Adaptive Output-Feedback for Wave PDE with Anti-damping – Application to Surface-based control of Oil Drilling Stick-Slip Instability

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***Abstract*— We develop an adaptive output-feedback controller for a wave PDE in one dimension with actuation and measurement on one boundary and with an unknown anti-damping dynamics on the opposite boundary. This model is representative of drill string torsional instabilities arising in deep oil drilling, for which the model of bottom interaction with the rock is poorly known. The key achievement of the proposed controller is that it requires only the measurements of top-boundary values and not of the entire distributed state of the system. Our approach is based on employing Riemann variables to convert the wave PDE into a cascade of two delay elements, with the first of the two delay elements being fed by control and the same element in turn feeding into a scalar ODE, and to reconstruct the opposite boundary delayed state. This enables us to employ a prediction-based design for systems with output and input delays, suitably converted to the adaptive output-feedback setting. The result's relevance and ability to suppress undesirable torsional vibrations of the drill string in oil well drilling systems is illustrated with simulation example.**

I. INTRODUCTION

For oil and gas exploration and production, wells are drilled with a rotating rock-crushing device, called a bit, driven by a rotatory table at the surface, equipped with an electric motor. The torque applied at the surface is transmitted at the bottom of the borehole through drill string, consisting in a succession of thin tubes. Due to its slenderness, the drill string is subject to various vibration phenomena [8]. One of them is the so-called stick-slip phenomenon, that is torsional vibrations which appear due to the friction of the bit with the rock. In details, due to this nonlinear interaction, the bit slows downs before finally stalling while the rotatory table is still in motion. This generates a torsional wave which propagates back to the surface and then reflects from the rotatory table. The corresponding limit cycle of the (distributed) drill string velocity should be suppressed to avoid damages of the devices.

Even if most of the approaches developed in the literature rely on finite dimensional models [9] [15] [20], the drill string torsional dynamics is more accurately represented by a linear wave equation subject to non-linear boundary conditions accounting for the top-drive and frictional processes [2] [16]. Lately, by neglecting the effect of damping along the structure, this model has been revisited and the bit dynamics recast as a neutral delay differential equation [19].

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In this paper, we follow this overture and propose to study a wave equation subject to linearized anti-damping boundary conditions with unmatched parametric uncertainty, in view of oil drilling application. Indeed, in addition to being nonlinear, the rock-on-the-bit friction term involved in the stick-slip modeling is highly uncertain as it depends among other factors [13] [15] on the nature of the rock which varies with time and operation and is therefore poorly known.

Following [3] and our recent studies [4] [5], we propose to employ modified Riemann variables to reformulate the plant as a linear input-delay model cascaded with a transport equation opposite of the input propagation direction. This frameworks allows then to reconstruct the delayed bottom velocity from top-boundary measurement. Recasting the problem as an output- and input-delay problem, we propose to use infinite-dimensional time-delay tools for control design, namely a prediction-based controller and a tailored Lyapunov methodology for stability analysis.

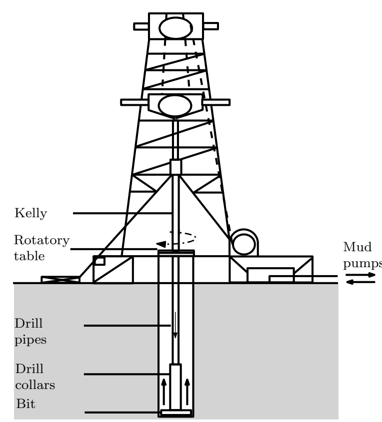


Fig. 1. Schematic view of a drilling system.

Both the controller and the parameter estimators that we design employ *only top-boundary measurements*. This is the main achievement of this paper. It is particularly relevant from an application point of view because, in a drilling system, bottom-hole sensors and actuators have a high risk of collapse due to their location[1]. While [18] also proposes for the same system an output-feedback control law stabilizing the bit velocity and [11] states an adaptive stabilization result for a similar unstable wave PDE with unmatched parametric uncertainty, the present paper is the first result on output-feedback adaptive control providing a mixed $\mathscr{L}_2/\mathscr{L}_{\text{inf}}$-norm stabilization result for the entire distributed state.

The paper is organized as follows. In Section II, we

[1]However, some recent works have investigated the potential of using the so-called weight-on-the-bit at the bottom of the borehole [6] to suppress stick-slip oscillations. Yet, such approachs are not commonly used and rotatory table feedback control is most often employed.

describe the linear dynamics under consideration before presenting our adaptive controller and the main stabilization result of the paper in Section III. Then, before providing its proof in Section V, we apply the proposed approach to surface-based stick-slip suppression in Section IV and illustrate its merits in simulation.

Notations. $|\cdot|$ is the Euclidian norm and $\|u(\cdot)\|$ is the spatial $\mathscr{L}_2$-norm of a signal $u(\cdot,x)$, $x\in[0,1]$, i.e.

$$\|u(t)\| = \sqrt{\int_0^1 u(x,t)^2 dx} \tag{1}$$

For $(a,b)\in\mathbb{R}^2$ such that $a<b$, we define the standard projection operator on the interval $[a,b]$ as a function of two scalar arguments f (denoting the parameter being updated) and g (denoting the nominal update law) as follows:

$$\mathrm{Proj}_{[a,b]}(f,g) = g\begin{cases} 0 \text{ if } f=a \text{ and } g<0 \\ 0 \text{ if } f=b \text{ and } g>0 \\ 1 \text{ otherwise}\end{cases} \tag{2}$$

II. Problem statement

We investigate closed-loop regulation toward a trajectory $u^r(x)=dx+u_0$ ($u_0\in\mathbb{R}$) for the following system

$$\begin{aligned} u_{xx} =& u_{tt} && (3)\\ u_x(1,t) =& U(t) && (4)\\ u_{tt}(0,t) =& aqu_t(0,t)+a[u_x(0,t)-d] && (5)\end{aligned}$$

in which $U(t)$ is the scalar input, (u,u_t) is the system state with $(u(.,0),u_t(.,0))\in H_1([0,1])\times L_2([0,1])$, $a>0$ is a scalar constant parameter and both the anti-damping parameter $q>0$ and the trajectory coefficient $d\in\mathbb{R}$ are unknown. We assume that *only the signal* $u_t(1,\cdot)$ *is measured* for all time and we aim at developing an adaptive control law.

Despite the uncertainties, a key control challenge here is the instability arising from the anti-damping dynamics (5), which acts on the opposite boundary from the one that is controlled. To deal with parameter uncertainties, as always in indirect adaptive control, we formulate an a priori assumption on the parameter values knowledge.

Assumption 1: There exist known constants $\underline{q}$, $\overline{q}$, $\underline{d}$ and $\overline{d}$ such that $\underline{q}<\overline{q}$, $\underline{d}<\overline{d}$, $q\in[\underline{q},\overline{q}]$ and $d\in[\underline{d},\overline{d}]$.

As a first step in our design, we reformulate plant (3)–(5) by introducing the following modified Riemann variables and transformed control variable

$$\begin{aligned} \zeta =& u_t+u_x-\hat{d}(t) && (6)\\ \omega =& u_t-u_x+\hat{d}(t) && (7)\\ W(t) =& U(t)+u_t(1,t)-\hat{d}(t) && (8)\end{aligned}$$

in which $\hat{d}(t)$ is an estimate of d. This yields the following new dynamics, in which we introduce $\tilde{d}(t)=d-\hat{d}(t)$,

$$\begin{aligned} u_{tt}(0,t) =& a(q-1)u_t(0,t)+a[\zeta(0,t)-\tilde{d}(t)] && (9)\\ \zeta_t =& \zeta_x-\dot{\hat{d}}(t) && (10)\\ \zeta(1,t) =& W(t) && (11)\\ \omega_t =& -\omega_x+\dot{\hat{d}}(t) && (12)\\ \omega(0,t) =& 2u_t(0,t)-\zeta(0,t) && (13)\end{aligned}$$

Specifically, in this new framework, the wave equation is represented as the cascade of two opposite transport PDEs with source term (10) and (12) with one ODE (9) being driven by the first PDE and feeding the second one. The boundary condition (13) accounts for the reflection of the wave at $x=0$.

Remark 1: From the transport equations (10) and (12), we have that $\zeta(x,t)=\zeta(y,t+x-y)-\hat{d}(t)+\hat{d}(t+x-y)$ and $\omega(x,t)=\omega(y,t-x+y)+\hat{d}(t)-\hat{d}(t-x+y)$ for any $(x,y)\in[0,1]^2$ and $t\geq 0$.

Remark 2: Note that $\zeta(1,t)$ and $\omega(1,t)$ are known for all time, as $u_t(1,t)$ is measured and $U(t)=u_x(1,t)$ and $\hat{d}(t)$ are known from (6)–(7). Further, following Remark 1 $\zeta(0,t-1)=\zeta(1,t-1)-\hat{d}(t)+\hat{d}(t-1)$ and $\omega(0,t-1)=\omega(1,t)+\hat{d}(t)-\hat{d}(t-1)$. Consequently, $u_t(0,t-1)=\frac{1}{2}(\zeta(0,t-1)+\omega(0,t-1))$ is also known as, replacing with (6)–(7),

$$u_t(0,t-1)=\frac{1}{2}[u_t(1,t)+u_t(1,t-2)-u_x(1,t)+u_x(1,t-2)] \tag{14}$$

Following [10], when $\dot{\hat{d}}(t)=0$, (9)–(11) can be interpreted as an input-delay ODE with one-unit of time delay. This motivates our control design.

III. Control design

Consider the control law

$$\begin{aligned} U(t) =& -u_t(1,t)+\hat{d}(t)-(c_0+\hat{q}(t)-1)\Big(e^{2a(\hat{q}(t)-1)}\mu(t)\\ &+a\int_{t-2}^{t} e^{a(\hat{q}(t)-1)(t-\tau)}[\eta(\tau)-\hat{d}(t)]d\tau\Big) \qquad (15)\end{aligned}$$

in which $c_0>0$ is a given constant, $\hat{q}(t)$ is an estimate of the unknown parameter q and

$$\begin{aligned} \mu(t) =& \frac{1}{2}[u_t(1,t)+u_t(1,t-2)-u_x(1,t)+u_x(1,t-2)] && (16)\\ \eta(t) =& U(t)+u_t(1,t) && (17)\end{aligned}$$

The parameter estimate update laws are chosen as

$$\begin{aligned} \dot{\hat{q}}(t) =& \frac{a\gamma_q}{1+N(t)}\mathrm{Proj}_{[\underline{q},\overline{q}]}\Big\{\hat{q}(t),\mu(t)\Big(\mu(t) && (18)\\ &+b_1(c_0+\hat{q}(t)-1)\int_{t-2}^{t} e^{(a(\hat{q}(t)-1)+1)(\tau-t+2)}w(\tau,t)d\tau\Big)\Big\}\\ \dot{\hat{d}}(t) =& -\frac{a\gamma_d}{1+N(t)}\mathrm{Proj}_{[\underline{d},\overline{d}]}\Big\{\hat{d}(t),\mu(t) && (19)\\ &+b_1(c_0+\hat{q}(t)-1)\int_{t-2}^{t} e^{(a(\hat{q}(t)-1)+1)(\tau-t+2)}w(\tau,t)d\tau\Big\}\end{aligned}$$

$$N(t) = \mu(t)^2 + \frac{b_1}{2}\int_{t-2}^{t} e^{\tau-t+2} w(\tau,t)^2 d\tau$$
$$+ b_2 \int_{t-1}^{t} e^{\tau-t+1}[2\mu(\tau) - \eta(\tau-2) + \hat{d}(t)]^2 d\tau \qquad (20)$$

in which the bounds $\underline{q}, \overline{q}, \underline{d}, \overline{d}$ are defined in Assumption 1, the update gains $\gamma_d, \gamma_q > 0$ and the normalization constants $b_1, b_2 > 0$ are tuning parameters, Proj is the standard projection operator and, for $t \geq 0$ and $t-2 \leq \tau \leq t$,

$$w(\tau,t) = \eta(\tau) - \hat{d}(t) + (c_0 + \hat{q}(t) - 1)\Big(e^{a(\hat{q}(t)-1)(\tau-t+2)}\mu(t)$$
$$+ \int_{t-2}^{\tau} e^{a(\hat{q}(t)-1)(\tau-\xi)}[\eta(\xi) - \hat{d}(t)]ds\Big) \qquad (21)$$

To further understand the meaning of this control strategy, we provide several comments next, before stating our main result.

The choice of the control law originates from the fact that (9)–(11) in addition to the considerations given in Remark 2 can be interpreted as an output- and input-delay ODE. Specifically, in the case of known parameters d and q, one can simply choose $\hat{q} = q$, $\hat{d} = d$ and obtain $W(t) = \eta(t) - d$. Then, the following prediction-based control law [1] [14] would exactly compensate the delay after one unit of time

$$W(t) = -(c_0 + q - 1)\Big(e^{2a(q-1)}u_t(0,t-1)$$
$$+ a\int_{t-2}^{t} e^{a(q-1)(t-\tau)}[\eta(\tau) - d]d\tau\Big) \qquad (22)$$

In other word, as $W(t) = u_t(0,t+1)$ is a two units of time ahead prediction starting from the delayed output $u_t(0,t-1) = \mu(t)$, after one unit of time, the closed loop dynamics writes $u_{tt}(0,t) = -c_0 u_t(0,t)$, which is exponentially stable. Then, from Remark 2 and applying the certainty equivalence principle, the control law (15) follows.

The update laws (18)–(19) originate from a Lyapunov design, as detailed in Section V. As usual in adaptive control [7] [12], we employ normalization and projection operators to guarantee global tracking.

Theorem 1: Consider the closed-loop system consisting in the plant (3)-(5), the control law (15) and the parameter update laws (18)–(21). Define the functional

$$\Gamma(t) = \int_{t-1}^{t} u_t(0,s)^2 ds + \max_{s\in[t-1,t]} \int_0^1 [u_x(x,s) - d]^2 dx$$
$$+ \max_{s\in[t-1,t]} \int_0^1 u_t(x,s)^2 dx + (q - \hat{q}(t))^2 + (d - \hat{d}(t))^2 \qquad (23)$$

For any $c_0 > 0$, there exist $\gamma^* > 0$ and $R, \rho > 0$ such that, for any $\gamma \in (0, \gamma^*)$, then

$$\forall t \geq 0,\ \Gamma(t) \leq R(e^{\rho\Gamma(0)} - 1) \qquad (24)$$

and the regulation in maximum norm follows, i.e.,

$$\lim_{t\to\infty} \max_{x\in[0,1]} |u_t(x,t)| = \lim_{t\to\infty} \max_{x\in[0,1]} |u_x(x,t) - d|$$
$$= \lim_{t\to\infty}(d - \hat{d}(t)) = 0 \qquad (25)$$

Before providing the proof of this theorem, we detail its application in the context of oil drilling vibrations stabilization.

IV. Application to surface-based oil-drilling stick-slip stabilization

Following [18], after normalization (see [17]) and neglecting damping, the torsion dynamics of an oil well drill string such as the one pictured in Fig. 1 writes

$$u_{tt}(x,t) = u_{xx}(x,t) \qquad (26)$$
$$u_x(1,t) = U(t) \qquad (27)$$
$$u_{tt}(0,t) = aF(u_t(0,t)) + au_x(0,t) \qquad (28)$$

in which u is the angular displacement of the drill string, U is the scalar input,$a > 0$ is constant and F is a given nonlinear function. In details, the boundary conditions account for two different phenomena: (27) represents the torque actuation of the rotatory table at $x = 1$ and (28) models the dynamics of the drill bit, subject to friction while interacting with the rock. The function F represents the rock-on-bit friction and is pictured in Fig. 2. This function is highly uncertain as it depends among other things on the nature of the rock, which varies with operation and is also poorly known.

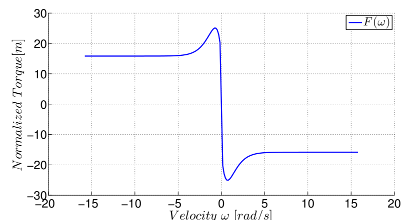
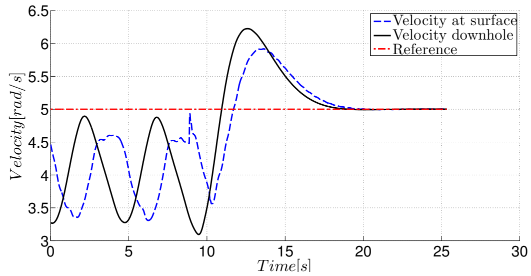


Fig. 2. Rock-on-bit friction term, as a function of the bit angular velocity.

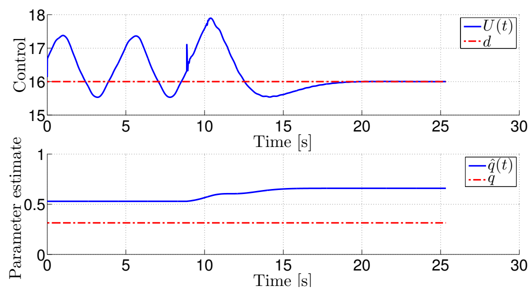
The control objective is to stabilize the angular velocity $u_t(\cdot,t)$ toward a given uniform rotatory speed u_t^r. Corresponding steady-state angular displacement profiles are therefore $u^r(x,t) = u_t^r t - F(u_t^r)x + u_0$ ($u_0 \in \mathbb{R}$) and the corresponding steady-state control law is $U^r = -F(u_t^r)$. Considering that $u_t^r = 0$ and the linearized version of (26)–(28), one obtains (3)–(5) with corresponding unknown parameter $d = -F(u_t^r)$ and $q = \frac{\partial F}{\partial u_t}(u_t^r)$. Therefore, Theorem 1 holds locally and we propose to apply the adaptive control law (15)–(21) for regulation, i.e. with $\mu(t) - u_t^r$ in lieu of $\mu(t)$.

To give physical insight on its performances, simulations results are provided in physical coordinate, i.e. using the model proposed in [18] which is equivalent to (26)–(28) (see [17]). The model parameters used in simulations are taken from [17] to ease performance comparisons and gathered in Table I. Velocity reference is chosen as $\theta_t^r = 5$ rad/s (or equivalently $u_t^r \approx 3\ \mathrm{s}^{-1}$). Corresponding unknown parameter are therefore $d = 16$ and $q = 0.31$. Initial parameters estimates are obtained with an incorrect rock-on-bit friction function and are $\hat{d}(0) = 16.15$ and $\hat{q}(0) = 0.53$. The control gain is chosen such that $c_0 a = 1$.

The corresponding closed-loop simulations are pictured in Fig. 3. The controller is turned on at $t = 9$ s. One can observe that the open-loop system not only exhibits a oscillatory behavior, as previously discussed, but is also biased because of the uncertainty of the rock-on-bit friction



(a) Velocity evolution. The adaptive controller is turned on after 9 sec.



(b) Control input and parameter estimate evolutions. The adaptive controller is turned on after 9 sec.

Fig. 3. Stabilization of plant (26)–(28) using the output-feedback adaptive controller proposed in Theorem 1 for the unknown parameters $d = -F(u_t^r)$ and $q = \frac{dF}{du_t}(u_t^r)$ ($c_0 = 1/a$, $\gamma_d = 8$, $\gamma_q = 1$ and $b_1 = 1e-3$, $b_2 = 0.1$).

term which is used as feedforward[2]. The proposed closed-loop strategy efficiently suppresses both of these effects. The control starts acting on the bit velocity around 9.5 s, which is consistent with the time of propagation of the physical system , i.e., $T \approx .6s$ (see [17] and Table I). From there, the bit velocity converges in an exponential manner to its reference. The velocity of the rotatory table follows a similar trend, delayed by T s, which corresponds to the time needed to the control law to propagate back to the surface and is therefore consistent with the theory. Fig. 3(b) pictures the variation of the parameter estimates. As stated in Theorem 1, the rock-on-bit friction term d is asymptotically estimated. On the other hand, the estimate of the anti-damping coefficient does converge but not to the unknown parameter, even if stabilization is achieved. This behavior well-known in adaptive control [7] is consistent with the error equations.

The obtained performance favorably compare to the ones recently obtained in the literature [17] [20] and can be easily tuned through the choice of the feedback gain c_0 which represents the desired closed-loop eigenvalue. However, as our approach aims at compensating the delay arising from the wave propagation from the surface to the bit and on relying on the stablity of the reverse transport phenomenon, it suffers from the fact that a minimum settling time exists, which is equal to twice the propagation delay and therefore superior to the one obtained in [17] [18]. Nevertheless, contrary to these ones, our controller does not require neither the knowledge of the distributed state nor the one of the bottom velocity.

[2]The input oscillations that can be observed in Fig. 3 correspond to a constant rotatory table velocity in the physical coordinate (see [17]). The rotatory table constant value is chosen according to the poorly known rock-on-the-bit friction term and therefore biased.

Symbol	Description	Value
G	Shear modulus of the drill pipe	79.6 $e10$ N/m^2
I	Drill pipe moment of inertia per unit of length	0.095 kg.m
I_B	Moment of inertia of the BHA	311 kg.m^2
J	Drill pipe second moment of area	1.19 e-5 m^4
L	Length of the drill pipe	2000 m
T_{tob}	Torque on the bit parameter	7500 N.m
α_1, α_2, α_3	Friction parameters	5.5; 2.2;3500
γ	Damping parameter	0.03 N.m.s /rad

BHA stands for Bottom Hole Assembly (bit and drill collars, see Fig. 1).

TABLE I

LIST OF THE PARAMETERS USED IN SIMULATIONS (SEE [17] FOR PHYSICAL MODELING AND NORMALIZATION).

V. PROOF OF THEOREM 1

A. Delayed variables and backstepping transformation

To account for the fact that we can reconstruct a one unit of time-delayed version of the bottom velocity (see Remark 2), we perform our stability analysis with a delayed set of variables, which are the following (in the sequel, we consider that $u_t(0,t-1) = \mu(t)$ is known),

$$X(t) = u_t(0,t-1) \tag{29}$$

$$\alpha(x,t) = \begin{cases} \zeta(2x,t-1) + \hat{d}(t-1) - \hat{d}(t), & x \in [0,1/2] \\ \zeta(2x,t), & x \in [1/2,1] \end{cases} \tag{30}$$

$$\beta(x,t) = \omega(x,t-1) - \hat{d}(t-1) + \hat{d}(t) \tag{31}$$

and satisfy the following dynamics, using (9)–(13)

$$\dot{X}(t) = a(q-1)X(t) + a[\alpha(0,t) - \tilde{d}(t)] \tag{32}$$

$$2\alpha_t = \alpha_x - 2\dot{\hat{d}}(t) \tag{33}$$

$$\alpha(1,t) = W(t) \tag{34}$$

$$\omega_t = -\omega_x + \dot{\hat{d}}(t) \tag{35}$$

$$\omega(0,t) = 2g(t) - \alpha(1/2,t) \tag{36}$$

$$\beta_t = -\beta_x + \dot{\hat{d}}(t) \tag{37}$$

$$\beta(0,t) = 2X(t) - \alpha(0,t) \tag{38}$$

with

$$g(t) = e^{a(q-1)}X(t) + a\int_0^1 e^{a(q-1)(1-x)}[\alpha(x/2,t) - \tilde{d}(t+x)]dx \tag{39}$$

In details, α accounts for the (modified) history of ζ over a time window of two-units of time, while β account for the (modified) history of ω over one-unit of time. Note that ω is not necessary to study the system stability and is only included in the analysis to be able to reconstruct and express our stability result in terms of the physical variables u_t and $u_x - \hat{d}$. Overall, the representation (32)–(38) accounts for the system state (u,u_t) over a time window of one unit of time.

Now, we consider the backstepping transformation of the distributed variable α

$$z(x,t) = \alpha(x,t) + (c_0 + \hat{q}(t) - 1)\left(e^{2a(\hat{q}(t)-1)x}X(t) + 2a\int_0^x e^{2a(\hat{q}(t)-1)(x-y)}\alpha(y,t)dy\right) \tag{40}$$

Following Remark 1 and with a suitable change of variable, one can observe that this transformation is closely related to (21), namely $z(x,t) = w(t+2(x-1),t)$ for $x \in [0,1]$ and $t \geq 0$. Using this relation, the control law (15) can be reformulated as

$$W(t) = -(c_0+\hat{q}(t)-1)\Big(e^{2a(q-1)}X(t) + 2a\int_0^1 e^{2a(\hat{q}(t)-1)(1-x)}\alpha(x,t)dx\Big) \tag{41}$$

The plant (32)–(38) can then be reformulated as the target system

$$\dot{X}(t) = -c_0X(t) + a\left(z(0,t)+\tilde{q}(t)X(t)-\tilde{d}(t)\right) \tag{42}$$

$$2z_t = z_x + \dot{\hat{q}}(t)g_q(x,t) + \dot{\hat{d}}(t)g_d(x,t) + [\tilde{q}(t)y(t)-\tilde{d}(t)]h(x,t) \tag{43}$$

$$z(1,t) = 0 \tag{44}$$

$$\omega_t = -\omega_x + \dot{\hat{d}}(t) \tag{45}$$

$$\omega(0,t) = 2g(t) + z(1/2,t) - (c_0+\hat{q}(t)-1) \times \Big(e^{-ac_0}X(t) + 2a\int_0^{1/2} e^{-ac_0(1-2y)}z(y,t)dy\Big) \tag{46}$$

$$\beta_t = -\beta_x + \dot{\hat{d}}(t) \tag{47}$$

$$\beta(0,t) = (1+c_0+\hat{q}(t))X(t) - z(0,t) \tag{48}$$

in which $\tilde{q}(t) = q - \hat{q}(t)$ is the anti-damping parameter estimation error, g is given in (39) and

$$g_q(x,t) = e^{2a(\hat{q}(t)-1)x}X(t) + 2a\int_0^x e^{2a(\hat{q}(t)-1)(x-y)}\alpha(y,t)dy + (c_0+\hat{q}(t)-1)\Big(2axe^{(\hat{q}(t)-1)x}X(t) + 4a^2\int_0^x (x-y)e^{2a(\hat{q}(t)-1)(x-y)}\alpha(y,t)dy\Big) \tag{49}$$

$$g_d(x,t) = -2 - 2a(c_0+\hat{q}(t)-1)\int_0^x e^{2a(\hat{q}(t)-1)(x-y)}dy \tag{50}$$

$$h(x,t) = 2a(c_0+\hat{q}(t)-1)e^{2a(\hat{q}(t)-1)x} \tag{51}$$

This target system is the one which is used in the Lyapunov analysis, as it presents the suitable boundary condition $z(1,t) = 0$.

B. Proof of stability

1) Lyapunov analysis: Consider the Lyapunov functional candidate $V = V_1 + V_2$, with

$$V_1(t) = \log(1+N(t)) + \frac{\tilde{q}(t)^2}{\gamma_q} + \frac{\tilde{d}(t)^2}{\gamma_d} \tag{52}$$

$$V_2(t) = \int_0^1 e^{1-x}\omega(x,t)^2dx \tag{53}$$

in which, following Remark 1, the normalization factor originally defined in (20) can be expressed as

$$N(t) = X(t)^2 + 2b_1\int_0^1 e^x z(x,t)^2dx + b_2\int_0^1 e^{1-x}\beta(x,t)^2dx \tag{54}$$

Taking a time-derivative and using integrations by parts, one obtains

$$\begin{aligned}\dot{V}_1(t) = \frac{1}{1+N(t)}\Big(&-2c_0X(t)^2 - b_1(z(0,t)^2+\|z(t)\|^2) \\ &+2a(z(0,t)+\tilde{q}(t)X(t)-\tilde{d}(t))X(t) + 2b_1[\tilde{q}(t)X(t)-\tilde{d}(t)] \\ &\times\int_0^1 e^x h(x,t)z(x,t)dx + 2b_1\dot{\hat{q}}(t)\int_0^1 e^x g_q(x,t)z(x,t)dx \\ &+2b_1\dot{\hat{d}}(t)\int_0^1 e^x g_d(x,t)z(x,t)dx + b_2\Big(e\beta(0,t)^2 - \|\beta(t)\|^2 \\ &-2\dot{\hat{d}}(t)\int_0^1 e^x\beta(x,t)dx\Big)\Big) - \frac{2\tilde{q}(t)\dot{\hat{q}}(t)}{\gamma_q} - \frac{2\tilde{d}(t)\dot{\hat{d}}(t)}{\gamma_d}\end{aligned} \tag{55}$$

Using the properties of the projection operator and the update laws (18)–(19) reformulated in terms of the backstepping transformation $z(x,\cdot)$ with the help of Remark 1, one gets

$$\begin{aligned}\dot{V}_1(t) = \frac{1}{1+N(t)}\Big(&-2c_0X(t)^2 + 2az(0,t)X(t) - b_1z(0,t)^2 \\ &-b_1\|z(t)\|^2 + 2b_1\dot{\hat{q}}(t)\int_0^1 e^x g_q(x,t)z(x,t)dx \\ &+2b_1\dot{\hat{d}}(t)\int_0^1 e^x g_d(x,t)z(x,t)dx + b_2\Big(e\beta(0,t)^2 - \|\beta(t)\|^2 \\ &-2\dot{\hat{d}}(t)\int_0^1 e^x\beta(x,t)dx\Big)\Big)\end{aligned} \tag{56}$$

Further, applying Cauchy-Schwartz and Young inequalities and using the inverse backstepping transformation of (40) which is

$$\alpha(x,t) = z(x,t) - (c_0+\hat{q}(t)-1)\Big(e^{-2ac_0x}X(t) + 2a\int_0^x e^{-2ac_0(x-y)}z(y,t)dy\Big) \tag{57}$$

one obtains the existence of $M > 0$ and $\tilde{M}(b_1) > 0$ such that

$$\left|2\dot{\hat{q}}(t)\int_0^1 e^x g_q(x,t)z(x,t)dx\right| \leq \gamma_q\tilde{M}(b_1)(X(t)^2+\|z(t)\|^2) \tag{58}$$

$$\left|2\dot{\hat{d}}(t)\int_0^1 e^x g_d(x,t)z(x,t)dx\right| \leq \gamma_d\tilde{M}(b_1)(X(t)^2+\|z(t)\|^2) \tag{59}$$

$$\left|2\dot{\hat{d}}(t)\int_0^1 e^x\beta(x,t)dx\right| \leq \gamma_d\tilde{M}(b_1)(X(t)^2+\|z(t)\|^2+\|\beta(t)\|^2) \tag{60}$$

$$\beta(0,t)^2 \leq M(X(t)^2+z(0,t)^2) \tag{61}$$

Hence, applying Young inequality, one gets

$$\begin{aligned}\dot{V}_1(t) \leq \frac{1}{1+N(t)}\Big(&-\Big(c_0 - [(\gamma_q+\gamma_d)b_1+\gamma_db_2]\tilde{M}(b_1) \\ &-eb_2M\Big)X(t)^2 - \Big(b_1 - \frac{a^2}{c_0} - eb_2M\Big)z(0,t)^2 \\ &-(b_1 - [(\gamma_q+\gamma_d)b_1+\gamma_db_2)\tilde{M}(b_1))\|z(t)\|^2 \\ &-b_2\|\omega(t)\|^2 - b_2(1-\gamma_d\tilde{M}(b_1))\|\beta(t)\|^2\Big)\end{aligned} \tag{62}$$

Choosing $b_1 > \frac{a^2}{c_0}$, $b_2 < \min\left\{\frac{c_0}{eM}, \frac{1}{eM}\left(b_1 - \frac{a^2}{c_0}\right)\right\}$ and

$$\gamma_q + \gamma_d < \min\left\{\frac{c_0 - eb_2M}{(b_1+b_2)\tilde{M}(b_1)}, \frac{b_1}{(b_1+b_2)\tilde{M}(b_1)}\right\} = \gamma^* \tag{63}$$

one obtains the existence of a constant $\eta > 0$ such that

$$\dot{V}_1(t) \leq -\frac{\eta}{1+N(t)}\left(X(t)^2 + \|z(t)\|^2 + \|\beta(t)\|^2\right) \tag{64}$$

and finally that

$$\forall t \geq 0, \quad V_1(t) \leq V_1(0). \tag{65}$$

Now, consider the functional V_2 defined in (53). Taking a time-derivative and using integration by parts and Young inequality, one obtains

$$\dot{V}_2(t) \leq -\frac{1}{2}V_2(t) + e\omega(0,t)^2 + 2e^2\dot{\hat{d}}(t)^2 \tag{66}$$

Therefore, from Gronwall inequality, it follows that

$$V_2(t) \leq e^{-t/2}V_2(0) + \int_0^t e^{\frac{s-t}{2}}\left(e\omega(0,s)^2 + 2e^2\dot{\hat{d}}(s)^2\right)ds \tag{67}$$

Further, using (19) and (36) and applying Young inequality, one gets the existence of $M_0 > 0$ such that

$$\begin{aligned} &2e\omega(0,s)^2 + 2e\dot{\hat{d}}(s)^2 \\ &\leq \begin{cases} M_0\left(V_1(s) + \int_0^1 V_1(s+x)dx + \alpha(1/2,s)^2\right) & \text{if } s \leq 1 \\ M_0\left(V_1(s) + \int_0^1 V_1(s+x)dx + V_1(s-1)\right) & \text{otherwise} \end{cases} \end{aligned} \tag{68}$$

noticing that $\alpha(1/2,s) = W(s-1)$ for $s \geq 1$. Hence, it follows

$$\begin{aligned} V_2(t) &\leq e^{-t/2}V_2(0) \\ &+ M_0\int_0^1 e^{\frac{s-t}{2}}\left(V_1(s) + \int_0^1 V_1(s+x)dx + \alpha\left(\frac{1+s}{2},0\right)^2\right)ds \\ &+ M_0\int_1^t e^{\frac{s-t}{2}}\left(V_1(s) + \int_0^1 V_1(s+x)dx + V_1(s-1)\right)ds \end{aligned} \tag{69}$$

Using (65), it therefore follows that

$$V_2(t) \leq V_2(0) + 6M_0V_1(0) \tag{70}$$

Matching (65) and (70), one finally gets that

$$V(t) \leq (1+7M_0)V(0), \; t \geq 0 \tag{71}$$

2) Stability in terms of the functional Γ *(23):* To prove the stability result (24), we define the intermediate functionals

$$\begin{aligned} V_0(t) =& X(t)^2 + \|\alpha(t)\|^2 + \|\omega(t)\|^2 + \|\beta(t)\|^2 \\ &+ \tilde{q}(t)^2 + \tilde{d}(t)^2 \end{aligned} \tag{72}$$

$$\begin{aligned} \Gamma_0(t) =& \int_{t-1}^t u_t(0,s)^2ds + \max_{s\in[t-1,t]}\|\zeta(t)\|^2 + \max_{s\in[t-1,t]}\|\omega(t)\|^2 \\ &+ \tilde{q}(t)^2 + \tilde{d}(t)^2 \end{aligned} \tag{73}$$

We start by observing that, for $s \in [t-1,t]$,

$$\begin{aligned} u_t(0,s) =& e^{a(q-1)(s-t+1)}X(t) \\ &+ 2a\int_0^{\frac{s-t+1}{2}} e^{a(q-1)(s-2x-t+1)}\alpha(x,t)dx \end{aligned} \tag{74}$$

Therefore, using Young inequality, there exists $M_1 > 0$ such that, for all $s \in [t-1,t]$,

$$\int_{t-1}^t u_t(0,s)^2ds \leq M_1(X(t)^2 + \|\alpha(t)\|^2) \tag{75}$$

Similarly, one gets, for $s \in [t-1,t]$,

$$X(t) = e^{-a(q-1)(s-t+1)}u_t(0,s) + 2a\int_0^{\frac{s-t+1}{2}} e^{-a(q-1)2x}\alpha(x,t)dx \tag{76}$$

which gives, integrating this last expression for s between $t-1$ and t and applying Young and Cauchy-Schwartz inequalities, the existence of $M_2 > 0$ such that

$$X(t)^2 \leq M_2\left(\int_{t-1}^t u_t(0,s)^2ds + \|\alpha(t)\|^2\right) \tag{77}$$

Second, using Remark 1 and after some derivations, one can observe that

$$\begin{aligned} &2\|\alpha(t)\|^2 + \|\omega(t)\|^2 + \|\beta(t)\|^2 \\ &= \|\zeta(t)\|^2 + \|\zeta(t-1)\|^2 + \|\omega(t)\|^2 + \|\omega(t-1)\|^2 \end{aligned} \tag{78}$$

and that, for any $(y_1,y_2) \in [0,1]$,

$$\begin{aligned} &2\|\alpha(t)\|^2 + \|\omega(t)\|^2 + \|\beta(t)\|^2 \\ &\geq \int_{y_1}^{1+y_1}[\zeta(1,t+x-2) - \hat{d}(t) + \hat{d}(t+x-2)]^2dx \\ &+ \int_{y_2}^{1+y_2}[\omega(1,t+x-2) + \hat{d}(t) - \hat{d}(t+x-2]^2dx \\ &= \|\zeta(t-1+y_1)\|^2 + \|\omega(t-1+y_2)\|^2 \end{aligned} \tag{79}$$

Hence, from (78)–(79), it follows that

$$\begin{aligned} &\max_{s\in[t-1,t]}\|\zeta(s)\|^2 + \max_{s\in[t-1,t]}\|\omega(s)\|^2 \\ &\leq 2\|\alpha(t)\|^2 + \|\omega(t)\|^2 + \|\beta(t)\|^2 \leq \\ &2\max_{s\in[t-1,t]}\|\zeta(s)\|^2 + 2\max_{s\in[t-1,t]}\|\omega(s)\|^2 \end{aligned} \tag{80}$$

Further, from the backstepping transformation (40) and its inverse (57), using Cauchy-Schwartz and Young inequalities, one obtains the existence of strictly positive constants r_1, r_2, s_1 and s_2 such that

$$\|\alpha(t)\|^2 \leq r_1X(t)^2 + r_2\|z(t)\|^2 \tag{81}$$

$$\|z(t)\|^2 \leq s_1X(t)^2 + s_2\|\alpha(t)\|^2 \tag{82}$$

and therefore the one of M_3 and M_4 such that

$$\frac{1}{M_4}V(t) \leq V_0(t) \leq M_3(e^{V(t)} - 1) \tag{83}$$

Consequently, from (71) and gathering (75), (77), (80) and (83), it follows that

$$\begin{aligned} \Gamma_0(t) \leq& (M_1+2)M_3(e^{(1+7M_0)V(0)} - 1) \\ \leq& (M_1+2)M_3(e^{(1+7M_0)M_4(M_2+2)\Gamma_0(0)} - 1) \end{aligned} \tag{84}$$

The stability in terms of Γ follows straightforwardly, using (6)–(7) and the inverse transformations, for $s \in [t-1,t]$,

$$u_t(x,s) = \frac{\zeta(x,s)+\omega(x,s)}{2} \tag{85}$$

$$u_x(x,s)-\hat{d}(s) = \frac{\zeta(x,s)-\omega(x,s)}{2} \tag{86}$$

C. Convergence in $\mathscr{L}_2$-norm

Lemma 1: Consider the closed-loop system consisting in the plant (3)-(5), the control law (15) and the parameter update laws (18)–(20). Then, under the conditions stated in Theorem 1,

$$\lim_{t\to\infty} u_t(0,t) = \lim_{t\to\infty} \|z(t)\| = \lim_{t\to\infty} \|\omega(t)\| = 0 \tag{87}$$

Proof: From (65), one easily gets that $\tilde{q}(t)$, $\tilde{d}(t)$ and $N(t)$ are are uniformly bounded for $t \geq 0$, and therefore $X(t)$, $\|z(t)\|$ and $\|\beta(t)\|$ are also uniformly bounded for $t \geq 0$. Consequently, so is $\|\omega(t)\|$ for $t \geq 1$. Further, from (81), $\|\alpha(t)\|$ is also uniformly bounded for $t \geq 0$ and so is $\|\zeta(t)\|$ from (80).

From there, applying Cauchy-Schwartz inequality to (15), one can obtain that $\alpha(1,t)$ is uniformly bounded for $t \geq 0$. Further, using (30) and Remark 1, $\alpha(x,t)$ is also uniformly bounded for $t \geq 2(1-x)$ and in particular $\alpha(1/2,t)$ and $\alpha(0,t)$ are uniformly bounded for $t \geq 1$ and $t \geq 2$ respectively. From (40),

$$z(0,t) = \alpha(0,t) + (c_0 + \hat{q}(t) - 1)X(t) \tag{88}$$

and, consequently, $z(0,t)$ is also uniformly bounded for $t \geq 2$. From there, applying Young inequality to (18)–(19), one can obtain that $\dot{\hat{q}}$ and $\dot{\hat{d}}$ are uniformly bounded for $t \geq 2$. Further, from (42)-(46),

$$\frac{d}{dt}X(t)^2 = 2X(t)\Big(-c_0 X(t) + a(z(0,t) + \tilde{q}(t)X(t) - \tilde{d}(t))\Big) \tag{89}$$

$$\begin{aligned}\frac{d}{dt}\|z(t)\|^2 = \|z(t)\| \Bigg(&-\frac{1}{2}z(0,t)^2 + \int_0^1 z(x,t)\Big((\tilde{q}(t)X(t) \\ &- \tilde{d}(t))h(x,t) + \dot{\hat{q}}(t)h_q(x,t) + \dot{\hat{d}}(t)h_d(x,t)\Big)dx\Bigg)\end{aligned} \tag{90}$$

$$\begin{aligned}\frac{d}{dt}\|\beta(t)\|^2 = \|\beta(t)\| \Bigg(&-(2X(t-1) - \alpha(0,t-1) + \hat{d}(t))^2 \\ &+ \beta(0,t)^2 + \dot{\hat{d}}(t)\int_0^1 \beta(x,t)dx\Bigg)\end{aligned} \tag{91}$$

in which $\beta(0,t) = 2X(t) - \alpha(0,t)$. Applying Cauchy-Schwartz inequality and the previous considerations, it is straightforward that the second terms in the previous equations are all uniformly bounded for $t \geq 3$.

Finally, integrating (64) from 0 to ∞, it follows that $X(t)$, $\|z(t)\|$ and $\|\beta(t)\|$ are square integrable. Following Barbalat Lemma, $X(t)$, $\|z(t)\|$ and $\|\beta(t)\|$ tend to zero as t tends to ∞. Consequently, $u_t(0,t) = X(t+1)$ and $\|\omega(t)\| = \|\beta(t+1)\|$ also tend to zero asymptotically. ■

D. Pointwise convergence

Using Agmon inequality, one gets

$$\max_{x\in[0,1]} \alpha(x,t)^2 \leq \alpha(1,t)^2 + 2\|\alpha(t)\|\,\|\alpha_x(t)\| \tag{92}$$

in which, applying Young inequality to (15),

$$\alpha(1,t)^2 = W(t)^2 \leq M_5\left(X(t)^2 + \|\alpha(t)\|^2\right) \tag{93}$$

for some positive constant M_5. Further, from (33)-(34), the spatial-derivative of the distributed variable α_x satisfies the following equations

$$2\alpha_{xt} = \alpha_{xx} \tag{94}$$

$$\begin{aligned}\alpha_x(1,t) = &-\dot{\hat{q}}(t)(c_0+\hat{q}(t)-1)\Bigg(2ae^{2a(\hat{q}(t)-1)}X(t) \\ &+ 4a^2\int_0^1 e^{2a(\hat{q}(t)-1)(1-x)}(1-x)\alpha(x,t)dx\Bigg) \\ &- \dot{\hat{q}}(t)\left(e^{2a(\hat{q}(t)-1)}X(t) + 2a\int_0^1 e^{2a(\hat{q}(t)-1)(1-x)}\alpha(x,t)dx\right) \\ &- (c_0+\hat{q}(t)-1)\Bigg(e^{2a(\hat{q}(t)-1)}(a(q-1)X(t) + a\alpha(0,t)) \\ &+ 2a\left(\alpha(1,t) - e^{2a(\hat{q}(t)-1)}\alpha(0,t)\right) \\ &+ 4a^2(\hat{q}(t)-1)\int_0^1 e^{2a(\hat{q}(t)-1)(1-x)}\alpha(x,t)dx\Bigg)\end{aligned} \tag{95}$$

Consequently, using integration by parts, one can deduce that

$$\frac{d}{dt}\left[2\int_0^1 e^x \alpha_x(x,t)^2 dx\right] \leq e\alpha_x(1,t)^2 - \int_0^1 e^x\alpha_x(x,t)^2 dx \tag{96}$$

and, using Gronwall inequality,

$$\|\alpha_x(t)\|^2 \leq e^{1-t/2}\left\|\alpha_x(0)^2\right\| + \int_0^t e^{-\frac{t-s}{2}} e\alpha_x(1,s)^2 ds \tag{97}$$

Applying Young's and Cauchy-Schwartz's inequality to (95), one can further obtain that $\alpha_x(1,t)^2$ is bounded, using the conclusions obtained in the previous section. Therefore, it follows that $\|\alpha_x(t)\|$ is bounded. Furthermore, as $X(t)$ and $\|z(t)\|$ tend to zero asymptotically from Lemma 1, $\|\alpha(t)\|$ also tends to zero as t tends to infinity and, from (93), so does $\alpha(1,t)$. From (92), one concludes that $\max_{x\in[0,1]}\alpha(x,t)^2$ tends to zero as t tends to infinity.

Similar arguments give the convergence of $\max_{x\in[0,1]}\beta(x,t)$ and therefore of $\max_{x\in[0,1]}\omega(x,t) = \max_{x\in[0,1]}\beta(x,t+1)$.

Finally, from the inverse transformations (85)-(86), applying the triangle inequality, one can obtain that $\max\limits_{x\in[0,1]}|u_x(x,t)-\hat{d}(t)|$ and $\max\limits_{x\in[0,1]}|u_t(x,t)|$ tend to zero asymptotically. This concludes the proof, using (5) and (19).

VI. Conclusion

In this paper, we proposed an adaptive output-feedback controller for a wave PDE in one dimension with uncertain anti-damping dynamic boundary and actuation and measurement at the opposite boundary and show the relevance of our result for suppressing stick-slip oscillations in oil-drilling facilities. The main interest of our approach is to require only to measure the top boundary velocities. Extension of our approach to fusion with a bottom velocity sensor, transmitting its signal via the mud system flowing back to the surface or via dedicated acoustic waves, and therefore generating a (time-varying) transport delay [9] is a direction of future work.

REFERENCES

[1] Z. Artstein. Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, 27(4):869–879, 1982.

[2] A. G. Balanov, N. B. Janson, P. V. Mc Clintock, R. W. Tucker, and C. H. T. Wang. Bifurcation analysis of a neutral delay differential equation modelling the torsional motion of a driven drill-string. *Chaos, Solitons & Fractals*, 15(2):381–394, 2003.

[3] N. Bekiaris-Liberis and M. Krstic. Compensation of wave actuator dynamics for nonlinear systems. *IEEE Transactions on Automatic Control*, to appear.

[4] D. Bresch-Pietri and M. Krstic. Output-feedback adaptive control of a wave PDE with boundary anti-damping. *Automatica*, to appear.

[5] D. Bresch-Pietri and M. Krstic. Adaptive output feedback for oil drill-string stick-slip instability modeled by wave PDE with anti-damped dynamic boundary. In *Proc. of the American Control Conference*, to appear, 2014.

[6] C. Canudas-de Wit, F. R. Rubio, and M. A. Corchero. D-oskil: A new mechanism for controlling stick-slip oscillations in oil well drillstrings. *IEEE Transactions on Control Systems Technology*, 16(6):1177–1191, 2008.

[7] P. A. Ioannou and B. Fidan. *Adaptive Control Tutorial*. Society for Industrial Mathematics, 2006.

[8] J. D. Jansen and L. Van den Steen. Active damping of self-excited torsional vibrations in oil well drillstrings. *Journal of Sound and Vibration*, 179(4):647–668, 1995.

[9] R. B. Jijón, C. Canudas-de Wit, S. Niculescu, and J. Dumon. Adaptive observer design under low data rate transmission with applications to oil well drill-string. In *American Control Conference*, 2010.

[10] M. Krstic. *Boundary Control of PDEs: a Course on Backstepping Designs*. Society for Industrial and Applied Mathematics Philadelphia, PA, USA, 2008.

[11] M. Krstic. Adaptive control of an anti-stable wave PDE. *Dynamics of Continuous, Discrete and Impulsive Systems*, 17:853–882, 2010.

[12] M. Krstic, I. Kanellakopoulos, and P.V. Kokotovic. *Nonlinear and Adaptive Control Design*. John Wiley & Sons New York, 1995.

[13] L. Li, Q. Zhang, and N. Rasol. Time-varying sliding mode adaptive control for rotary drilling system. *Journal of Computers*, 6(3):564–570, 2011.

[14] A. Manitius and A. Olbrot. Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, 24(4):541–552, 1979.

[15] E. M Navarro-López and D. Cortés. Sliding-mode control of a multi-dof oilwell drillstring with stick-slip oscillations. In *American Control Conference*, 2007.

[16] P. Rouchon. Flatness and stick-slip stabilization. Technical report, MINES ParisTech, 1998.

[17] C. Sagert, F. Di Meglio, M. Krstic, and P. Rouchon. Backstepping and flatness approaches for stabilization of the stick-slip phenomenon for drilling. In *IFAC Symposium on System, Structure and Control*, 2013.

[18] M. B. Saldivar, S. Mondié, J.-J. Loiseau, and V. Rasvan. Stick-slip oscillations in oillwell drilstrings: distributed parameter and neutral type retarded model approaches. In *IFAC 18th World Congress*, pages 283–289, 2011.

[19] M. B. Saldivar, A. Seuret, and S. Mondié. Exponential stabilization of a class of nonlinear neutral type time-delay systems, an oilwell drilling model example. In *International Conference on Electrical Engineering Computing Science and Automatic Control*, pages 1–6. IEEE, 2011.

[20] A. Serrarens, M. Van de Molengraft, J. Kok, and L. Van den Steen. H_∞ control for suppressing stick-slip in oil well drillstrings. *Control Systems, IEEE*, 18(2):19–30, 1998.