# The Strang and Fix Conditions

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**Abstract**

Then Strang and Fix conditions are recalled and proved.

The Strang and Fix conditions caracterize the approximating properties of a shift invariant localized operator by its ability to reconstruct polynomials. In particular, it is used to relate the number of vanishing moments of a wavelet to the order of approximation provided by the corresponding multiresolution analysis. Article [1] is generally given as a reference for this result. This note is intented to be an alternative to it, since the original paper has probably disappeared from most bookshelves.

## 1 Notations

- $\mathbf{L^2}(\mathbb{R})$ is the space of integrable functions with a finite energy. $\mathbf{H}^N(\mathbb{R})$ is defined recursively by $\mathbf{H}^0(\mathbb{R}) = \mathbf{L^2}(\mathbb{R})$ and $\mathbf{H}^N(\mathbb{R})$ is the space of functions in $\mathbf{L^2}(\mathbb{R})$ with derivatives in $\mathbf{H}^{N-1}(\mathbb{R})$. If $f \in \mathbf{H}^N(\mathbb{R})$, $f^{(N)}$ denotes the $n^{th}$ derivative of $f$.

- In what follows, $K \in \mathbf{L^2}_{\mathrm{Loc}}(\mathbb{R} \times \mathbb{R})$ is a kernel such that

$$K(x+1, y+1) = K(x, y) \text{ a.e.} \tag{1}$$

$$\exists M \text{ s.t. } K(x, y) = 0 \text{ if } |x - y| \geq M, \tag{2}$$

  that is, $K$ *is a localized kernel and it commutes with integer shifts.* Because of the localization property (2), $K$ is of finite energy with respect to the separate variables $x$ and $y$. Because of the shift invariance property (1), it cannot be of finite energy with respect to the joint variables $(x, y)$ (unless it is zero).

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- for $\delta \neq 0$, the operator $U_\delta$ is defined by

$$U_\delta f(x) = \frac{1}{\sqrt{\delta}} f\left(\frac{x}{\delta}\right)$$

Observe that

$$\|U_\delta f\|_{\mathbf{L2}(\mathbb{R})} = \|f\|_{\mathbf{L2}(\mathbb{R})}, \tag{3}$$

and, more generally,

$$\left\|[U_\delta f]^{(N)}\right\|_{\mathbf{L2}(\mathbb{R})} = \frac{1}{\delta^N}\|f\|_{\mathbf{L2}(\mathbb{R})}. \tag{4}$$

- To such an kernel $K$ and any real $\delta \leq 1$ is associated an operator $P_\delta$ defined by

$$P_\delta f(x) = \frac{1}{\delta}\int_{\mathbb{R}} K\left(\frac{x}{\delta}, \frac{y}{\delta}\right) f(y)dy \tag{5}$$

Observe that

$$P_\delta = U_\delta P_1 U_{\frac{1}{\delta}} \tag{6}$$

and hence,

$$\|P_\delta f\|_{\mathbf{L2}(\mathbb{R})} = \left\|P_1 U_{\frac{1}{\delta}} f\right\|_{\mathbf{L2}(\mathbb{R})}$$

# 2 Preliminary results

## 2.1 Continuity

**Theorem 1** *There exists $C \geq 0$ such that, for any $f \in \mathbf{L}^{\mathbf{2}}(\mathbb{R})$ and $\delta \leq 1$,*

$$\|P_\delta f\|_{\mathbf{L2}(\mathbb{R})} \leq C\|f\|_{\mathbf{L2}(\mathbb{R})} \tag{7}$$

**Proof:** Equations (3) and (6) imply that (7) is satisfied if and only if it is valid for $\delta = 1$. Then

$$\begin{aligned}
\int\left|\int K(x,y)f(y)dy\right|^2 dx &\leq \int\left[\int |f(y)|^2 1_{[-M,M]}(x-y)dy\right]\left[\int |K(x,y)|^2 dy\right]dx \\
&\leq \sum_{k\in\mathbb{Z}}\left[\int_k^{k+1}\int |K(x,y)^2|dxdy\right] \\
&\qquad \left[\sup_{x\in[k,k+1]}\int |f(y)|^2 1_{[-M,M]}(x-y)dy\right] \\
&\leq \left[\int_0^1\int |K(x,y)^2|dxdy\right]
\end{aligned}$$

$$\times \int |f(y)|^2 \sum_{k\in\mathbb{Z}} \left[\int_k^{k+1} 1_{[k-M,k+M+1]}(y)dy\right]$$

$$\leq \quad (2M+2)\left[\int_0^1 \int |K(x,y)^2|dxdy\right] \int |f(y)|^2 dy$$

which proves the result with $C^2 = (2M+2)\left[\int_0^1 \int |K(x,y)^2|dxdy\right]$.

## 2.2 Convergence and rescaling

**Theorem 2** *Let $P$ a continuous operator over $\mathbf{L^2}(\mathbb{R})$ and define*

$$P_\delta = U_\delta P U_{1/\delta}.$$

*If there exists an integer $N$ and a real number $C$ such that, for any $f \in \mathbf{H}^{N+1}(\mathbb{R})$,*

$$\|Pf\|_{\mathbf{L^2}(\mathbb{R})} \leq C \left\|f^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})} \tag{8}$$

*then*

$$\|P_\delta f\|_{\mathbf{L^2}(\mathbb{R})} \leq C\delta^{N+1} \left\|f^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})} \tag{9}$$

*and*

$$\delta^{-N}\|P_\delta f\|_{\mathbf{L^2}(\mathbb{R})} \to 0 \text{ when } \delta \to 0 \tag{10}$$

**Proof:** equations (4), (6) and (8) imply that

$$\|P_\delta f\|_{\mathbf{L^2}(\mathbb{R})} = \|P_1 U_{\frac{1}{\delta}} f\|_{\mathbf{L^2}(\mathbb{R})} \leq C \left\|\left[U_{\frac{1}{\delta}} f\right]^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})} = C\delta^{N+1} \left\|f^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})}$$

which proves (9).

Equations (3) and (6) also imply that the family $(P_\delta)_{\delta\leq 1}$ is equicontinuous over $\mathbf{L^2}(\mathbb{R})$. Since $\mathbf{H}^1(\mathbb{R})$ is dense in $\mathbf{L^2}(\mathbb{R})$, this proves (10) for $N=0$. For $N>0$, the family $\left(\delta^{-N}P_\delta\right)_{\delta\leq 1}$ is proved to be equicontinuous over $\mathbf{H}^N$ in the following way: $f \in \mathbf{H}^{N+1}$ is decomposed as $f = f_1 + f_2$, the Fourier transform $\hat{f}_1$ of $f_1$ being defined by $\hat{f}_1 = \hat{f} \times 1_{[-1,1]}$. From (9) we derive

$$\|P_1 f_1\|_{\mathbf{L^2}(\mathbb{R})} \leq C \left\|f^{(N+1)}\right\|_{\mathbf{L^2}(\mathbb{R})} \leq C \left\|f_1^{(N)}\right\|_{\mathbf{L^2}(\mathbb{R})} \tag{11}$$

On the other hand, thorem 1 implies

$$\|P_1 f_2\|_{\mathbf{L^2}(\mathbb{R})} \leq C \|f_2\|_{\mathbf{L^2}(\mathbb{R})} \leq C \left\|f_2^{(N)}\right\|_{\mathbf{L^2}(\mathbb{R})} \tag{12}$$

After rescaling, (11) and (12) imply

$$\delta^{-N} \|P_\delta f - f\|_{\mathbf{L^2}(\mathbb{R})} \leq \left\|f^{(N)}\right\|_{\mathbf{L^2}(\mathbb{R})}$$

which, together with the density of $\mathbf{H}^{N+1}$ in $\mathbf{H}^N$, proves the equicontinuity over $\mathbf{H}^N$. Then (9) and the density of $\mathbf{H}^{N+1}$ in $\mathbf{H}^N$ imply (10).

## 2.3 Vanishing moments

**Lemma 1** *Assume that there exists an integer $N$ such that*

$$\int K(x,y)y^p dy = 0 \text{ for } 0 \le p \le N \tag{13}$$

*Then*

*(L1) there exists a constant $C$ such that, for any $f \in \mathbf{H}^{N+1}(\mathbb{R})$ and $\delta \le 1$,*

$$\|P_\delta f\|_{\mathbf{L2}(\mathbb{R})} \le C\delta^{N+1} \left\| f^{(N+1)} \right\|_{\mathbf{L2}(\mathbb{R})} \tag{14}$$

*(L2) For any $f \in \mathbf{H}^N(\mathbb{R})$,*

$$\delta^{-N} \|P_\delta f\|_{\mathbf{L2}(\mathbb{R})} \to 0 \text{ when } \delta \to 0 \tag{15}$$

**Proof:** theorem 2 proves that equation (14) implies ($L2$).

Let us concentrate on the proof of (14). As noted before, (14) holds if and only if it is valid for $\delta = 1$.

Observe that (14) with $p = 0$ implies

$$(P_1 f)(x) = \int K(x,y)(f(y) - f(x))dy \tag{16}$$

Moreover, if $N > 0$, then (13) implies

$$\int K(x,y)(x-y)^p dy = 0 \text{ for } 1 \le p \le N \tag{17}$$

Using a Taylor expansion in (16) together with (17) yields

$$\int |P_1 f|^2 dx \le \int \left( \int |K(x,y)| \left| \int_{[x,y]} f^{(N+1)}(z)(y-z)^N dz \right| dy \right)^2 dx \tag{18}$$

Observe that

$$\begin{aligned}
& \int |K(x,y)| \left| \int_{[x,y]} f^{(N+1)}(z)(y-z)^N dz \right| dy \\
\le \quad & \left[ \int |K(x,y)| dy \right] \sup_{|x-y| \le M} \left| \int_{[x,y]} f^{(N+1)}(z)(y-z)^N dz \right| \\
\le \quad & \left[ \int |K(x,y)| dy \right] \sup_{|x-y| \le M} \int_{[x,y]} \left| f^{(N+1)}(z) \right| |y-z|^N dz \\
\le \quad & M^N \left[ \int |K(x,y)| dy \right] \int_{x-M}^{x+M} \left| f^{(N+1)}(z) \right| dz
\end{aligned}$$

Hence (18) implies

$$\begin{aligned}
\int |P_1 f|^2 dx &\leq \int \left[\int |K(x,y)|dy\right]^2 \left[M^N \int_{x-M}^{x+M} \left|f^{(N+1)}(z)\right| dz\right]^2 dx \\
&\leq 4M^{2(N+1)} \int \left[\int |K(x,y)|^2 dy\right] \left[\int_{x-M}^{x+M} \left|f^{(N+1)}(z)\right|^2 dz\right] dx \\
&\leq 4M^{2(N+1)} \sum_{k\in\mathbb{Z}} \left[\int_k^{k+1} \int |K(x,y)|^2 dy dx\right] \left[\int_{k-M}^{k+M+1} \left|f^{(N+1)}(z)\right|^2 dz\right] \\
&= 4M^{2(N+1)} \left[\int_0^1 \int |K(x,y)|^2 dy dx\right] \\
&\quad \times \int \left|f^{(N+1)}(z)\right|^2 \sum_{k\in\mathbb{Z}} 1_{[k-M,k+M+1]}(z) dz \\
&\leq 8(M+1)M^{2(N+1)} \left[\int_0^1 \int |K(x,y)|^2 dy dx\right] \left[\int \left|f^{(N+1)}\right|^2 dz\right]
\end{aligned}$$

# 3 The Strang and Fix conditions

**Theorem 3 (Stang and Fix [1])** *The three following statements are equivalent:*

*(A1) For any $f \in \mathbf{H}^N(\mathbb{R})$,*

$$\delta^{-N} \|P_\delta f - f\|_{\mathbf{L2}(\mathbb{R})} \to 0 \text{ when } \delta \to 0$$

*(A2) For any $f \in \mathbf{H}^{N+1}(\mathbb{R})$ and $\delta \leq 1$,*

$$\|P_\delta f - f\|_{\mathbf{L2}(\mathbb{R})} \leq C\delta^{N+1} \left\|f^{(N+1)}\right\|_{\mathbf{L2}(\mathbb{R})}$$

*(A3) For any integer $p$, $0 \leq p \leq N$,*

$$\int_{\mathbb{R}} K(x,y) y^p dy = x^p \tag{19}$$

*for almost every $x$.*

**Proof.**

- Proof of $(A3) \Rightarrow (A2)$

  As before, rescaling implies that $(A2)$ holds if and only if it is valid for $\delta = 1$.

Observe that $(A3)$ implies

$$(P_1 f - f)(x) = \int K(x,y)(f(y) - f(x))dy \tag{20}$$

Moreover, if $N > 0$, then Property $(A3)$ implies

$$\int K(x,y)(x-y)^p dy = 0 \text{ for } 1 \le p \le N \tag{21}$$

Using a Taykor expansion in (20) and (21) yields

$$\int |P_1 f - f|^2 dx \le \int \left( \int |K(x,y)| \left| \int_{[x,y]} f^{(N+1)}(z)(y-z)^N dz \right| dy \right)^2 dx$$

The rest of the proof is then identical to the proof of lemma 1 after equation (18).

- Proof of $(A2) \Rightarrow (A1)$

  Using lemma 1 with $P = P_1 - Id$ proves the result.

- Proof of $(A1) \Rightarrow (A3)$: for $0 \le p \le N$, let $\omega_p$ an infinitely differentable function with compact support such that

$$\int x^k \omega_p(x) dx = \delta_{k,p} \tag{22}$$

and define

$$\mu_p(x) = \int K(x,y)(x-y)^p dy \text{ for } 0 \le p \le N,$$

$$\tilde{K}(x,y) = K(x,y) - \sum_{k=0}^{k=N} \mu_k(x)\omega_k(x-y)$$

and

$$\tilde{P}_\delta f(x) = \frac{1}{\delta} \int \tilde{K}\left(\frac{x}{\delta}, \frac{y}{\delta}\right) f(y) dy$$

Observe that $\tilde{K}$ satisfies (1) and (2). In order to use lemma 1, let us prove that $\tilde{K}$ has $N$ vanishing moments. If $\Sigma$ denotes the difference $K - \tilde{K}$, this is equivalent to proving that $\Sigma$ satisfies condition $(A3)$. Reordering the integrations in the computation of the moments of $\Sigma$ gives

$$\begin{aligned} \int \mu_k(x)\omega_k(x-y)y^p dy &= \int \left[ \int K(x,z)(x-z)^k dz \right] \omega_k(x-y) y^p dy \\ &= \int K(x,z)(x-z)^k dz \int \omega_k(x-y) y^p dy \end{aligned}$$

The inner integral satisfies

$$\begin{aligned}
\int \omega_k(x-y)y^p dy &= \int \omega_k(y)(x-y)^p dy \\
&= \sum_{i-0}^{i=p} \left( \begin{array}{c} p \\ i \end{array} \right) x^{p-i} \int \omega_k(y)(-y)^i \\
&= \left\{ \begin{array}{ccc} 0 & \text{if} & p<k \\ \left( \begin{array}{c} p \\ k \end{array} \right)(-1)^k x^{p-k} & \text{if} & p \geq k \end{array} \right.
\end{aligned}$$

Hence the moments of $\Sigma$ satisfy

$$\begin{aligned}
\sum_{k=0}^{k=N} \int \mu_k(x)\omega_k(x-y)y^p dy &= \sum_{k=0}^{k=p} \int K(x,z)(x-z)^k dz \left( \begin{array}{c} p \\ k \end{array} \right)(-1)^k x^{p-k} \\
&= \int K(x,z) \sum_{k=0}^{k=p} \left( \begin{array}{c} p \\ k \end{array} \right)(z-x)^k x^{p-k} dz \\
&= \int K(x,z) z^p dz
\end{aligned}$$

which proves

$$\int \tilde{K}(x,y)y^p dy = 0 \text{ for } 0 \leq p \leq N$$

Therefore, lemma 1 can be applied to $\tilde{K}$, and property $(A2)$ for $K$ implies that $\Sigma = K - \tilde{K}$ also satisfies $(A2)$, and hence, $(A1)$. This means

$$\delta^{-N} \left\| \Pi_\delta f - f \right\|_{\mathbf{L2}(\mathbb{R})} \to 0 \text{ when } \delta \to 0 \tag{23}$$

with

$$\Pi_\delta f(x) = \frac{1}{\delta} \int \Sigma\left(\frac{x}{\delta}, \frac{y}{\delta}\right) f(y) dy$$

For $0 \leq p \leq N$, let $f_p$ an infinitely differentiable, compactly supported function with $\mathrm{supp}(f_p) \subset [-2,3]$, and $f_p(x) = x^p$ if $x \in [-1,2]$. Then, for $\delta$ small enough and $x \in [0,1]$

$$\begin{aligned}
\frac{1}{\delta} \int \omega_k \left(\frac{x-y}{\delta}\right) f_p(y) dy = \delta_{k,p} &= \frac{1}{\delta} \int \omega_k \left(\frac{x-y}{\delta}\right) y^p dy \\
&= \delta^p \int \omega_k(x-y) y^p dp \\
&= \delta^p \sum_{i=0}^{i=p} \omega_k(x-y) \left( \begin{array}{c} p \\ i \end{array} \right)(y-x)^i x^{p-i} dy \\
&= \left\{ \begin{array}{ll} 0 & \text{if } k>p \\ (-1)^k \delta^p \left( \begin{array}{c} p \\ k \end{array} \right) x^{p-k} & \text{if } k \leq p \end{array} \right.
\end{aligned}$$

For such a $\delta$,

$$\begin{aligned}
\|\Pi_\delta f_p - f_p\|^2_{\mathbf{L2}(\mathbb{R})} &= \int_{\mathbb{R}} \left| \sum_{k=0}^{k=N} \mu_k \left(\frac{x}{\delta}\right) \frac{1}{\delta} \int \omega_k \left(\frac{x-y}{\delta}\right) f_p(y)dy - f_p(x) \right|^2 dx \\
&\geq \int_0^1 \left| \sum_{k=0}^{k=N} \mu_k \left(\frac{x}{\delta}\right) \frac{1}{\delta} \int \omega_k \left(\frac{x-y}{\delta}\right) f_p(y)dy - f_p(x) \right|^2 dx \\
&= \int_0^1 \left| \sum_{k=0}^{k=p} \mu_k \left(\frac{x}{\delta}\right) (-1)^k \delta^p \begin{pmatrix} p \\ k \end{pmatrix} x^{p-k} - x^p \right|^2 dx \qquad (24)
\end{aligned}$$

On the other hand, $\mu_k$ is 1-periodic, so, if $\Delta$ denotes the integer value of $1/\delta$,

$$\begin{aligned}
&\int_0^1 \left| \sum_{k=0}^{k=p} \mu_k \left(\frac{x}{\delta}\right) (-1)^k \delta^p \begin{pmatrix} p \\ k \end{pmatrix} x^{p-k} - x^p \right|^2 dx \\
\geq &\sum_{i=0}^{i=\Delta} \int_{i\delta}^{(i+1)\delta} \left| \sum_{k=0}^{k=p} \mu_k \left(\frac{x}{\delta}\right) (-1)^k \delta^p \begin{pmatrix} p \\ k \end{pmatrix} x^{p-k} - x^p \right|^2 dx \\
= &\sum_{i=0}^{i=\Delta} \int_{i\delta}^{(i+1)\delta} \left| \sum_{k=0}^{k=p} \mu_k \left(\frac{x-i\delta}{\delta}\right) (-1)^k \delta^p \begin{pmatrix} p \\ k \end{pmatrix} x^{p-k} - x^p \right|^2 dx \\
= &\sum_{i=0}^{i=\Delta} \int_0^{\delta} \left| \sum_{k=0}^{k=p} \mu_k \left(\frac{x}{\delta}\right) (-1)^k \delta^p \begin{pmatrix} p \\ k \end{pmatrix} (x+i\delta)^{p-k} - (x+i\delta)^p \right|^2 dx \qquad (25)
\end{aligned}$$

For $p = 0$, equations (24) and (25) yield

$$\begin{aligned}
\|\Sigma_\delta f_p - f_p\|^2_{\mathbf{L2}(\mathbb{R})} &\geq \int_0^\delta \sum_{i=0}^{i=\Delta} \left| \mu_0 \left(\frac{x}{\delta}\right) - 1 \right|^2 dx \\
&= \Delta\delta \int_0^1 |\mu_0(x) - 1|^2 \, dx \\
&\geq (1-\delta) \int_0^1 |\mu_0(x) - 1|^2 \, dx \qquad (26)
\end{aligned}$$

Since $N \geq 0$, equation (26) together with the convergence condition (23) implies that $\mu_0 = 1$ almost everywhere and (19) is satisfied for $p = 0$.

Let us prove by recursion on $p$ that, if $N > 0$, (19) holds for $0 \leq p \leq N$. To do so, (19) is assumed to be valid for $0 \leq p < n$ with $n \leq N$. This implies that, if $N > 1$, $\mu_k$ vanishes almost everywhere for $0 < j < n$.

Taking $p = n$ in (24) and (25) gives

$$\begin{aligned}
\|\Sigma_\delta f_n - f_n\|^2_{\mathbf{L2}(\mathbb{R})} &\geq \int_0^\delta \left| \sum_{i=0}^{i=\Delta} \mu_n \left(\frac{x}{\delta}\right) (-1)^n \delta^n \right|^2 dx \\
&= \Delta \delta^{n+1} \int_0^1 |\mu_n(x)|^2 \, dx \\
&\geq \delta^n (1-\delta) \int_0^1 |\mu_n(x)|^2 \, dx
\end{aligned} \tag{27}$$

Since $N \geq n$, equation (27) together with the convergence condition (23) implies that $\mu_n = 0$ almost everywhere. Because of the recursion assumption,

$$0 = \mu_n(x) = \int K(x,y)(x-y)^n dy = x^n - \int K(x,y) y^n dy \text{ a.e.}$$

which proves that (19) holds for $p = n$.

## References


[1] Strang G. and Fix G., *A Fourier analysis of the finite element variational method*, Construct. Aspects of Funct. Anal., pp. 796-830, 1971.