$$

Now consider the following looped-functional

$$\mathcal{V}_k(\tau,\chi_k) = p_5(T_k-\tau)\chi_k(\tau)^2 \tag{33}$$

with $\tau \in [0,T_k]$ and $\chi_k(\tau) = \beta(t_k+\tau^-,1) - \rho(1-k_0)\alpha(t_k+\tau^-,1)$ for all $k\in\mathbb{N}$. Let us remark that $\chi_k(T_k) = \beta(t_{k+1}^-,1) - \rho(1-k_0)\alpha(t_{k+1}^-,1)$. In addition, note that the assumption $T_k = T\in\mathcal{T}$ guarantees that $\alpha(t,1)$ and $\beta(t,1)$ are continuous between the sampling times, since the discontinuities reflected by the boundary condition at $x=1$ coincide always with a sampling instant. Taking the derivative with respect to τ of $\mathcal{V}_k(\tau,\chi_k)$, one gets

$$\dot{\mathcal{V}}_k(\tau,\chi_k) = -p_5\chi_k(\tau)^2 + 2p_5(T_k-\tau)\chi_k(\tau)\dot\chi_k(\tau). \tag{34}$$

The time-derivative of $\beta(1,t)$ is given by $\beta_t(1,t) = N_0\zeta(t)$ where N_0 is given in (15) and

$$\zeta(t) = [\beta(t,0)\ \ \beta(t,1)\ \ \alpha(t,1)\ \ \beta_x(t,0)\ \ \alpha_x(t,1)\ \ \psi(t)]^\top \tag{35}$$

with $\psi(t) = \int_0^1 \lambda_1 L_x^\alpha(x)\alpha(t,x) - \lambda_2 L_x^\beta(x)\beta(t,x)dx$. Using the definitions in (15) into equations (31), (32), it is obtained

$$\dot V(t) \leqslant -\gamma V(t) + \zeta(t)^\top\Pi_1\zeta(t) \tag{36}$$

and using equation (34), we get

$$\begin{aligned}&\dot{\mathcal{V}}_k(\tau,\chi_k) \leqslant -\eta\mathcal{V}_k(\tau,\chi_k)\\ &+\zeta(t_k+\tau)^\top(-\Pi_2+(T_k-\tau)(\eta\Pi_2+\Pi_3))\zeta(t_k+\tau)\end{aligned} \tag{37}$$

with $\eta=\gamma-p_6$. In addition, note that the term $\psi(t)$ can be simply bounded by $V_1(t)$ as follows $\psi(t)^2 \leqslant MV_1(t)$ where M is given in (15). Let us remark that the kernels of the inverse backstepping transformation are differentiable and defined in a compact set, and thus functions L_x^α and L_x^β used in M are bounded. Adding $\frac{p_6}{M}(\psi(t)^2-\psi(t)^2)$ to (36) and considering the bound of $\psi(t)^2$, we obtain

$$\begin{aligned}\dot V(t_k+\tau)+\dot{\mathcal{V}}_k(\tau,\chi_k) &\leqslant -\eta(V(t_k+\tau)+\mathcal{V}_k(\tau,\chi_k))\\ &+\zeta(t_k+\tau)^\top\Psi(\tau,T_k)\zeta(t_k+\tau),\end{aligned} \tag{38}$$

where $\Psi(\tau,T_k) = (\Pi_1-\Pi_2+(T_k-\tau)(\eta\Pi_2+\Pi_3)-\frac{p_6}{M}e_6^6e_6^{6\top})$. Note that if $\Psi(\tau,T_k)\leqslant 0$ then it follows

$$\dot V(t_k+\tau)+\dot{\mathcal{V}}_k(\tau,\chi_k) \leqslant -\eta\left(V(t_k+\tau)+\mathcal{V}_k(\tau,\chi_k)\right) \tag{39}$$

for all $\tau\in[0,T_k]$, and thus, we get

$$\begin{aligned}&V(t_k+\tau) \leqslant V(t_k+\tau)+\mathcal{V}_k(\tau,\chi_k)\\ &\leqslant e^{-\eta(t_k+\tau)}\left(V(0)+\mathcal{V}_0(0,\chi_0)\right) = e^{-\eta(t_k+\tau)}V(0),\end{aligned} \tag{40}$$

which implies (note that $\mathcal{V}_0(0,\chi_0)=0$, since t_0 is the first sampling time)

$$\|\bar s(t,\cdot)\|^2_{L_2([0,1],\mathbb{R}^4)} \leqslant Ce^{-\eta t}\|(\alpha^0,\beta^0,\alpha_x^0,\beta_x^0))^\top\|^2_{L_2([0,1],\mathbb{R}^4)} \tag{41}$$

with $\bar s(t,\cdot) = (\alpha(t,\cdot),\beta(t,\cdot),\partial_x^-\alpha(t,\cdot),\partial_x^+\beta(t,\cdot))^\top$.
In addition, the derivative of (4) with respect to x is given by

$$\begin{aligned}u_x(t,x) &= \alpha_x(t,x) - L^{\alpha\alpha}(x,x)\alpha(t,x) - L^{\alpha\beta}(x,x)\beta(t,x)\\ &-\int_0^x\left(L_x^{\alpha\alpha}(x,y)\alpha(t,y)+L_x^{\alpha\beta}(x,y)\beta(t,y)\right)dy,\\ v_x(t,x) &= \beta_x(t,x) - L^{\beta\alpha}(x,x)\alpha(t,x) - L^{\beta\beta}(x,x)\beta(t,x)\\ &-\int_0^x\left(L_x^{\beta\alpha}(x,y)\alpha(t,y)+L_x^{\beta\beta}(x,y)\beta(t,y)\right)dy.\end{aligned} \tag{42}$$

Using (4) and (42) into (41), we get (17). Finally, note that $\Psi(\tau,T_k)\leqslant 0$ from $T_k=T$, conditions (19)-(21) and a convexity argument on τ and T. Therefore, the first item in Lemma 1 is proved. The proof of the second item follows directly by noting that $k_0=0$ implies that $\chi_k(\tau)$ is continuous, and thus, the looped-functional $\mathcal{V}_k$ is continuous between the sampling instants for any $T_k\in(0,T^*]$. That is, assumption $T_k=T\in\mathcal{T}$ is not necessary. Finally, in order to prove the third item, note that $\rho=0$ implies that assumption $T_k=T\in\mathcal{T}$ is again not necessary, since the looped-functional $\mathcal{V}_k$ is continuous between the sampling instants for any $T_k\in(0,T^*]$. In addition, we get

$$\begin{aligned}\dot V_1(t_k+\tau)+\dot{\mathcal{V}}_k(\tau,\chi_k) &\leqslant -\eta(V_1(t_k+\tau)+\mathcal{V}_k(\tau,\chi_k))\\ &+\zeta(t_k+\tau)^\top E^\top\Psi(\tau,T_k)E\zeta(t_k+\tau)\end{aligned} \tag{43}$$

for $\rho=0$ with $E=[e_1^6\ e_2^6\ e_3^6\ e_6^6]^\top$. Since $\Psi(\tau,T_k)$ is negative-semidefinite, then $E^\top\Psi(\tau,T_k)E$ is also negative-semidefinite. The rest of the proof follows as the proof of the first item.

4) Proof of Theorems 1 and 2: Note that if the right term in (20) is strictly negative then there exists a small enough T^* such that (21) holds, which directly proves Theorems 1 and 2 whenever condition (19) is satisfied. Therefore, let us find constants $\gamma>0$ and p_i, $i\in\{1,\dots 6\}$ such that condition (20) is strict and (19) holds. First, from the definition of N_0 it follows the following inequality

$$N_0^\top N_0 \leqslant (1+\epsilon)\rho^2\lambda_1^2e_5^6e_5^{6\top} + (1+\frac{1}{\epsilon})\tilde N_0^\top\tilde N_0 \tag{44}$$

for all $\epsilon>0$ with

$$\tilde N_0 = [\lambda_2L^\beta(0)-\lambda_1L^\alpha(0)q\ \ -\lambda_2L^\beta(1)\ \ \lambda_1L^\alpha(1)\ \ \ 0\ \ 0\ \ -1]. \tag{45}$$

Taking into account the above inequality and after some calculations, condition (20) is strict if the following conditions hold

$$\frac{p_4}{\lambda_2^3}e^{\frac{\gamma}{\lambda_2}}(1+\frac{1}{\epsilon})\|\tilde N_0\|^2 + \frac{p_1q^2}{\lambda_1} - \frac{p_2}{\lambda_2} < 0 \tag{46}$$

$$\frac{p_3q^2\lambda_2^2}{\lambda_1^3} - \frac{p_4}{\lambda_2} < 0 \tag{47}$$

$$\frac{p_4}{\lambda_2^3}e^{\frac{\gamma}{\lambda_2}}(1+\frac{1}{\epsilon})\|\tilde N_0\|^2 - \frac{p_6}{M} < 0 \tag{48}$$

$$\frac{p_4}{\lambda_2^3}e^{\frac{\gamma}{\lambda_2}}(1+\epsilon)\rho^2\lambda_1^2 - \frac{p_3}{\lambda_1}e^{\frac{-\gamma}{\lambda_1}} < 0 \tag{49}$$

$$\begin{bmatrix} \varrho + \frac{p_2}{\lambda_2} e^{\frac{\gamma}{\lambda_2}} - p_5 & p_5 \rho(1-k_0) \\ \star & \varrho - \frac{p_1}{\lambda_1} e^{\frac{-\gamma}{\lambda_1}} - p_5 \rho^2 (1-k_0)^2 \end{bmatrix} < 0 \tag{50}$$

with $\varrho = \frac{p_4}{\lambda_2^3} e^{\frac{\gamma}{\lambda_2}} (1+\frac{1}{\epsilon}) \|\tilde{N}_0\|^2$. First, let us pick γ and ϵ such that

$$\gamma < -\frac{\log(\rho q)^2 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2}, \qquad \epsilon < \frac{e^{\gamma(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})}}{(\rho q)^2} - 1. \tag{51}$$

Now consider any $p_2 > 0$, and pick $p_1 < \frac{p_2}{q^2 \lambda_2}$ and $p_6 < \gamma$, thus (19) holds. By choosing p_4 small enough and p_5 sufficiently large with respect to p_2, conditions (46), (48), and (50) are satisfied. For the given p_4, conditions (47) and (49) hold by choosing p_3 as follows

$$\frac{p_4(1+\epsilon)\rho^2 \lambda_1^3 e^{\gamma(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})}}{\lambda_2^3} < p_3 < \frac{p_4 \lambda_1^3}{\lambda_2^3 q^2}. \tag{52}$$

Equation (51) guarantees that the above p_3 exists. Therefore, the proof is complete.

V. Illustrative example

In this section, a numerical example is provided in order to illustrate the proposed stability analysis. Consider the system (1)-(2) with $\lambda_1 = 1$, $\lambda_2 = 2$, $\sigma_1 = 2$, $\sigma_2 = 2$, and $q = 0.5$. Let us analyze the following two cases:

1) System with boundary reflection: Consider for instance $\rho = 0.5$, then the existence of T^*, such that the exponential estimation (17) holds, follows directly from Theorem 1. In addition, solving the conditions in Lemma 1 with a line search in the variable p_6 and γ, we obtain that $T^* = 1.5/12$ is the maximum sampling period belonging to $\mathcal{T} = (\frac{1.5}{j})_{j=1}^{\infty}$ for which the exponential convergence of the solution is guaranteed for all $k_0 \in [0.3, 1]$. For $k_0 \in [0, 0.3)$ the maximum sampling period is $T^* = 1.5/13$. For the particular case $k_0 = 0$, exponential convergence is guaranteed with aperiodic sampling satisfying $T_k \in (0, T^*]$.

Fig. 1 shows the evolution of the component $v(t,x)$ when the system is controlled by the continuous controller (left) and the sampled-data controller (right) with $k_0 = 0.8$ and the initial condition $u^0(x) = qv^0(x)$, $v^0(x) = 10(1-x)$ for all $x \in [0,1]$. In addition, the control signal U is plotted in Fig. 2 for both cases. Note how the discontinuities, introduced by the sampled-data controller, propagate across the spatial domain.

2) System without boundary reflection, $\rho = 0$: Theorem 2 guarantees that there exists T^* such that the exponential convergence given by (18) holds. In addition, from Lemma 1, we can obtain a decay rate $\eta = 1.3 \times 10^{-2}$ for the parameter $T^* = 0.12$. As we may expect it is possible to obtain larger decay rates (upper bounded by the decay rate of the continuous case) by reducing T^*. On the other hand, the computation of the upper bound T^* of the inter-event times for the case $k_0 = 0$ by using the results in [5] leads to $T^* < 0.06$ independently of the decay rate, suggesting that the proposed method is less conservative in terms of the upper bound computation.

VI. Conclusions

This work provides results on the existence of small enough inter-sampling times such that the global exponential stability

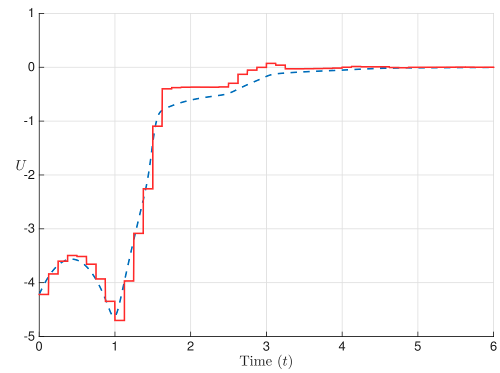

Fig. 2. Time-evolution of the continuous-time controller (blue dashed line) and the sampled-data controller (red line).

of a 2×2 linear hyperbolic system with a predesigned controller is preserved in presence of sampling. The controller under study is the modified version, proposed in [1], of the full state boundary feedback controller propounded in [25] by means of the backstepping method. Two different cases depending on the boundary conditions are analyzed. Firstly, we consider that the two PDEs are interconnected on both boundaries (system with boundary reflection). In this case, we show that there exists a sufficiently small sampling period, under a commensurability assumption with the transport velocities, for which the exponential stability, in terms of the L_2-norm of the states and their spatial derivatives, is guaranteed. Secondly, we assume that no boundary reflection occurred, which allows us to guarantee exponential stability, in terms of the L_2-norm of only the states, under small enough inter-sampling times (encompassing both periodic and aperiodic sampling). In order to easily compute an upper bound of the maximum inter-sampling time or sampling period, we provide tractable sufficient stability conditions in the form of matrix inequalities. As a future work, it could be interesting to generalize the obtained results to a more general class of systems such as $n \times n$ linear hyperbolic systems and also to relax equation (16) for $k_0 \neq 0$. Finally, the proposed method might be combined with an event-triggering mechanism, in order to design an event-triggering strategy with a greater dwell-time, or a periodic event-triggered controller following the ideas in [7].

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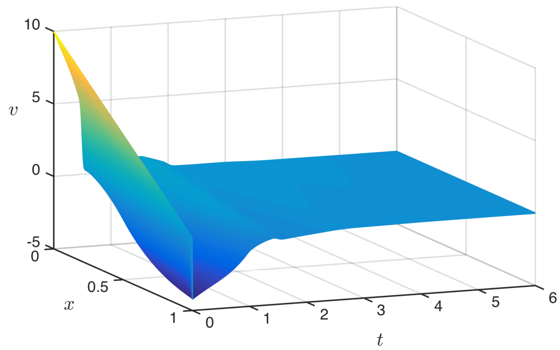

Fig. 1. Numerical solution of the component v of the closed-loop system with the continuous-time controller (left) and with the sampled-data controller (right).

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Appendix

This section contains the Steps 1 and 2 of the proof given in Section IV.

1) Existence of solution: Consider the system (5)-(6) with $U(t) = K$ for all $t \geqslant 0$ and some $K \in \mathbb{R}$. Let us prove that for all piecewise continuous initial condition $(\alpha^0, \beta^0)^\top$, there exists $T > 0$ such that the above system has a unique solution for all $t \in [0, T)$. By the method of characteristics we get

$$\alpha(t,x) = \begin{cases} \alpha(0, x - \lambda_1 t), & x > \lambda_1 t \\ \alpha\left(t - \frac{x}{\lambda_1}, 0\right), & x \leqslant \lambda_1 t \end{cases} \tag{53}$$

$$\beta(t,x)=\begin{cases}\beta(0,x+\lambda_2 t), & x<1-\lambda_2 t\\ \beta\left(t-\frac{1-x}{\lambda_2},1\right), & x\geqslant 1-\lambda_2 t\end{cases}\tag{54}$$

Let us consider $T=\min(\frac{1}{\lambda_1},\frac{1}{\lambda_2})$, the solutions $\alpha(t,x)$ and $\beta(t,x)$ are given as follows:

$$\alpha(t,x)=\begin{cases}\alpha(0,x-\lambda_1 t) & x>\lambda_1 t\\ q\beta\left(0,\lambda\left(t-\frac{x}{\lambda_1}\right)\right) & x\leqslant\lambda_1 t\end{cases}\tag{55}$$

$$\beta(t,x)=\ \beta(0,x+\lambda_2 t)\quad x<1-\lambda_2 t\tag{56}$$

and defining $\bar{t}=t-\frac{1-x}{\lambda_2}$, the solution $\beta(t,x)$ for $x\geqslant 1-\lambda_2 t$ is given by

$$\beta(t,x)=d(\bar{t})+\int_{1-\lambda_2\bar{t}}^{1}L^{\beta}(y)\beta\left(\bar{t}-\tfrac{1-y}{\lambda_2},1\right)dy\tag{57}$$

with

$$d(\bar{t})=K+\rho\alpha(0,1-\lambda_1\bar{t})+\int_0^{\lambda_1\bar{t}}qL^{\alpha}(y)\beta\left(0,\lambda_2\left(\bar{t}-\tfrac{y}{\lambda_1}\right)\right)dy$$
$$+\int_{\lambda_1\bar{t}}^{1}L^{\alpha}(y)\alpha(0,y-\lambda_2\bar{t})dy+\int_0^{1-\lambda_2\bar{t}}L^{\beta}(y)\beta(0,y+\lambda_2\bar{t})dy\tag{58}$$

In order to prove that there exits $\beta(\bar{t},1)$ that satisfies (57), let us consider the change of variable $z=\bar{t}-\frac{1-y}{\lambda_2}$, then it follows

$$\beta(\bar{t},1)=d(\bar{t})+\int_0^{\bar{t}}\lambda_2 L^{\beta}(1-\lambda_2(\bar{t}-z))\beta(z,1)dz\tag{59}$$

Since $d(\bar{t})$ and $L^{\beta}(1-\lambda_2(\bar{t}-z))$ are well-defined for $\bar{t}\in[0,t]$ and $t<\min(\frac{1}{\lambda_1},\frac{1}{\lambda_2})$, and in addition $L^{\beta}(1-\lambda_2(\bar{t}-z))$ is also continuous in the interval, then equation (59) is a linear Volterra integral equation with a unique solution (see Theorem 6 in [9]). Therefore, there exists a unique solution to (5)-(6) with $U(t)=K$ for all $t\geqslant 0$ in the interval $[0,T)$. Since U defined in Section II is a piecewise constant function with jump discontinuities at $\{t_k\}_{k\in\mathbb{N}}$, the solution to system (5)-(6) is simply made up of an initial condition $(\alpha^0,\beta^0)^\top\in C([0,1],\mathbb{R}^2)$ and a sequence of segments of the solution to the initial-value problem (5)-(6) with constant $U(t)=K$ for all $t\geqslant 0$ and different values of K. Finally, the inverse backstepping transformation (4) leads to the existence of a unique solution to system (1)-(2) with a piecewise constant signal U.

2) Backstepping transformation: In this part, we aim at proving that for any piecewise continuous signal $U:\mathbb{R}_{\geqslant}\to\mathbb{R}$ with jump discontinuities at $\{t_k\}_{k\in\mathbb{N}}$, the backstepping transformation (3) maps system (1)-(2) to system (5)-(6). This part of the proof is added for the sake of completeness, and it is similar to the proof in [2, 25] but with an explicit treatment of the discontinuities. Consider the sets given in (12). Note that $T_k\in(0,T^*]$ implies that there is a finite number of elements in $\widehat{\mathbb{X}}(t,x)$ and $\check{\mathbb{X}}(t,x)$ for all $t\geqslant 0$ and $x\in[0,1]$. Since the elements in $\widehat{\mathbb{X}}(t,x)$ and $\check{\mathbb{X}}(t,x)$ can be ordered, we define the sequences $(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_{\hat{N}(t,x)})$ and $(\check{x}_1,\check{x}_2,\ldots,\check{x}_{\check{N}(t,x)})$ for any instant $t\in\mathbb{R}_{\geqslant}$ and $x\in[0,1]$, where $\hat{N}(t,x)$ and $\check{N}(t,x)$ are the number of elements in the sets $\widehat{\mathbb{X}}(t,x)$ and $\check{\mathbb{X}}(t,x)$, respectively. The elements in sequences evolve with dynamics given by equation (13).

Consider the backstepping transformation (3), without loss of generality let us assume $\hat{N}(t,x)>0$ and $\check{N}(t,x)>0$, then the transformation of u (similarly for v) can be rewritten as follows:

$$\alpha(t,x)=u(t,x)-\int_0^{\hat{x}_1^-}K^{uu}(x,y)u(t,y)dy$$
$$-\int_{\hat{x}^+_{\hat{N}(t,x)}}^{x}K^{uu}(x,y)u(t,y)dy-\sum_{i=1}^{\hat{N}(t,x)-1}\int_{\hat{x}_i^+}^{\hat{x}_{i+1}^-}K^{uu}(x,y)u(t,y)dy$$
$$-\int_0^{\check{x}_1^-}K^{uv}(x,y)v(t,y)dy-\int_{\check{x}^+_{\check{N}(t,x)}}^{x}K^{uv}(x,y)v(t,y)dy$$
$$-\sum_{i=1}^{\check{N}(t,x)-1}\int_{\check{x}_i^+}^{\check{x}_{i+1}^-}K^{uv}(x,y)v(t,y)dy\tag{60}$$

Taking a time-derivative of $\alpha(t,x)$ on $\mathcal{D}_1$ and using equation (1), we get

$$\alpha_t(t,x)=u_t(t,x)-\int_0^{x}K^{uu}(x,y)(-\lambda_1\partial_y^-u(t,y)$$
$$+\sigma_1(y)v(t,y))dy+\sum_{i=1}^{\hat{N}(t,x)}\lambda_1K^{uu}(x,\hat{x}_i)\left(u(t,\hat{x}_i^+)-u(t,\hat{x}_i^-)\right)$$
$$-\int_0^{x}K^{uv}(x,y)(\lambda_2\partial_y^+v(t,y)+\sigma_2(y)u(t,u))dy$$
$$+\sum_{i=1}^{\check{N}(t,x)}\lambda_2K^{uv}(x,\check{x}_i)\left(v(t,\check{x}_i^+)-v(t,\check{x}_i^-)\right)\tag{61}$$

From (1) and using integration by parts, it is obtained

$$\int_0^{x}\lambda_1K^{uu}(x,y)\partial_y^-u(t,y)dy=\lambda_1K^{uu}(x,x)u(t,x)$$
$$-\lambda_1K^{uu}(x,0)u(t,0)-\int_0^{x}K_y^{uu}(x,y)u(t,y)dy\tag{62}$$
$$-\sum_{i=1}^{\hat{N}(t,x)}\lambda_1K^{uu}(x,\hat{x}_i)\left(u(t,\hat{x}_i^+)-u(t,\hat{x}_i^-)\right)$$

and

$$\int_0^{x}\lambda_2K^{uv}(x,y)\partial_y^+v(t,y)dy=\lambda_2K^{uv}(x,x)v(t,x)$$
$$-\lambda_1K^{uv}(x,0)v(t,0)-\int_0^{x}K_y^{uv}(x,y)v(t,y)dy\tag{63}$$
$$-\sum_{i=1}^{\check{N}(t,x)}\lambda_2K^{uv}(x,\check{x}_i)\left(v(t,\check{x}_i^+)-v(t,\check{x}_i^-)\right).$$

Replacing (62) and (63) into (61), it follows

$$\alpha_t(t,x)=u_t(t,x)+\lambda_1K^{uu}(x,x)u(t,x)-\lambda_1K^{uu}(x,0)u(t,0)$$
$$-\int_0^{x}K_y^{uu}(x,y)u(t,y)dy-\int_0^{x}K^{uu}(x,y)\sigma_1(y)v(t,y)dy$$
$$+\lambda_2K^{uv}(x,x)v(t,x)-\lambda_2K^{uv}(x,0)v(t,0)$$
$$-\int_0^{x}K_y^{uv}(x,y)v(t,y)dy-\int_0^{x}K^{uv}(x,y)\sigma_2(y)u(t,y)dy\tag{64}$$

Note that despite the discontinuities introduced by U, the obtained time-derivative, $\alpha_t(t,x)$, is the same as in [2] (similarly for β). Hence, following the procedure in [2], we get the target system (5)-(6) as in the case of a continuous U.