Prediction-based control of linear input-delay system subject to state-dependent state delay – Application to suppression of mechanical vibrations in drilling

Delphine Bresch-Pietri [*] and **Florent Di Meglio** [**]

[*] *CNRS at GIPSA-lab, Control Department, 11 rue des Mathématiques, 38000 Grenoble, France (e-mail: delphine.bresch-pietri@gipsa-lab.fr)*
[**] *CAS - Centre Automatique et systèmes, MINES ParisTech, PSL Research University, 60 bd St-Michel, 75006 Paris, France (e-mail: florent.di_meglio@mines-paristech.fr).*

Abstract:
In this paper, we consider linear dynamics subject to a distributed state-dependent delay and a pointwise input-delay. We propose a prediction-based controller which exponentially stabilizes the plant. The controller design is based on a backstepping approach where delays are reformulated as hyperbolic transport PDEs. Infinity-norm stability analysis of the corresponding closed-loop system is addressed. We show that this result is of interest to suppress mechanical vibrations arising in drilling facilities, which have been attributed recently to a coupling between torsional and vertical displacement involving an implicit state delay equation. Numerical simulations illustrate the merits of our controller in this context.

1. INTRODUCTION

Mechanical instabilities are an important source of damage for drilling equipment, particularly in the oil industry. These undesirable dynamical behaviors cause wear-and-tear on the installations and sometimes lead to complete and premature failures. Three main types of instability arise: vertical vibrations, leading to pressure oscillation in the surrounding mud, whirl oscillations, due to an unbalanced drillstring, and torsional vibrations. All these vibration phenomena often degenerate into an oscillatory behavior referred to as stick slip phenomenon. It consists of a phase during which the drill bit velocity decreases, potentially up to a point where the bit stops rotating, followed by a phase where the angular velocity suddenly increases, up to twice the rotating velocity imposed on the rotating table at the top of the drill pipe.

This is why this behavior has been the focus of many studies, aiming at identifying the mechanisms of self-excitation to suppress them. Conventionally, axial and torsional vibrations are considered as decoupled problems. The torsional dynamics is assimilated to an inverted pendulum excited by a rock-on-the-bit friction term which decreases with the bit angular velocity, acting as an "anti-damping" term (see Navarro-López and Suarez-Cortez [2004], Dankowicz and Nordmark [2000]). Lately, an alternative interpretation has emerged. The model proposed in Depouhon and Detournay [2014] instead attributes stick slip to a coupling between torsional and vertical displacements. Inspired by studies of tool chatter in metal machining, this work proposes to represent the torque acting on the angular bit velocity as a function of the vertical displacement. The model characterizing this displacement is an implicit state-dependent delay equation and therefore leads to the study of an input-delay dynamics subject to distributed state-dependent state delay.

For this reason, in this paper, we focus on linear systems subject to constant (pointwise) input delay and state-dependent (distributed) state delay. We aim at designing a prediction-based control strategy for this problem. This class of controllers, more commonly known as Smith Predictor (see Smith [1959], Artstein [1982], Manitius and Olbrot [1979]), is grounded on the use of a prediction of the system state on a time horizon equal to the input delay and aims at compensating it, which notably improves the transient performances. However, while its use is state-of-the-art for systems subject to a single constant input time-delay, its applicability to systems with both input and state delays has seldom and only recently been studied: a nominal prediction-based controller has been proposed in Kharitonov [2013] for a linear systems subject to pointwise state and input delays (see Bresch-Pietri and Petit [2014] for a delay-robustness version of this result) and extended in Bekiaris-Liberis [2014] to encompass nonlinear dynamics and (potentially) distributed state delays. All these results consider the state delay as constant.

In this paper, we extend this methodology to tackle the case of a state-dependent state delay. This type of dependency has been considered in Bekiaris-Liberis et al. [2012] for a specific class of state-dependent state delay systems which are not subject to input delay and in Bekiaris-Liberis and Krstic [2013] which consider state-dependency of the input delays, resulting in a very intricate relation between the system state and the control inputs which is not involved here. These have inspired the proposed prediction design.

Our stability analysis is grounded on PDEs tools that were proposed lately to address input delay compensation (see Krstic and Smyshlyaev [2008], Krstic [2008]) and were extended

recently in Bresch-Pietri et al. [2015] to handle the case of an additional distributed state-delay. In this paper, we build on those previous contributions to propose a PDE framework accounting for state-dependency of the state delay. Modeling both actuator and state delays as transport PDEs coupled with the original Ordinary Differential Equation (ODE), we rely on a backstepping transformation of the distributed input to analyze the closed-loop stability. To formulate the corresponding target system, one needs to study an implicit functional PDE. While state delay is responsible for the implicit nature, the functional one originates from the fact that this delay is distributed. We then carry out an $\mathscr{L}_\infty$ analysis for the closed-loop system. This along with the prediction design is the main contribution of the paper.

The paper is organized as follows. In Section 2, we introduce the problem under consideration before providing the prediction-based control we propose. Then, we present the stability analysis of the closed-loop dynamics in Section 3. Finally, Section 4 is devoted to the application of this control strategy to the suppression of mechanical vibrations in drilling. We conclude with directions of future work.

2. PROBLEM STATEMENT AND CONTROL DESIGN

We consider the problem of stabilizing the following (controllable) linear system subject to a distributed [2] space-dependent state delay and a constant input delay

$$\dot{X}(t) = A_0 X(t) + A_1 \int_{t-D_1(X_t)}^{t} X(s)ds + BU(t-D_2) + Ch(t) \tag{1}$$

in which $X \in \mathbb{R}^n$, U is the scalar input [3], h is a *known* function of time, the state-delay $D_1 : \mathscr{C}([-\overline{D},0],\mathbb{R}^n) \to [0,\overline{D}]$ ($\overline{D} > 0$) is a continuously differentiable function, D_2 is a constant input-delay and X_t denotes the function $X_t : s \in [-\overline{D},0] \mapsto X(t+s)$. Similarly, in the following, we denote $X_{t,1} : s \in [-D_1(X_t),0] \mapsto X(t+s)$ and $U_{t,2} : s \in [-D_2,0] \mapsto U(t+s)$. In the sequel, we consider [4] that $D_2 \geq \overline{D}$.

Before presenting our strategy to handle the input-delay, we make an assumption on the nominal input-delay free system stabilization.

Assumption 1. There exists $\kappa : \mathscr{C}^0([-D_1,0],\mathbb{R}^n) \mapsto \mathbb{R}$ which is a linear and class $\mathscr{C}^1$ feedback law such that the dynamics

$$\dot{X}(t) = A_0 X(t) + A_1 \int_{t-D_1(X_t)}^{t} X(s)ds + B\kappa(X_{t,1}) + Ch(t) \tag{2}$$

is globally exponentially stable, i.e. (see Kolmanovskii and Myshkis [1999], Pepe and Karafyllis [2013]), there exist a continuous functional $V_0 : \mathscr{C}([-\overline{D},0],\mathbb{R}^n) \to \mathbb{R}$ and constants $C_1, C_2, C_3 > 0$ such that

$$C_1 \|\varphi\|_\infty \leq V_0(\varphi) \leq C_2 \|\varphi\|_\infty \tag{3}$$
$$|\partial_\varphi V_0(\varphi)| \leq C_3 \tag{4}$$

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[2] Note that the methodology we propose can be straightforwardly extended to handle pointwise state delay, provided that we know a suitable nominal feedback law, namely, that the analog of Assumption 1 holds in this context.

[3] This assumption is only made for the sake of clarity of the exposition. One can easily observe that the proposed strategy straightforwardly extends to the case of multi-input subject to the same delay.

[4] This assumption is also made for the sake of simplicity. When the state delay can potentially be larger than the input delay, the proposed prediction-based control strategy remains unchanged. Only the prediction computation (and the stability analysis) will need to be updated.

and, moreover, the functional V_0 is differentiable along the trajectories of the closed-loop system (2) and

$$\dot{V}_0(t) \leq -V_0(t) \tag{5}$$

It is worth noting that requiring a linear feedback map is not demanding as we consider linear dynamics.

Even if this assumption can seem quite restrictive at first glance, it actually encompasses a large class of systems. Indeed, all systems under a strict-feedforward form for example will satisfy it. More generally, all systems for which the backstepping methodology can be applied will fold under this assumption, as the following example illustrates it.

Example: Consider the plant

$$\dot{x}_1(t) = x_1(t) + \int_{t-D_1(X_t)}^{t} x_1(s)ds + x_2(t) \tag{6}$$
$$\dot{x}_2(t) = 2x_1(t) + x_2(t) + U(t) \tag{7}$$

which is under the form (1) with $X = [x_1 \quad x_2]^T$ and $h = 0$. The state delay D_1 is a given known state-dependent function. Taking x_2 as a virtual input in (6) to map it into the target dynamics $\dot{x}_1(t) = -x_1(t)$ leads to the choice

$$\kappa(X_t) = -[x_2(t) - v(t)] - 2x_1(t) - x_2(t) + \dot{v}(t) \tag{8}$$
$$= -7x_1(t) - 2x_2(t) + x_1(t-D_1(X_t)) - 3\int_{t-D_1(X_t)}^{t} x_1(s)ds$$

in which

$$v(t) = -2x_1(t) - \int_{t-D_1(X_t)}^{t} x_1(s)ds \tag{9}$$

The control law (8) satisfies Assumption 1 as it is linear, of class $\mathscr{C}_\infty$ and the closed-loop system corresponding to $U(t) = \kappa(X_t)$ is

$$\dot{x}_1(t) = -x_1(t) + x_2(t) - v(t) \tag{10}$$
$$\dot{x}_2 - \dot{v}(t) = -(x_2(t) - v(t)) \tag{11}$$

which is exponentially stable. ∎

We are now ready to carry out the prediction-based control design. With this aim in view, consider the state prediction history

$$P^t(\tau) = \tag{12}$$
$$\begin{cases} X(\tau + D_2) & \text{if } t - \overline{D} - D_2 \leq \tau \leq t - D_2 \\ e^{A_0(\tau+D_2-t)}X(t) + \int_{t}^{\tau+D_2} e^{A_0(\tau+D_2-s)}Ch(s)ds + \int_{t-D_2}^{\tau} e^{A_0(\tau-s)} \\ \quad \times \left[A_1 \int_{s-D_1(P_s)}^{s} P^t(\xi)d\xi + BU(s) \right] ds & \text{if } t - D_2 \leq \tau \leq t \end{cases}$$

for $t \geq 0$ and $\tau \in [t - \overline{D} - D_2, t]$. We now use this prediction as argument for the nominal input-delay free control law in lieu of the original distributed state

$$U(t) = \kappa(P_{t,1}^t) \tag{13}$$

Theorem 1. Consider the closed-loop system consisting of (1) satisfying Assumption 1 and the control law (13) involving the prediction (12). Define the functional

$$\Gamma(t) = \max_{s \in [-\overline{D},0]} |X(t+s)| + \max_{s \in [-D_2,0]} |U(t+s)| \tag{14}$$

There exist $R, \rho > 0$ such that, for $(X_0, U_{0,2}) \in \mathscr{C}^0([-\overline{D},0],\mathbb{R}^n) \times \mathscr{C}^0(-D_2,0),\mathbb{R})$,

$$\Gamma(t) \leq R\Gamma(0)e^{-\rho t}, \quad t \geq 0 \tag{15}$$
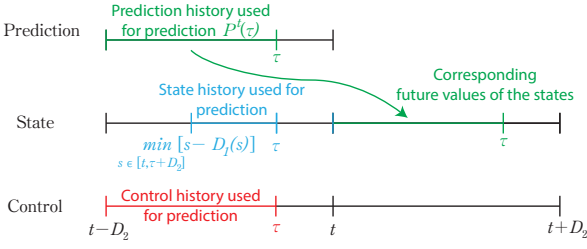
Fig. 1. Schematic view of the variables involved in the computation of the prediction $P^t(\tau)$ in (12).
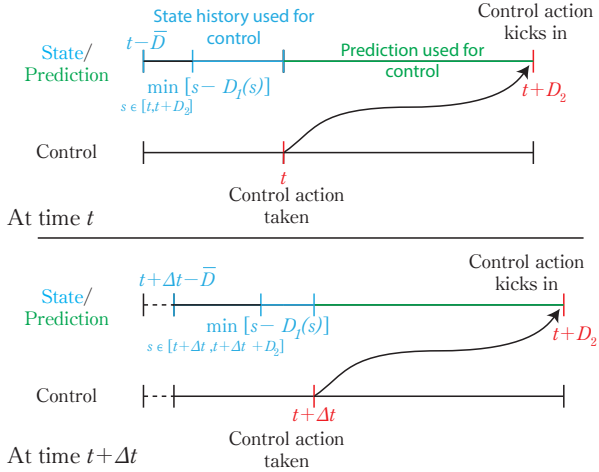

Fig. 2. Schematic view of the state history involved in the control computation for two different time instances. At time $t + \Delta t$, the involved state history horizon has shrunken compared to time t for this example.

In order to properly understand the choice of the control law, we provide several comments next.

First, it is worth noticing that the function P^t_t defined through (12) is a D_2 units of time ahead prediction of X_t, the state history over a time horizon. Indeed, integrating (1) between t and $\tau + D_2$ with the use of the variation of constant formula, one obtains

$$X(\tau+D_2) = e^{A_0(\tau+D_2-t)}X(t) + \int_t^{\tau+D_2} e^{A_0(\tau+D_2-s)}$$
$$\times \left[A_1 \int_{s-D_1(X_s)}^s X(\xi)d\xi + BU(s-D_2) + Ch(s) \right] ds$$
$$= e^{A_0(\tau+D_2-t)}X(t) + \int_t^{\tau+D_2} e^{A_0(\tau+D_2-s)}Ch(s)ds \quad (16)$$
$$+ \int_{t-D_2}^{\tau} e^{A_0(\tau-s)} \left[A_1 \int_{s-D_1(X_{s+D_2})}^s X(\xi+D_2)d\xi + BU(s) \right] ds$$

in which the last expression has been obtained performing a change of variable under the integral. One can observe that (16) is formally equivalent to the definition proposed in (12) and thus can obtain formally that $P^t(\tau) = X(\tau+D_2)$ for all $\tau \in [t-\bar{D},t]$. Consequently, plugging the control law (13) into the original dynamics, one naturally infers that the resulting closed-loop dynamics is exponentially stable, from Assumption 1. This is indeed the result stated in Theorem 1.

Second, note that we define the prediction as a function of two arguments: $P^{t_1}(\tau)$ is the prediction $X(\tau+D_2)$ computed at time t_1, using $X(t_1)$ as a starting point. This aims at emphasizing the choice we make to compute this prediction by incorporating

measured delayed states in the definition (12) instead of relying on an open-loop integration of (1) as in Bekiaris-Liberis [2014]. Of course, without dynamics uncertainty, the two formulations are equivalent. However, in presence of uncertainty, $P^{t_1}(\tau)$ is likely to be different from $P^{t_2}(\tau)$ for $t_2 \neq t_1$ and, in all likelihood, this formulation should improve the robustness of the prediction-based controller to model mismatch.

Third, even if the equation (12) may seem implicit at first glance, this prediction is actually well-defined and the solution always exists and is unique, as the solution of the differential equation (1). Further, more interestingly, it is also practically computable, relying on suitable discretization scheme of the integral (see Van Assche et al. [1999] for a study on the effect of this discretization scheme on the closed-loop stability of linear systems and Karafyllis and Krstic [2014] where nonlinear dynamics are addressed and a time-varying discretization methodology is proposed. Alternatively, one can rely on a low-pass filter addition as proposed in Mondié and Michiels [2003] for linear systems or on an approximate predictor as done in Karafyllis [2011]), and only requires the knowledge of past values of the state and the input, as illustrated in Fig. 1 and 2.

Finally, it is worth understanding that, contrary to Bekiaris-Liberis and Krstic [2013], we do not need to impose any restriction on the state-dependent delay rate here and thus to limit our result to a local one. Indeed, we consider that this state-dependency only affects the state delay. Consequently, the prediction (12) is not impacted by the state-delay rate, which can be arbitrarily large a priori and in particular can vary faster than the absolute time.

We now provide the proof of this theorem.

3. STABILITY ANALYSIS – PROOF OF THEOREM 1

In the sequel, for the sake of conciseness, we sometimes write $D_1(t) = D_1(X_t)$ and $\dot{D}_1(t) = \frac{dD_1}{dX_t} \cdot \dot{X}_t$

3.1 PDEs reformulations

Consider the distributed variables

$$\zeta(x,t) = X(t+D_1(t)(x-1)) \quad (17)$$
$$\bar{\zeta}(x,t) = X(t+\bar{D}(x-1)) \quad (18)$$
$$u(x,t) = U(t+D_2(x-1)) \quad (19)$$

Those variables encompass the history of the state over a variable time horizon $D_1(X_t)$ and a fixed on $\bar{D}$ and the history of the input over a time horizon of length D_2, respectively. The plant (1) can then be reformulated as the following PDE-ODE cascade

$$\begin{cases} \dot{X}(t) = A_0X(t) + A_1D_1(t)\int_0^1 \zeta(x,t)dx + Bu(0,t) + Ch(t) \\ D_1(t)\partial_t\zeta(x,t) = (1+\dot{D}_1(t)(x-1))\partial_x\zeta(x,t) \\ \zeta(1,t) = X(t) \\ D_2\partial_t u(x,t) = \partial_x u(x,t) \\ u(1,t) = U(t) \end{cases} \quad (20)$$

and, in addition, one can obtain

$$\begin{cases} \bar{D}\partial_t\bar{\zeta}(x,t) = \partial_x\bar{\zeta}(x,t) \\ \bar{\zeta}(1,t) = X(t) \end{cases} \quad (21)$$

Now, define the following distributed predictions, for $(x,y) \in [0,1]^2$,

$$p(x,t) = e^{A_0 D_2 x} X(t) + \int_t^{t+D_2 x} e^{A_0(t+D_2 x - s)} Ch(s)ds + D_2 \quad (22)$$

$$\times \int_0^x e^{A_0 D_2 (x-y)} \left[Bu(y,t) + A_1 D_1^0(\bar{\chi}(y,\cdot,t)) \int_0^1 \chi(y,\xi,t)d\xi \right] dy$$

$$\bar{\chi}(x,y,t) = \begin{cases} \bar{\zeta}\left(y + \dfrac{D_2}{\bar{D}}x, t\right) & \text{if } x D_2 + \bar{D}(y-1) \leq 0 \\[2mm] p\left(x + \dfrac{\bar{D}}{D_2}(y-1), t\right) & \text{if } x D_2 + \bar{D}(y-1) \geq 0 \end{cases} \quad (23)$$

$$\chi(x,y,t) = \begin{cases} X(t + D_2 x + D_1^0(\bar{\chi}(x,\cdot,t))(y-1)) \\ \qquad \text{if} \quad x D_2 + D_1^0(\bar{\chi}(x,\cdot,t))(y-1) \leq 0 \\[2mm] p\left(x + \dfrac{D_1^0(\bar{\chi}(x,\cdot,t))}{D_2}(y-1), t\right) \\ \qquad \text{if} \quad x D_2 + D_1^0(\bar{\chi}(x,\cdot,t))(y-1) \geq 0 \end{cases} \quad (24)$$

in which we introduced the function D_1^0 defined as $D_1^0(\bar{\chi}(x,\cdot,t)) = D_1(X_{t+xD_2})$. In (22), $p(x,t)$ simply accounts for the prediction $X(t+xD_2)$, in accordance with (12), while $\bar{\chi}(x,\cdot,t)$ and $\chi(x,\cdot,t)$ represent history of $p(x,t)$ over time horizon of $\bar{D}$ and $D_1(X_t)$, respectively.

Finally, define the following backstepping transformation of u

$$w(x,t) = u(x,t) - \kappa^0(\chi(x,\cdot,t)), \quad 0 \leq x \leq 1 \quad (25)$$

in which we introduced $\kappa^0(\zeta(\cdot,t)) = \kappa(X_{t,1})$. We have the following result which is proven in Appendix A.

Lemma 1. The infinite-dimensional backstepping transformation (25) together with the control law (13) transform (20) into the target system

$$\begin{cases} \dot{X}(t) = A_0 X(t) + A_1 D_1(t) \int_0^1 \zeta(x,t)dx + Ch(t) + B\kappa_0(\zeta(\cdot,t)) \\ \qquad + Bw(0,t) \\ D_1(t)\partial_t \zeta(x,t) = (1 + \dot{D}_1(t)(x-1))\partial_x \zeta(x,t) \\ \zeta(1,t) = X(t) \\ D_2 \partial_t w(x,t) = \partial_x w(x,t) \\ w(1,t) = 0 \end{cases} \quad (26)$$

The aim of this backstepping transformation is to ease the Lyapunov analysis by introducing the suitable boundary condition $w(1,t) = 0$. This will appear clearly in the following section.

3.2 Lyapunov analysis

We are now ready to carry out the Lyapunov analysis. Consider the following Lyapunov functional candidate

$$V_p(t) = \left(\mu_0 + 1 - \frac{1}{2p}\right) V_0(t)^{2p} + b^{2p} D_2 \int_0^1 e^{2p\mu_0 x} w(x,t)^{2p} dx \quad (27)$$

in which V_0 has been introduced in Assumption 1, $p \in \mathbb{N}^*$ and $\mu_0 > 1$. Taking a time-derivative, one gets

$$\dot{V}_p(t) = -(2p(\mu_0 + 1) - 1)V_0(t)^{2p}$$
$$+ (2p(\mu_0 + 1) - 1)V_0(t)^{2p-1}\partial_\varphi V_0(X_t)Bw(0,t)$$
$$- b^{2p}w(0,t)^{2p} - b^{2p}2p\mu_0 \int_0^2 e^{2p\mu_0 x} w^{2p}(x,t)dx \quad (28)$$

Using Young inequality and (4), one obtains

$$\dot{V}_p(t) = -2p\mu_0 V_0(t)^{2p} - \left(b^{2p} - (\mu_0 + 1)^{2p}|C_3 B|^{2p}\right)w(0,t)^{2p}$$
$$- b^{2p}2p\mu_0 \int_0^2 e^{2p\mu_0 x} w^{2p}(x,t)dx \quad (29)$$

Consequently, choosing

$$b > (\mu_0 + 1)|C_3 B| \quad (30)$$

it follows that

$$\dot{V}_p(t) \leq -2p\eta V_p(t) \quad (31)$$

in which $\eta = \min\left\{\dfrac{\mu_0}{\mu_0 + 1}, \dfrac{1}{D_2}\right\}$ and thus

$$V_p(t)^{\frac{1}{2p}} \leq e^{-\eta t} \left(\left(\mu_0 + 1 - \frac{1}{2p}\right)^{\frac{1}{2p}} V_0(0) \right.$$
$$\left. + b\left(D_2 \int_0^1 e^{2p\mu_0 x} w(x,0)^{2p} dx\right)^{\frac{1}{2p}} \right) \quad (32)$$

This gives

$$\left(\mu_0 + 1 - \frac{1}{2p}\right)^{\frac{1}{2p}} V_0(t) + b\left(D_2 \int_0^1 e^{2p\mu_0 x} w(x,0)^{2p} dx\right)^{\frac{1}{2p}} \leq$$
$$2e^{-\eta t}\left(\left(\mu_0 + 1 - \frac{1}{2p}\right)^{\frac{1}{2p}} V_0(0) + b\left(D_2 \int_0^1 e^{2p\mu_0 x} w(x,0)^{2p} dx\right)^{\frac{1}{2p}}\right) \quad (33)$$

Taking the limit as p tends to infinity, one obtains

$$V_0(t) + b \max_{x\in[0,1]} e^{\mu_0 x}|w(x,t)| \leq e^{-\eta t}\left(V_0(0) + b \max_{x\in[0,1]} e^{\mu_0 x}|w(x,0)|\right) \quad (34)$$

Finally, using (3), the fact that κ is linear from Assumption 1 and applying Young and Cauchy-Schwarz inequalities to the backstepping transformation (25), one obtains the desired result.

4. APPLICATION TO SUPPRESSION OF MECHANICAL VIBRATIONS IN DRILLING

Mechanical vibrations are an important source of Non-Productive Time (NPT) and failure in the oil drilling industry, causing major financial losses. Consider the drilling facilities schematically depicted on Fig. 3. The operator imposes a force and
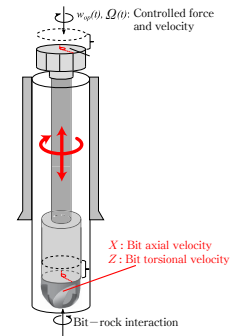

Fig. 3. Schematic view of drilling facilities.

rotating velocity at the surface. These are transmitted to the Bottom Hole Assembly (BHA) several kilometers downhole, which holds the drill bit that chatters and cuts the rock, thus creating the borehole. Axial and torsional displacement waves travel up and down the drillstring at a finite velocity, while the BHA is considered as a lumped oscillating mass. A nonlinear law describes the interaction of the drillbit with the rock, akin to a cutting process: both the torque and weight-on-bit are proportional to the *depth of cut*, defined as the vertical displacement of the bit over one revolution. More precisely, the following equations describe the deviation of the system states from an equilibrium (see Germay et al. [2009]).

- Topside actuation

$$\lambda(0,t) = \mu(0,t) + 2\tilde{w}_{op}(t) \tag{35}$$
$$\varphi(0,t) = \psi(0,t) + 2\tilde{\Omega}(t) \tag{36}$$

- Propagation of axial and torsional waves ($0 \le x \le L_p$)

$$\lambda_t(x,t) + c_a\lambda_x(x,t) = 0, \quad \mu_t(x,t) - c_a\mu_x(x,t) = 0 \tag{37}$$
$$\varphi_t(x,t) + c_\tau\varphi_x(x,t) = 0, \quad \psi_t(x,t) - c_\tau\psi_x(x,t) = 0 \tag{38}$$

- Velocity continuity at the drillstring – BHA junction

$$\mu(L_p,t) = -\lambda(L_p,t) + 2V(t) \tag{39}$$
$$\psi(L_p,t) = -\varphi(L_p,t) + 2\Omega(t) \tag{40}$$

- Dynamics of the BHA

$$\dot{V}(t) = \alpha\left[\lambda(L_p,t) - V(t)\right] - \beta\int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} V(s)ds - \gamma\tilde{t}_N(t) \tag{41}$$

$$\dot{\Omega}(t) = \alpha'\left[\varphi(L_p,t) - \Omega(t)\right] - \beta'\int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} V(s)ds - \gamma'\tilde{t}_N(t) \tag{42}$$

In each right-hand-side, the first term represents the interaction with the drillstring, while the second and third terms represent the interaction with the drillbit (see Germay et al. [2009]).

- Implicit definition of the delay

$$\int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} \Omega(s)ds + \Omega_0\tilde{t}_N(t) = 0 \tag{43}$$

All states and parameters are defined in Table 1, with their dependence on time t and space x. We consider that the top and bottom velocities [5] are measured and that the actuation act on the torque, as modeled by (35)–(36).

Symbol	Unit	Description
$\lambda(x,t)$	m.s^{-1}	Downward axial displacement wave
$\mu(x,t)$	m.s^{-1}	Upward axial displacement wave
$\varphi(x,t)$	rad.s^{-1}	Downward torsional displacement wave
$\psi(x,t)$	rad.s^{-1}	Upward torsional displacement wave
c_a	m.s^{-1}	Axial wave velocity
c_τ	m.s^{-1}	Torsional wave velocity
$\tilde{w}_{op}(t)$	m.s^{-1}	(Scaled) weight applied by the operator
$\tilde{\Omega}(t)$	rad.s^{-1}	Rotational velocity applied by the operator
Ω_0	rad.s^{-1}	Nominal rotational velocity
$V(t)$	m.s^{-1}	Bit axial velocity
$\Omega(t)$	rad.s^{-1}	Bit torsional velocity
$\alpha, \alpha', \beta, \beta', \gamma, \gamma'$		Bit-rock interaction law parameters (positive)
$\bar{t}_N$	s	Nominal state-delay at the bit
$\tilde{t}_N$	s	Deviation of the state-delay at the bit
L_p	m	Drillstring length

Table 1. States and parameters of the mechanical vibrations model (35)–(43)

Using the fact that $L_p/c_a < L_p/c_\tau$ (since $c_a = 5000$ m.s^{-1} and $c_\tau \approx 3100$ m.s^{-1}), one can easily show that (35)–(43) can be reformulated as (1) defining

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[5] While the top velocity is conventionally measured, the bottom one may actually not be available in practice. Future works will investigate the interest of extending the techniques employed in Bresch-Pietri and Krstic [2014] in this case.
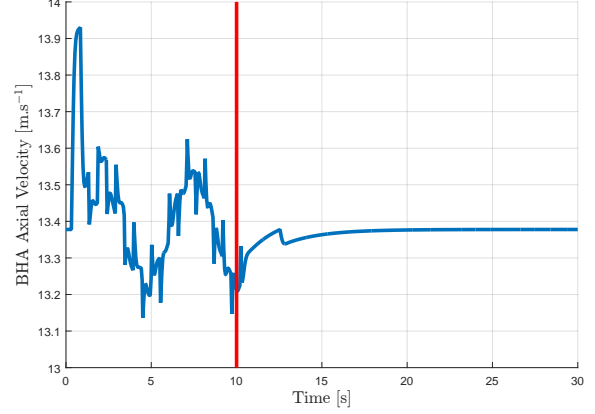
Fig. 4. Axial velocity of the BHA. The red line corresponds to the time instant when the controller is switched on.

$$D_2 = L_P/c_\tau, \quad D_1(X_t) = \bar{t}_N + \tilde{t}_N(t) \tag{44}$$
$$X(t) = [V(t) \quad \Omega(t)]^T \tag{45}$$
$$U(t) = [\lambda(0,t - L_p(1/c_\tau - 1/c_a)) \quad \varphi(0,t)]^T \tag{46}$$
$$A_0 = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha' \end{pmatrix}, \quad A_1 = \begin{pmatrix} -\beta & 0 \\ -\beta' & 0 \end{pmatrix} \tag{47}$$
$$B = -A, \quad C = -\begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \text{ and } h(t) = \tilde{t}_N(t) \tag{48}$$

The choice of U in (46) follows by noticing that $\lambda(L_p,t) = \lambda(0,t - L_p/c_a)$ and $\varphi(L_p,t) = \varphi(0,t - L_p/c_\tau)$ due to the transport equations (37)–(38).

It is worth noticing that, contrary to what could seem at first glance, the control choice (46) is causal. Indeed, we simply choose to voluntarily introduce an additional actuation delay in the control path: instead of controlling $\lambda(0,t)$, we define $\lambda(0,t) = \lambda_0(t + L_p(1/c_\tau - 1/c_a))$ and control $\lambda_0(t)$. This procedure leads to the formal control definition $U(t) = \lambda(0,t - L_p(1/c_\tau - 1/c_a))$ which is nevertheless causal. It aims at complying with the formulation (1) which considers a unique input delay. One can reasonably expect that this should limit the control performance and should be investigated in future works.

Finally, Assumption 1 is satisfied with the feedback law

$$\kappa(X_t) = -A_1\int_{t-D_1(X_t)}^{t} X(s)ds - Ch(t) - K_0X(t) \tag{49}$$

in which K_0 is a given matrix such that the closed-loop matrix dynamics $A_0 + BK_0$ is Hurwitz.

Figures 4 and 5 picture simulations where the proposed controller is used to stabilize the equilibrium corresponding to a nominal rotational velocity $\Omega_0 = 120$ rev/min. We pick K_0 as a zero matrix as the parameters α and α' are already positive. The controller is turned on after 10 seconds. One can observe that, first, the system exhibits an oscillatory behavior in open-loop (before 10s). Then, when the controller is switched on, the systems exponentially converges to its equilibrium, as expected from Theorem 1. Better performance could be obtained by tuning the feedback gain. This along with comparison with other controllers will be the focus of future works.

5. CONCLUSION

We have presented a predictor-based control design for system with state-dependent state delay and constant input delay. The
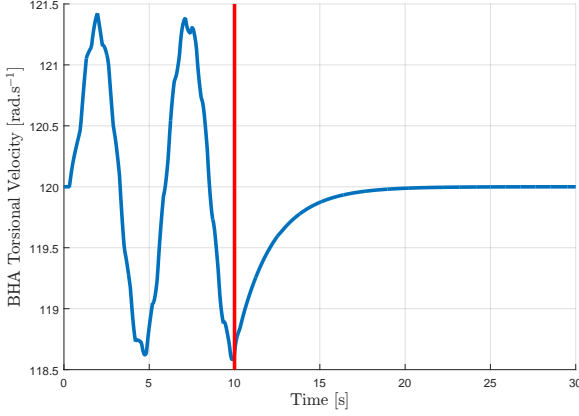
Fig. 5. Torsional velocity of the BHA. The red line corresponds to the time instant when the controller is switched on.

proposed controller effectively stabilizes models of coupled axial and torsional mechanical vibrations of drilling systems.

The controller uses measurements of the "downhole" states (i.e. uncollocated), which is unrealistic in practice and fixing this will be the topic of future works. Similarly, extending results from the recent literature, the authors believe it is possible to handle multiple input with multiple delays with the same methods employed here.

Appendix A. PROOF OF LEMMA 1

We start the proof by noticing $p(x,t) = P^t(t + D_2(x-1))$, that $\chi(x,y,t) = P^t(t + D_2(x-1) + D_1(t)(y-1))$ and that $\bar{\chi}(x,y,t) = P^t(t + D_2(x-1) + \bar{D}(y-1))$ for $(x,y) \in [0,1]^2$. From (25) evaluated for $x = 1$ and $x = 0$, one thus directly gets that $w(1,t) = 0$ and that $u(0,t) = w(0,t) + \kappa_0(\zeta(\cdot,t))$. For the remaining of this proof, we define

$$q(x,t) = D_2 \partial_t p(x,t) - \partial_x p(x,t) \tag{A.1}$$
$$r(x,y,t) = D_2 \partial_t \chi(x,y,t) - \partial_x \chi(x,y,t) \tag{A.2}$$
$$\bar{r}(x,y,t) = D_2 \partial_t \bar{\chi}(x,y,t) - \partial_x \bar{\chi}(x,y,t) \tag{A.3}$$

Now, taking space- and time-derivatives of (22), one gets

$$\partial_t p(x,t) = e^{A_0 D_2 x}[A_0 X(t) + A_1 D_1(t) \int_0^1 \zeta(x,t)dx + Bu(0,t)]$$
$$+ D_2 \int_0^x e^{A_0 D_2(x-y)} \left[A_1 \frac{dD_1^0}{d\bar{\chi}} \cdot \partial_t \bar{\chi}(y,\cdot,t) \int_0^x \chi(y,\xi,t)d\xi \right.$$
$$\left. + A_1 D_1^0(\bar{\chi}(y,\cdot,t) \int_0^x \partial_t \chi(y,\xi,t) + B\partial_t u(y,t) \right] dy \quad (A.4)$$

and

$$\partial_x p(x,t) = A_0 D_2 e^{A_0 D_2 x} X(t) + D_2 A_1 D_1^0(\bar{\chi}(x,\cdot,t)) \int_0^1 \chi(x,\xi,t)d\xi$$
$$+ D_2 Bu(x,t) + D_2 \int_0^x A_0 D_2 e^{A_0 D_2(x-y)} \left[Bu(y,t) \right.$$
$$\left. + A_1 D_1^0(\bar{\chi}(y,\cdot,t)) \int_0^1 \chi(y,\xi,t)d\xi \right] dy \quad (A.5)$$

$$= D_2 e^{A_0 D_2 x} \left[A_0 X(t) + A_1 D_1^0(\bar{\chi}(0,\cdot,t)) \int_0^1 \chi(0,\xi,t)d\xi \right.$$
$$\left. + Bu(0,t) \right] + D_2 \int_0^x e^{A_0 D_2(x-y)} \left[B\partial_x u(y,t) \right.$$
$$\left. + A_1 \frac{dD_1^0}{d\bar{\chi}} \cdot \partial_x \bar{\chi}(y,\cdot,t) \int_0^1 \chi(y,\xi,t)d\xi \right.$$

$$+ A_1 D_1^0(\bar{\chi}(y,\cdot,t) \int_0^1 \partial_x \chi(y,\xi,t)d\xi \right] dy \tag{A.6}$$

in which we used an integration by parts. Observing that $\chi(0,\xi,t) = \zeta(\xi,t)$, that $\bar{\chi}(0,\cdot,t) = \bar{\zeta}(\cdot,t)$ and thus that $D_1^0(\bar{\chi}(0,\cdot,t) = D_1(t)$, one obtains

$$q(x,t) = \int_0^x [\varphi_1(x,y,t,\chi(y,\cdot,t)) \cdot \bar{r}(y,\cdot,t) + \varphi_2(x,y,t) \cdot \bar{r}(y,\cdot,t)]dy \tag{A.7}$$

in which φ_1 and φ_2 are given in Appendix A. Now, taking time- and space-derivatives of (24) and using the dynamics of $\bar{\zeta}$, one gets

$$\bar{r}(x,y,t) = \begin{cases} 0 & \text{if } xD_2 + \bar{D}(y-1) \le 0 \\ q\left(x + \frac{\bar{D}}{D_2}(y-1),t\right) & \text{if } xD_2 + \bar{D}(y-1) \ge 0 \end{cases} \tag{A.8}$$

Similarly, taking time- and space-derivatives of (24), using (A.7) and the fact that $D_1(X_t) \le \bar{D}$, it follows that

$$r(x,y,t) = \tag{A.9}$$
$$\begin{cases} 0 & \text{if } xD_2 + D_1^0(\bar{\chi}(x,\cdot,t))(y-1) \le 0 \\ q\left(x + \frac{D_1^0(\bar{\chi}(x,\cdot,t))}{D_2}(y-1),t\right) & \\ \quad - (y-1)\frac{dD_1^0}{d\bar{\chi}} \cdot r(x,\cdot,t) \cdot \partial_x p\left(x + \frac{D_1^0(\bar{\chi}(x,\cdot,t))}{D_2}(y-1),t\right) & \\ & \text{if } xD_2 + D_1^0(\bar{\chi}(x,\cdot,t))(y-1) \ge 0 \end{cases}$$

Plugging together (A.7),(A.8) and (A.9), one gets

$$\begin{pmatrix} r(x,y,t) \\ \bar{r}(x,y,t) \end{pmatrix} = \tag{A.10}$$
$$\begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \\ \quad \text{if } xD_2 + D_1^0(\bar{\chi}(x,\cdot,t))(y-1) \le 0 & \\ \begin{pmatrix} \int_0^{x + \frac{D_1^0(\bar{\chi}(x,\cdot,t))}{D_2}(y-1)} \varphi_3(x,y,\xi_1,t) \cdot r(\xi_1,\cdot,t)d\xi_1 \\ 0 \end{pmatrix} & \\ \quad \text{if } D_1^0(\bar{\chi}(x,\cdot,t))(1-y) \le xD_2 \le \bar{D}(1-y) & \\ \begin{pmatrix} \int_0^{x + \frac{D_1^0(\bar{\chi}(x,\cdot,t))}{D_2}(y-1)} [\varphi_1(\varphi_0(x,y,t),\xi,t,\chi(\xi,\cdot,t)) \cdot \bar{r}(\xi,\cdot,t) \\ \quad + \varphi_2(\varphi_0(x,y,t),\xi,t) \cdot r(\xi,\cdot,t)]d\xi + \\ \varphi_3(x,y,t,u([0,\varphi_0(x,y,t)],t),\chi([0,\varphi_0(x,y,t)],t)) \cdot \bar{r}(x,\cdot,t) \\ \int_0^{x + \frac{\bar{D}}{D_2}(y-1)} [\varphi_1(\bar{\varphi}_0(x,y),\xi,t,\chi(\xi,\cdot,t)) \cdot \bar{r}(\xi,\cdot,t) \\ \quad + \varphi_2(\bar{\varphi}_0(x,y),\xi,t) \cdot r(\xi,\cdot,t)]d\xi \end{pmatrix} & \\ \quad \text{if } xD_2 + \bar{D}(y-1) \ge 0 \end{cases}$$

in which $\varphi_0, \bar{\varphi}_0, \varphi_1, \varphi_2$ and φ_3 are given in Appendix A. Define

$$\forall t \ge 0, \forall x \in [0,1], \forall s \in [-\bar{D}/D_2, 0] \tag{A.11}$$
$$r_x(s,\cdot,t) = \begin{cases} r(s+x,\cdot,t) & \text{if } s+x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

and similarly $\bar{r}_x$, $\bar{\chi}_x$ and $\bar{r}_x$. Then, taking a space-derivative of (A.10) and using the smoothness properties of the φ_i functionals gathered in Appendix A, one can obtain the existence of continuously differentiable functionals $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that

$$\begin{cases} \partial_x \bar{r}(x,\cdot,t) = \\ \quad \alpha_1(x,\cdot,t,\chi_x(\circ,\diamond,t)) \cdot \bar{r}_x(\circ,\diamond,t) + \alpha_2(x,\cdot,t) \cdot r_x(\circ,\diamond,t) \\ \bar{r}_0(\circ,\diamond,t) = 0 \end{cases}$$

$$(A.12)$$

$$\begin{cases} \partial_x r(x,\cdot,t) = \alpha_3(x,\cdot,t,\chi_x(\circ,\diamond,t),u(\circ,\diamond,t),\partial_x\chi_x(\circ,\diamond,t), \\ \quad \partial_x u_x(\circ,\diamond,t)) \cdot \bar{r}_{x,1}(\cdot,\diamond,t) + \alpha_4(x,\cdot,t) \cdot r_x(\circ,\diamond,t) \\ \bar{r}_0(\cdot,\diamond,t) = 0 \end{cases}$$

As the dynamics (1) is linear, its solution does not escape in finite time and is infinitely continuously differentiable. Thus, from the smoothness properties of the functional gathered in Appendix A, it follows that the solution of these delayed differential equations is unique. One thus easily obtains that $r(x,y,t) = 0$ and $\bar{r}(x,y,t) = 0$ for all $(x,y,t) \in [0,1]^2 \times \mathbb{R}_+$. Consequently, it follows that

$$D_2\partial_t w(x,t) - \partial_x w(x,t) = D_2\partial_t u(x,t) - \partial_x u(x,t) + \frac{d\kappa^0}{d\chi} \cdot r(x,\cdot,t)$$

$$= 0 \qquad (A.13)$$

which completes the proof.

Appendix B. EXPRESSIONS AND SMOOTHNESS PROPERTIES OF INTERMEDIATE FUNCTIONS AND FUNCTIONALS

The functionals φ_1 and φ_2 introduced in (A.7) are defined as, for $\psi \in \mathscr{C}([0,1],\mathbb{R}^n)$,

$$\varphi_1(x,y,t,\chi(y,\cdot,t)) \cdot \psi =$$
$$D_2 e^{A_0 D_2(x-y)} A_1 \int_0^1 \chi(y,\xi,t) d\xi \frac{dD_1^0}{d\bar{\chi}}(\bar{\chi}(y,\cdot,t)) \cdot \psi$$
$$\varphi_2(x,y,t) \cdot \psi =$$
$$D_2 e^{A_0 D_2(x-y)} A_1 D_1^0(\bar{\chi}(y,\cdot,t)) \int_0^1 \psi(\xi) d\xi$$

The functions φ_0, $\bar{\varphi}_0$ and the functional φ_3 introduced in (A.10) are defined as, for $\psi \in \mathscr{C}([0,1],\mathbb{R}^n)$,

$$\varphi_0(x,y,t) = x + \frac{D_1^0(\bar{\chi}(x,\cdot,t))}{D_2}(y-1)$$

$$\bar{\varphi}_0(x,y) = x + \frac{\bar{D}}{D_2}(y-1)$$

$$\varphi_3(x,y,t,u([0,\varphi_0(x,y,t)],t),\chi([0,\varphi_0(x,y,t)],t)) \cdot \psi =$$
$$-(y-1)\partial_x p(\varphi_0(x,y,t),t) \frac{dD_1^0}{d\bar{\chi}}(\bar{\chi}(x,\cdot,t)) \cdot \psi$$

in which $\partial_x p(x,t)$ is a function which depends on x, t, $\chi([0,x],\cdot,t)$ and $u([0,x],\cdot,t)$ according to (A.5).

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