Adaptive Tracking Controller for Systems With Unknown Long Delay and Unknown Parameters in the Plant

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Abstract— In a recent paper we presented the first adaptive control design for an ODE system with a possibly large actuator delay of unknown length. We achieved global stability under full state feedback. In this paper we generalize the design to the situation where, besides the unknown delay value, the ODE also has unknown parameters, and where trajectory tracking (rather than equilibrium regulation) is pursued.

I. INTRODUCTION

Until recently, the only results on adaptive control of systems with actuator delays dealt only with uncertain parameters in the ODE part of the system [4], [16], [17] but not with uncertainty in the delay value itself. The importance of designing adaptive controllers for unknown delay was recognized in [3], [9], however only approximation-based ideas for limited classes of plants were dealt with.

In a recent paper [2] we presented the first results on delay-adaptive control for a general class of plants, under full state feedback. This result is global—including being global in the initial delay value estimate (one can arbitrarily underestimate or overestimate the delay value initially, and still achieve stabilization adaptively).

However, the result in [2] contains two limitations, one made for pedagogical reasons and the other which is fundamental. The limitation for which the reason is pedagogical is in assuming that there are no unknown parameters in the ODE. This limitation is being removed by this paper. This limitation was imposed in [2] to prevent the novel ideas on how to develop global adaptivity in the infinite-dimensional (delay) context from being buried under standard but nevertheless complicated details of ODE adaptive control.

The other limitation in [2], which is fundamental, is in assuming that the full actuator state is measured, though the delay value is completely unknown. The physical meaning of this is that the actuator delay is modeled as a transport process, to which the control designer has physical access for measurement but the speed of propagation of this transport process is completely unknown. As we explain in [2], the problem where the actuator state is not measurable and the delay value is unknown is not solvable globally, since the problem is not linearly parametrized. We show in [2] how one can solve it locally, however, this is not a very satisfactory result, since it is local both in the initial state and in the initial parameter error. In other words, the initial delay estimate needs to be sufficiently close to the true delay. (The delay can be long, but it needs to be known quite closely.) Under such an assumption, one might as well use a linear controller and rely on robustness of the feedback law to small errors in the assumed delay value.

So, for this reason, in this paper we continue with a full-state feedback design, specifically assuming the measurement of the actuator state. In [2] we discussed all the possible problems that one can consider with respect to the availability or unavailability of measurement of the ODE state, the actuator state, the knowledge (or lack of knowledge) of the delay value, and the knowledge (or lack thereof) of the ODE parameters. There is a total number of 14 distinct problem combinations. Here we focus on the most interesting one of them, with both the ODE parameters and the delay value unknown, but with full state measurement. An extension to the case where only an output (and not the complete state) of the ODE is available for measurement is easy (with the method of backstepping and Kreisselmeier observers). We don't pursue it here for consistency of concepts—since we must measure the actuator state, we might as well present a result with full measurement of the ODE state.

As in most of the research on control of unstable plants with a long actuator delay [1], [4], [5], [6], [7], [8], [11], [13], [14], [15], [16], [18], [19], [20], [21], the essence of our approach is "predictor feedback," which we recently showed in [8], [11] to be a form of backstepping boundary control for PDEs [12] and extended to nonlinear plants [7].

In this paper we generalize the design from [2] in two major ways: we extend it to ODEs with unknown parameters and extend it from equilibrium regulation to trajectory tracking. A significant number of new technical issues arise in this problem. The estimation error of the ODE parameters appears in the error models of both the ODE and of the infinite-dimensional (delay) subsystem, which is reflected also in the update law. The update law has to also deal appropriately with ensuring stabilizability with the parameter estimates, for which projection is employed. Finally, our approach for dealing with delay adaptation involves normalized Lyapunov-based tuning, a rather non-standard approach as compared to finite-dimensional adaptive control. In this framework, we need to bound numerous terms involving parameter adaptation rates (both for the delay and for the ODE parameters) in the Lyapunov analysis. Some of these terms are vanishing (when the tracking error is zero), while the others (which are due to the reference trajectory) are non-vanishing. These terms receive different treatment though both are bounded by normalization and their size is controlled with the adaptation gain.

We begin in Section II by defining the problem and present the adaptive control design and the main stability theorem in Section III. Simulations results are shown in Section IV, followed by the stability proof in Section V.

II. PROBLEM FORMULATION

We consider the following system

$$\dot{X}(t) = A(\theta)X(t) + B(\theta)U(t-D) \tag{1}$$
$$Y(t) = CX(t), \tag{2}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, $D > 0$ is an unknown constant delay, the system matrix $A(\theta)$ and the input vector $B(\theta)$ in (1) are linearly parametrized, i.e.,

$$A(\theta) = A_0 + \sum_{i=1}^{p} \theta_i A_i \tag{3}$$

$$B(\theta) = B_0 + \sum_{i=1}^{p} \theta_i B_i, \tag{4}$$

and θ is an unknown but constant parameter vector that belongs to the following convex set

$$\Pi = \{\theta \in \mathbb{R}^p | \mathscr{P}(\theta) \le 0\}, \tag{5}$$

where, by assuming that the convex function $\mathscr{P} : \mathbb{R}^p \to \mathbb{R}$ is smooth, we assure that the boundary $\partial\Pi$ of Π is smooth.

Assumption 1: The set Π is bounded and known. A constant $\bar{D}$ is known such that $D \in]0; \bar{D}]$.

Assumption 2: The pair $(A(\theta), B(\theta))$ is completely controllable for each θ and there exists a triple of vector/matrix-valued functions $(K(\theta), P(\theta), Q(\theta))$ such that $(K, P) \in C^1(\Pi)^2$, $Q \in C^0(\Pi)$, the matrices $P(\theta)$ and $Q(\theta)$ are positive definite and symmetric, and the following Lyapunov equation is satisfied for all $\theta \in \Pi$:

$$P(\theta)(A+BK)(\theta) + (A+BK)(\theta)^T P(\theta) = -Q(\theta). \tag{6}$$

Example 1: Consider the potentially unstable plant

$$\dot{X}_1(t) = \theta X_1(t) + X_2(t) \tag{7}$$
$$\dot{X}_2(t) = U(t-D) \tag{8}$$
$$Y(t) = X_1(t), \tag{9}$$

where we assume $\Pi = [-\underline{\theta}; \bar{\theta}]$ and define

$$A(\theta) = A_0 + \theta A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \theta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{10}$$

$$B = B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{11}$$

Using the backstepping method we construct (K, P, Q) as

$$K(\theta) = -\begin{pmatrix} 1+(\theta+1)^2 & \theta+2 \end{pmatrix} \tag{12}$$

$$P(\theta) = \frac{1}{2}Q(\theta) = \begin{pmatrix} 1+(1+\theta)^2 & 1+\theta \\ 1+\theta & 1 \end{pmatrix}, \tag{13}$$

which satisfies the Lyapunov equation (6).

Assumption 3: The quantities

$$\underline{\lambda} = \inf_{\theta \in \Pi} \min\{\lambda_{\min}(P(\theta)), \lambda_{\min}(Q(\theta))\} \tag{14}$$

$$\bar{\lambda} = \inf_{\theta \in \Pi} \lambda_{\max}(P(\theta)). \tag{15}$$

exist and are known.

Example 2: (Example 1 continued) One can show that

$$\lambda_{\max}(P(\theta)) = \frac{2+(\theta+1)^2+|\theta+1|\sqrt{(\theta+1)^2+1}}{2} \tag{16}$$

$$\lambda_{\min}(P(\theta)) = \frac{1}{\lambda_{\max}P(\theta)}, \tag{17}$$

from which $\underline{\lambda}$ and $\bar{\lambda}$ area readily obtained over the set $\Pi = [-\underline{\theta}; \bar{\theta}]$.

Assumption 4: For a given smooth function $Y^r(t)$, there exist known functions $X^r(t, \theta)$ and $U^r(t, \theta)$, which are bounded in t and continuously differentiable in the unknown argument θ on Π, and which satisfy

$$\dot{X}^r(t, \theta) = A(\theta)X^r(t, \theta) + B(\theta)U^r(t, \theta) \tag{18}$$
$$Y^r(t) = CX^r(t). \tag{19}$$

Example 3: (Example 2 continued) Take $Y^r(t) = \sin(t)$. Then, the reference trajectory pair for the state and input is

$$X^r(t, \theta) = \begin{pmatrix} \sin(t) \\ \cos(t) - \theta\sin(t) \end{pmatrix} \tag{20}$$

$$U^r(t, \theta) = -\sin(t+D) - \theta\cos(t+D), \tag{21}$$

bounded in t and continuously differentiable in θ.

III. CONTROL DESIGN

We first represent the plant as

$$\dot{X}(t) = A(\theta)X(t) + B(\theta)u(0,t) \tag{22}$$
$$Y(t) = CX(t) \tag{23}$$
$$Du_t(x,t) = u_x(x,t) \tag{24}$$
$$u(1,t) = U(t), \tag{25}$$

where the delay is represented as a transport PDE and $u(x,t) = U(t+D(x-1))$. We consider reference trajectories $X^r(t)$ and $U^r(t)$, such as described in Assumption 4. Let us introduce the following error variables

$$\tilde{X}(t) = X(t) - X^r(t, \hat{\theta}) \tag{26}$$
$$\tilde{U}(t) = U(t) - U^r(t, \hat{\theta}) \tag{27}$$
$$e(x,t) = u(x,t) - u^r(x,t,\hat{\theta}), \tag{28}$$

with an estimate $\hat{\theta}$ of the unknown θ. When D and θ are known, one can show that the control law

$$U(t) = U^r(t) - KX^r(t+D) \\ + K\left[e^{AD}X(t) + D\int_0^1 e^{AD(1-y)}Bu(y,t)dy\right] \tag{29}$$

achieves exponential stability of the equilibrium $(\tilde{X}, e) = 0$, compensating the effects of the delay D.

When D and θ are unknown, we employ the control law

$$U(t) = U^r(t, \hat{\theta}) - K(\hat{\theta})X^r(t+\hat{D}, \hat{\theta}) + K(\hat{\theta})\left[e^{A(\hat{\theta})\hat{D}(t)} \\ \times X(t) + \hat{D}(t)\int_0^1 e^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})u(y,t)dy\right] \tag{30}$$

based on the certainty equivalence principle. The update laws for the estimates $\hat{D}$ and $\hat{\theta}$ are chosen based on the Lyapunov analysis (presented in Section V) as

$$\dot{\hat{D}}(t) = \gamma_1 \text{Proj}_{[0,\bar{D}]}\{\tau_D(t)\} \tag{31}$$

$$\dot{\hat{\theta}}(t) = \gamma_2 \text{Proj}_\Pi\{\tau_\theta(t)\}, \tag{32}$$

with adaptation gains γ_1 and γ_2 chosen as positive and

$$\tau_D(t) = -\frac{\int_0^1 (1+x)w(x,t)K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}dx}{1+\tilde{X}(t)^T P(\hat{\theta})\tilde{X}(t)+b\int_0^1 (1+x)w(x,t)^2 dx} \times \left((A+BK)(\hat{\theta})\tilde{X}(t)+B(\hat{\theta})w(0,t)\right) \quad (33)$$

$$\tau_\theta(t) = \frac{\frac{2\tilde{X}(t)^T P(\hat{\theta})}{b} - \int_0^1 (1+x)w(x,t)K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}dx}{1+\tilde{X}(t)^T P(\hat{\theta})\tilde{X}(t)+b\int_0^1 (1+x)w(x,t)^2 dx} \times (A_i X(t)+B_i u(0,t))_{1\leq i\leq p} . \quad (34)$$

The matrix P is defined in Assumption 2, the standard projector operators are given by

$$\text{Proj}_{[0,\bar{D}]}\{\tau_D\} = \tau_D \begin{cases} 0, & \hat{D}=0 \text{ and } \tau_D < 0 \\ 0, & \hat{D}=\bar{D} \text{ and } \tau_D > 0 \\ 1, & \text{else} \end{cases} \quad (35)$$

$$\text{Proj}_\Pi\{\tau_\theta\} = \tau_\theta \begin{cases} 1, & \hat{\theta}\in\mathring{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^T\tau \leq 0 \\ \\ I - \frac{\nabla_{\hat{\theta}}\mathscr{P}\nabla_\theta\mathscr{P}^T}{\nabla_{\hat{\theta}}\mathscr{P}^T\nabla_\theta\mathscr{P}}, & \hat{\theta}\in\partial\Pi \text{ and } \nabla_{\hat{\theta}}\mathscr{P}^T\tau > 0. \end{cases} \quad (36)$$

The transformed state of the actuator is

$$w(x,t) = e(x,t) - \hat{D}(t)\int_0^x K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta}) \times e(y,t)dy - K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}\tilde{X}(t) \quad (37)$$

and the constant b is chosen such as

$$b \geq 4\sup_{\hat{\theta}\in\Pi}|PB|^2(\hat{\theta})\frac{\bar{D}}{\underline{\lambda}}. \quad (38)$$

Theorem 1: Let Assumptions 1–4 hold and consider the closed-loop system consisting of (22)–(25), the control law (30) and the update laws defined by (31)–(38). There exists $\gamma^* > 0$ such that for any $\gamma \in [0,\gamma^*[$, there exist positive constants R and ρ (independent of the initial conditions) such that, for all initial conditions satisfying $(X^0, u^0, \hat{D}^0, \theta^0) \in \mathbb{R}^n \times L_2(0,1)\times]0,\bar{D}] \times \Pi$, the following holds:

$$\Upsilon(t) \leq R\left(e^{\rho\Upsilon(0)}-1\right), \quad \forall t\geq 0, \quad (39)$$

where

$$\Upsilon(t) = |\tilde{X}(t)|^2 + \int_0^1 e(x,t)^2 dx + \tilde{D}(t)^2 + \tilde{\theta}(t)^T\tilde{\theta}(t). \quad (40)$$

Furthermore, asymptotic tracking is achieved, i.e.,

$$\lim_{t\to\infty}\tilde{X}(t) = 0, \quad \lim_{t\to\infty}\tilde{U}(t) = 0. \quad (41)$$

IV. SIMULATIONS

We return to the system from Examples 1–3. We focus on the issues arising from the large uncertainties in D and θ and from the tracking problem with the reference trajectory (20)–(21). We take $D=1$, $\theta=0.5$, $\bar{D}=2D=2$, $\underline{\theta}=0$, $\bar{\theta}=2\theta=1$. We pick the adaptations gains as $\gamma_1=10$, $\gamma_2=2.3$ and the normalization coefficient as $b=\frac{4|PB|^2\bar{D}}{\underline{\lambda}}=3200$. We show simulation results for $X_1(0)=X_2(0)=0.5$, $\hat{\theta}(0)=0$, and two different values of $\hat{D}(0)$. In Figures 1 and 2, the tracking of $X^r(t)$ is achieved for both simulations, as Theorem 1 predicts. In 2 we observe that $\hat{\theta}(t)$ converges to the true θ, whereas this is not the case with $\hat{D}(t)$. This is consistent
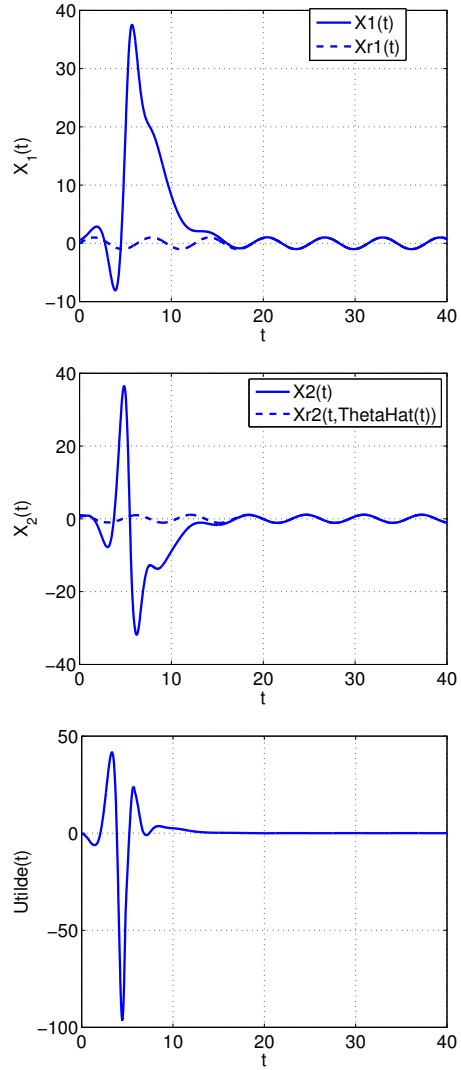

Fig. 1. The system respons of the system (1)–(2) with the reference trajectory (20)–(21) for $D=1$, $\theta=0.5$, $\hat{\theta}(0)=0$ and $\hat{D}(0)=0$.

with the theory. By examining the error systems (43), (44), with the help of persistency of excitation, we could infer the convergence of $\hat{\theta}(t)$ but not of $\hat{D}(t)$.

V. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 1. We start by considering the transformation (37) along with its inverse

$$e(x,t) = w(x,t) + \hat{D}(t)\int_0^x K(\hat{\theta})e^{(A+BK)(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta}) \times w(y,t)dy + K(\hat{\theta})e^{(A+BK)(\hat{\theta})\hat{D}(t)x}\tilde{X}(t). \quad (42)$$

Using these transformations and the models (1) and (18), the transformed system is written as

$$\dot{\tilde{X}}(t) = (A+BK)(\hat{\theta})\tilde{X}(t)+B(\hat{\theta})w(0,t)+A(\tilde{\theta})X(t) + B(\tilde{\theta})u(0,t)-\frac{\partial X^r}{\partial\hat{\theta}}(t,\hat{\theta})\dot{\hat{\theta}}(t) \quad (43)$$

$$Dw_t(x,t) = w_x(x,t)-\tilde{D}(t)p_0(x,t)-D\dot{\hat{D}}(t)q_0(x,t) - D\tilde{\theta}(t)^T p(x,t)-D\dot{\hat{\theta}}(t)^T q(x,t) \quad (44)$$
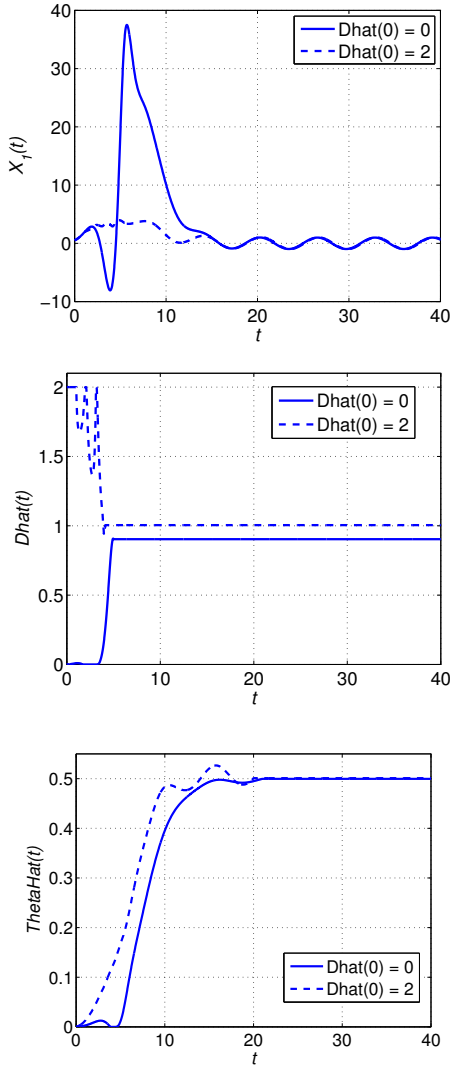
Fig. 2. The system response of the system (1)–(2) with the reference trajectory (20)–(21) for $D = 1$, $\theta = 0.5$, $\hat{\theta}(0) = 0$ and two dramatically different initial conditions for $\hat{D}$: $\hat{D}(0) = 0$ and $\hat{D}(0) = \bar{D} = 2D = 2$. Note that the solid plots in this figure correspond to the same simulation in Figure 1.

$$w(1,t) = 0, \tag{45}$$

where $\tilde{D}(t) = D - \hat{D}(t)$ is the estimation error of the delay, the quantities $A(\tilde{\theta}) = \sum_{i=1}^{p} \tilde{\theta}_i A_i = \sum_{i=1}^{p}(\theta_i - \hat{\theta}_i(t))A_i, B(\tilde{\theta}) = \sum_{i=1}^{p} \tilde{\theta}_i B_i$ are linear in the parameter error $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$,

$$
\begin{aligned}
p_0(x,t) &= K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}((A+BK)(\hat{\theta})\tilde{X}(t) \\
&\quad + B(\hat{\theta})w(0,t)) \tag{46}
\end{aligned}
$$

$$
\begin{aligned}
q_0(x,t) &= \int_0^x K(\hat{\theta})\left(I + A(\hat{\theta})\hat{D}(t)(x-y)\right)e^{A(\hat{\theta})\hat{D}(t)(x-y)} \\
&\quad \times B(\hat{\theta})e(y,t)dy + KA(\hat{\theta})xe^{A(\hat{\theta})\hat{D}(t)x}\tilde{X}(t) \tag{47}
\end{aligned}
$$

and the vector-valued functions $q(x,t)$ and $p(x,t)$ are defined through their coefficients as follows, for $1 \le i \le p$,

$$
\begin{aligned}
p_i(x,t) &= K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}(A_i X(t) + B_i u(0,t)) \tag{48} \\
&= K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}((A_i + B_i K(\hat{\theta}))\tilde{X}(t) + B_i w(0,t) \\
&\quad + A_i X^r(t,\hat{\theta}) + B_i u^r(0,t,\hat{\theta})) \tag{49}
\end{aligned}
$$

$$
\begin{aligned}
q_i(x,t) &= \hat{D}(t)\int_0^x \left(\left[\frac{\partial K}{\partial \hat{\theta}_i}(\hat{\theta}) + K(\hat{\theta})A_i\hat{D}(t)(x-y)\right] \right. \\
&\quad \times e^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta}) + K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)(x-y)}B_i\Big) \\
&\quad \times e(y,t)dy + \left(\frac{\partial K}{\partial \hat{\theta}_i} + K(\hat{\theta})A_i\hat{D}(t)x\right) \\
&\quad \times e^{A(\hat{\theta})\hat{D}(t)x}\tilde{X}(t) - K(\hat{\theta})e^{A(\hat{\theta})\hat{D}(t)x}\frac{\partial X^r}{\partial \hat{\theta}_i}(t,\hat{\theta}) \\
&\quad + \frac{\partial u^r}{\partial \hat{\theta}_i}(x,t,\hat{\theta}) - \hat{D}(t)\int_0^x K(\hat{\theta}) \\
&\quad \times e^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta})\frac{\partial u^r}{\partial \hat{\theta}_i}(y,t,\hat{\theta})dy \tag{50}
\end{aligned}
$$

Now, we define the following Lyapunov function

$$V(t) = D\log(N(t)) + \frac{b}{\gamma_1}\tilde{D}(t)^2 + \frac{bD}{\gamma_2}\tilde{\theta}(t)^T\tilde{\theta}(t), \tag{51}$$

where

$$N(t) = 1 + \tilde{X}(t)^T P(\hat{\theta})\tilde{X}(t) + b\int_0^1 (1+x)w(x,t)^2 dx. \tag{52}$$

Taking a time derivative of $V(t)$, we obtain

$$
\begin{aligned}
\dot{V}(t) &= -\frac{2b}{\gamma_1}\tilde{D}(t)(\dot{\hat{D}}(t) - \gamma_1\tau_D(t)) \\
&\quad -\frac{2bD}{\gamma_2}\tilde{\theta}(t)^T(\dot{\hat{\theta}}(t) - \gamma_2\tau_\theta(t)) \\
&\quad +\frac{D}{N(t)}\left(\sum_{i=1}^{p}\dot{\hat{\theta}}_i(t)\left(\tilde{X}(t)^T\frac{\partial P}{\partial \hat{\theta}_i}(\hat{\theta})\tilde{X}(t)\right.\right. \\
&\quad \left. -\tilde{X}(t)^T P(\hat{\theta})\frac{\partial X^r}{\partial \hat{\theta}_i}(t,\hat{\theta})\right) - \tilde{X}(t)^T Q(\hat{\theta})\tilde{X}(t) \\
&\quad +2\tilde{X}(t)^T PB(\hat{\theta})w(0,t) - \frac{b}{D}\|w\|^2 - \frac{b}{D}w(0,t)^2 \\
&\quad -2b\dot{\hat{D}}(t)\int_0^1(1+x)w(x,t)q_0(x,t)dx \\
&\quad \left. -2b\dot{\hat{\theta}}(t)^T\int_0^1(1+x)w(x,t)q(x,t)dx\right), \tag{53}
\end{aligned}
$$

where we have used an integration by parts. Using the assumptions that $\hat{D}(0) \in]0;\bar{D}]$ and $\hat{\theta}(0) \in \Pi$, the update laws (31)–(32) with the properties of the projection operator, while substituting the expressions of (31)–(32) and using (38) with the Young inequality, we obtain

$$
\begin{aligned}
\dot{V}(t) &\le -\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|\tilde{X}|^2 + \frac{b}{D}\|w\|^2 + 2\frac{b}{D}w(0,t)^2\right) \\
&\quad +2Db\gamma_1\frac{\int_0^1(1+x)|w(x,t)||p_0(x,t)|dx}{N(t)} \\
&\quad \times \frac{\int_0^1(1+x)|w(x)||q_0(x,t)|dx}{N(t)} \\
&\quad +D\gamma_2\sum_{i=1}^{p}\left(\frac{\int_0^1(1+x)|w(x,t)||p_i(x,t)|dx}{N(t)}\right. \\
&\quad \left. +\frac{2|\tilde{X}(t)^T P(\hat{\theta})/b||A_i X(t) + B_i u(0,t)|}{N(t)}\right)\frac{1}{N(t)} \\
&\quad \times \left(\left|\tilde{X}(t)^T\frac{\partial P}{\partial \hat{\theta}_i}(\hat{\theta})\tilde{X}(t)\right| + \left|\tilde{X}(t)^T P(\hat{\theta})\frac{\partial X^r}{\partial \hat{\theta}_i}(t,\hat{\theta})\right|\right.
\end{aligned}
$$

$$+2b\int_0^1(1+x)|w(x,t)||q_i(x,t)|dx\Bigg). \tag{54}$$

Fuhermore, each signal depending on $\hat{\theta}$, namely $A,B,K,P,$ $\partial P/\partial\hat{\theta}_i,\partial X^r/\partial\hat{\theta}_i$ and $\partial u^r/\partial\hat{\theta}_i$, is given as continuous in $\hat{\theta}$. Since $\hat{\theta}$ remains in Π, a closed and bounded subset of $\mathbb{R}^p$, each signal is bounded in terms of $\hat{\theta}$ and admits a finite upper bound. We denote $M_A=\sup_{\hat{\theta}\in\Pi}|A(\hat{\theta})|$ and define $M_P,M_B,M_K,M_{A+BK},M_{\partial K/\partial\hat{\theta}}$. Therefore, substituting the expression of $e(x,t)$ in (47) and (50) with the inverse transformation (42), we obtain using Cauchy-Schwartz and Young inequalities, along with (46)–(47) first, (48)–(50) then,

$$\int_0^1(1+x)|w(x,t)||p_0(x,t)|dx$$
$$\leq M_0(|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2) \tag{55}$$

$$\int_0^1(1+x)|w(x,t)||q_0(x,t)|dx\leq M_0(|\tilde{X}(t)|^2+\|w(t)\|^2) \tag{56}$$

$$\left(\int_0^1(1+x)|w(x,t)||p_i(x,t)|dx+2|\tilde{X}(t)^TP(\hat{\theta})/b|\right.$$
$$\left.\times|A_iX(t)+B_iu(0,t)|\right)$$
$$\leq M_i(|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2+\|w(t)\|) \tag{57}$$

$$\int_0^1(1+x)|w(x,t)||q_i(x,t)|dx$$
$$\leq M_i(|\tilde{X}(t)|^2+\|w(t)\|^2+\|w(t)\|), \tag{58}$$

where M_0 and M_i $(1\leq i\leq p)$ are sufficiently large constants given by

$$\begin{aligned}M_0&=&M_K\max\{M_{A+BK}+M_A,2M_K((1+M_A\bar{D})\\&&\times(M_B+M_BM_K(1+\bar{D}))+M_A)\}e^{(M_A+M_{A+BK})\bar{D}}\end{aligned}\tag{59}$$
$$\begin{aligned}M_i&=&\max\{|A_i|+|B_i|M_K+|B_i|+2M_P/b,\\&&2\sup_{(t,\hat{\theta})\in\mathbb{R}\times\Pi}(|A_i||X^r(t,\hat{\theta})|+|B_i||u^r(0,t,\hat{\theta})|),\\&&2M_K\sup_{(t,\hat{\theta})\in\mathbb{R}\times\Pi}\left|\frac{\partial X^r}{\partial\hat{\theta}}(t,\hat{\theta})\right|\\&&+2\sup_{(t,\hat{\theta})\in\mathbb{R}\times\Pi}\left|\frac{\partial u^r}{\partial\hat{\theta}}(t,\hat{\theta})\right|(1+\hat{D}M_KM_B),\\&&\left(\left(M_{\partial K/\partial\hat{\theta}}+M_K|A_i|\bar{D}\right)M_B+|A_i|M_K\right)\\&&\times(2\bar{D}+2\bar{D}M_KM_B+M_K)\}e^{(M_A+M_{A+BK})\bar{D}}.\end{aligned}\tag{60}$$

Consequently, if we define

$$M_P'=\max_{1\leq i\leq p}\sup_{\hat{\theta}\in\Pi}\left|\frac{\partial P(\hat{\theta})}{\partial\hat{\theta}_i}\right| \tag{61}$$

$$M_r=\max_{1\leq i\leq p}\sup_{\hat{\theta}\in\Pi,t\geq0}\left|\frac{\partial X^r(t,\hat{\theta})}{\partial\hat{\theta}_i}\right|, \tag{62}$$

using (55)–(58) in (54), we obtain

$$\dot{V}(t)\leq-\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|\tilde{X}|^2+\frac{b}{D}\|w\|^2+2\frac{b}{D}w(0,t)^2\right)$$
$$+2Db\gamma_1M_0^2\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2}{N(t)}$$

$$\times\frac{|\tilde{X}(t)|^2+\|w(t)\|^2}{N(t)}$$
$$+D\gamma_2\sum_{i=1}^pM_i\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2+\|w(t)\|}{N(t)}$$
$$\times\left(\frac{M_P'|\tilde{X}(t)|^2+M_r|\bar{P}||\tilde{X}(t)|}{N(t)}\right.$$
$$\left.+2bM_i\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+\|w(t)\|}{N(t)}\right). \tag{63}$$

Bounding the cubic and quadric terms with the help of $N(t)$, we arrive at

$$\dot{V}(t)\leq-\frac{D}{2N(t)}\left(\underline{\lambda}|\tilde{X}(t)|^2+\frac{b}{D}\|w(t)\|^2+2\frac{b}{D}w(0,t)^2\right)$$
$$+\frac{2Db\gamma_1M_0^2}{\min\{\underline{\lambda},b\}}\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2}{N(t)}$$
$$+D\gamma_2\sum_{i=1}^pM_i\left(M_P'(\frac{1}{\underline{\lambda}}+\frac{1}{2\min\{1,b\}})\right.$$
$$+M_PM_r(\frac{1}{2}+\frac{1}{2\min\{1,\underline{\lambda}\}})$$
$$\left.+2bM_i(\frac{1}{\min\{\underline{\lambda},b\}}+\frac{1}{2\min\{1,b\}}+1)\right)$$
$$\times\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2}{N(t)}. \tag{64}$$

Defining the following constants,

$$m=\frac{2\max\left\{bM_0^2,\sum_{i=1}^pM_i(M_P'+M_PM_r+3bM_i)\right\}}{\min\{1,\underline{\lambda},b\}} \tag{65}$$
$$\gamma^*=\frac{\min\{\underline{\lambda},b/D\}}{4bm} \tag{66}$$

we finally obtain

$$\dot{V}(t)\leq-\frac{D}{2N(t)}\left(\min\left\{\underline{\lambda},\frac{b}{D}\right\}-2(\gamma_1+\gamma_2)m\right)$$
$$\times(|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2). \tag{67}$$

Consequently, by choosing $(\gamma_1,\gamma_2)\in[0;\gamma^*[^2$, we make $\dot{V}(t)$ negative semidefinite and hence

$$V(t)\leq V(0),\qquad\forall t\geq0. \tag{68}$$

Starting from this result, we now prove the results stated in Theorem 1. From the transformation (37) and its inverse (42), we obtain these two inequalities

$$\|w(t)\|^2\leq r_1\|e(t)\|^2+r_2|\tilde{X}(t)|^2 \tag{69}$$
$$\|e(t)\|^2\leq s_1\|w(t)\|^2+s_2|\tilde{X}(t)|^2, \tag{70}$$

where r_1,r_2,s_1,s_2 are sufficiently large positive constants given by

$$r_1=3\left(1+\bar{D}^2M_K^2e^{2M_{A+BK}\bar{D}}M_B^2\right) \tag{71}$$
$$r_2=3M_K^2e^{2M_{A+BK}\bar{D}} \tag{72}$$
$$s_1=3\left(1+\bar{D}^2M_K^2e^{2M_A\bar{D}}M_B^2\right) \tag{73}$$
$$s_2=3M_K^2e^{2M_A\bar{D}}. \tag{74}$$

Furthermore, from (51) and (70), it follows that

$$\tilde{D}(t)^2 + \tilde{\theta}(t)^T \tilde{\theta}(t) \quad \leq \quad \frac{\gamma_1 + \gamma_2}{b} V(t) \tag{75}$$

$$\|\tilde{X}(t)\|^2 \quad \leq \quad \frac{1}{\underline{\lambda}}(\mathrm{e}^{V(t)/D} - 1) \tag{76}$$

$$\|e(t)\| \quad \leq \quad \frac{s_1}{b}(\mathrm{e}^{V(t)/D} - 1) + s_2\|\tilde{X}(t)\|. \tag{77}$$

Thus, from the definition of $\Upsilon(t)$, it is easy to show that

$$\Upsilon(t) \leq \left(\frac{1+s_2}{\underline{\lambda}} + \frac{s_1}{b} + \frac{(\gamma_1 + \gamma_2)D}{b} \right)(\mathrm{e}^{V(t)/D} - 1) \tag{78}$$

Besides, using (69), we also obtain

$$V(0) \leq \left(D(\bar{\lambda} + s_2 b + 2s_1 b) + b\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \right) \Upsilon(0). \tag{79}$$

Finally, if we define

$$R \quad = \quad \frac{1+s_2}{\underline{\lambda}} + \frac{s_1}{b} + \frac{(\gamma_1 + \gamma_2)D}{b} \tag{80}$$

$$\rho \quad = \quad \bar{\lambda} + s_2 b + 2s_1 b + \frac{b}{D}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right), \tag{81}$$

we obtain the global stability result given in Theorem 1.

We now prove tracking. From (68), we obtain the uniform boudedness of $\|\tilde{X}(t)\|$, $\|w(t)\|$, $\hat{D}(t)$ and $\|\hat{\theta}(t)\|$. From (42), we obtain that $\|e(t)\|$ is also uniformly bounded in time. From (30), we get the uniformly boudedness of $U(t)$ and consequently of $\tilde{U}(t)$ for $t \geq 0$. Thus, we get that $u(0,t)$ and $e(0,t)$ are uniformly bounded for $t \geq D$. Besides, from (32) and (48), we obtain the uniform boundedness of $\|\dot{\hat{\theta}}(t)\|$ for $t \geq D$. Finally, with (43), we obtain that $d\tilde{X}(t)^2/dt$ is uniformly bounded for $t \geq D$. As $|\tilde{X}(t)|$ is square integrable, from (67), we conclude from Barbalat's Lemma that $\tilde{X}(t) \to 0$ when $t \to \infty$.

Besides, from (67), we get the square integrability of $\|w(t)\|$. From (70), we obtain the square integrability of $\|e(t)\|$. Consequently, with (30), we obtain the square integrability of $\tilde{U}(t)$. Furthermore,

$$\frac{d\tilde{U}(t)^2}{dt} = 2\tilde{U}(t)\left(K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\dot{\tilde{X}}(t) + \dot{\hat{D}}(t)G_0(t) \right.$$
$$\left. + \sum_{i=1}^{p} \dot{\hat{\theta}}_i(t)G_i(t) + \frac{\dot{\hat{D}}}{D}H(t) \right) \tag{82}$$

with

$$G_0(t) = K(\hat{\theta})\left[A(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\tilde{X}(t) + \int_0^1 (I + A(\hat{\theta})\hat{D}(t) \right.$$
$$\left. \times (1-y))\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})e(y,t)dy \right] \tag{83}$$

$$G_i(t) = \frac{\partial K}{\partial \hat{\theta}}(\hat{\theta})\left[\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\tilde{X}(t) + \hat{D}(t)\int_0^1 \mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)} \right.$$
$$\left. \times B(\hat{\theta})e(y,t)dy \right] + K(\hat{\theta})\left[A_i\hat{D}(t)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\tilde{X}(t) \right.$$
$$+ \hat{D}(t)\int_0^1 \left[A_i\hat{D}(t)(1-y)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta}) \right.$$
$$\left. \left. + \mathrm{e}^{A(\hat{\theta})\hat{D}(t)}B_i \right] e(y,t)dy \right] \tag{84}$$

$$H(t) = K(\hat{\theta})\left[B(\hat{\theta})\tilde{U}(t) - \mathrm{e}^{A(\hat{\theta})\hat{D}(t)}B(\hat{\theta})e(0,t) \right.$$

$$\left. + \int_0^1 A(\hat{\theta})\hat{D}(t)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})e(y,t)dy \right]. \tag{85}$$

The signals $\dot{\hat{D}}$, $\dot{\hat{\theta}}_1, \ldots, \dot{\hat{\theta}}_p$ are uniformly bounded over $t \geq 0$, according to (31)–(32). By using the uniform boundedness of $\tilde{X}(t), \dot{\tilde{X}}(t), \|e(t)\|, \tilde{U}(t)$ over $t \geq 0$ and of $e(0,t)$ for $t \geq D$ and the uniform boundedness of all the signals which are functions of $\hat{\theta}$ for $t \geq 0$, we obtain the uniform boundedness of $d\tilde{U}(t)^2/dt$ over $t \geq D$. Then, with Barbalat's lemma, we conclude that $\tilde{U}(t) \to 0$ when $t \to \infty$.

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